Inhomogeneous Equations and Inhomogeneous Boundary Conditions[1]

Abstract evolution equations.

Recall that a (linear) equation $\mathcal{A}u = 0$, where $\mathcal{A}$ is a (linear) operator is called `homogeneous`; $\mathcal{A}u = f$ is the corresponding `inhomogeneous` equation.

An `evolution` equation is an equation involving time, and we often write evolution equations in the form

$$u_t = \mathcal{A}u + f(t), \tag{1}$$

that is, ignoring all other variables. The wave equation $u_{tt} = c^2\mathbf{\Delta}u$ can be written as (1) using the usual matrix-vector trick $u_t = v, v_t = c^2\mathbf{\Delta}u$ so that

$$\mathcal{A} = \begin{pmatrix} 0 & 1 \\ c^2\mathbf{\Delta} & 0 \end{pmatrix}.$$

The basic facts about linear evolutions equations are as follows.

(1) `Superposition principle`:The general solution of the inhomogeneous equation is equal to the sum of the general solution of the homogeneous equation and any one particular solution of the inhomogeneous equation;

(2) `Variation of parameters`: If the solution of the homogeneous equation

$$U_t = \mathcal{A}U,\ U|_{t=0} = h,$$

is written as

$$U(t) = e^{\mathcal{A}t}h \tag{2}$$

then the solution of (1) with the same initial condition $u|_{t=0} = f$ is

$$u(t) = e^{\mathcal{A}t}h + \int_0^t e^{\mathcal{A}(t-s)}f(s)\,ds. \tag{3}$$

The exponential notation in (2) is just a notation. It is motivated by the matrix exponential and is a well-known relation when $\mathcal{A}$ is a number [so that you are dealing with a first-order linear ODE]. In fact, when $\mathcal{A}$ is a matrix, then (2), (3) are also (well-known) formulas for ODEs. Similar to ODEs, (3) is often called the `variation of parameters formula`.

In the case of the heat equation in $\mathbb{R}^n$, $\mathcal{A} = a\mathbf{\Delta}$, and

$$e^{\mathcal{A}t}h = e^{\mathcal{A}t}h(x) = \frac{1}{(4\pi at)^{n/2}}\int_{\mathbb{R}^n} e^{-|x-y|^2/(4at)}h(y)\,dy$$

so that, according to (3), the solution of

$$u_t = au_{xx} + f(t,x),\ t>0,\ x\in\mathbb{R},$$

with initial condition $u(0,x) = h(x)$, is

$$u(t,x) = \frac{1}{\sqrt{4\pi at}}\int_{-\infty}^{\infty} e^{-(x-y)^2/(4at)}h(y)\,dy + \int_0^t \frac{1}{\sqrt{4\pi a(t-s)}}\left(\int_{-\infty}^{\infty} e^{-(x-y)^2/(4a(t-s))}f(s,y)\,dy\right)ds \tag{4}$$

In the same way, the solution of

$$u_{tt} = c^2u_{xx} + f(t,x),\ t>0,\ x\in\mathbb{R},$$

with initial condition $u(0,x) = h(x)$, $u_t(0,x) = g(x)$ is

$$u(t,x) = \frac{h(x+ct)+h(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy + \frac{1}{2c}\int_0^t\left(\int_{x-c(t-s)}^{x+c(t-s)} f(s,y)\,dy\right)ds. \tag{5}$$

On the one hand, both (4) and (5) are particular cases of (3) [of course, (5) requires some extra work, with back-and-forth between the original equation and its matrix-vector formulation]. On the other hand, (4)

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and (5) are important in their own rights and often go under the name `Duhamel's principle`. Note also a certain analogy between (5) and the corresponding ODE formula: if

$$y''(t) + c^2 y(t) = f(t)$$

then

$$y(t) = y(0)\cos(ct) + \frac{y'(0)}{c}\sin(ct) + \frac{1}{c}\int_0^t \sin(c(t-s))f(s)\,ds.$$

Inhomogeneous boundary conditions. So far, most of the the equations we solved completely had `homogenous` boundary conditions of the form $u|_{\partial G} = 0$. On the other hand, the interesting equation [for battle ropes and transatlantic cable] had something happening at one point of the boundary: $u(t,0) = f(t)$, that is, an `inhomogeneous` boundary condition.

The general rule in this regard is that you come up with a suitable function w that satisfies the boundary conditions, and then introduce a new unknown function $v = u - w$ so that v satisfies homogeneous boundary conditions. Depending on your choice of the function w, the equation for v might be different from the equation satisfied by u. Here are some examples (in addition to the battle ropes in the previous set of notes).

Example 1.

$$u_t = u_{xx},\ u(t,0) = A,\ u(t,L) = B$$

[heat in the rod with ends held at fixed, and different, temperatures]. Then we take

$$w(x) = A + \frac{B-A}{L}\,x$$

because $w(0) = A$, $w(L) = B$ and w is the easiest function interpolating between A and B. Then $w_t = w_{xx} = 0$ so that $u(t,x) = v(t,x) + w(x)$, where $v_t = v_{xx}$ and $v(t,0) = v(t,L) = 0$.

Example 2.

$$u_{tt} = c^2 u_{xx},\ x > 0,\ u(t,0) = f(t).$$

We start by writing $u(t,x) = v(t,x) + f(t)$, so that $v(t,0) = 0$, but we cannot stop here because the equation is still on the half-line, whereas we only know bounded intervals or the *whole* line. In other words, we have to extend the function v to the whole line while ensuring $v(t,0) = 0$. An odd extension, also known under a fancier name `the method of reflection`, does the trick:

(6)
$$v(t,x) = \begin{cases} u(t,x) - f(t), & x > 0 \\ -u(t,-x) + f(t), & x < 0. \end{cases}$$

Then, after some extra work, we get an *inhomogeneous* wave equation for v:

$$v_{tt} = c^2 v_{xx} - f''(t)\mathrm{sgn}(x),$$

where

$$\mathrm{sgn}(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0. \end{cases}$$

Then (5) leads to

$$V(t,x) = \begin{cases} f\left(t - \frac{x}{c}\right), & x < ct \\ 0 & x \geq ct. \end{cases}$$

Example 3.

$$u_t = a u_{xx},\ x > 0,\ t > 0;\ u(t,0) = f(t).$$

This time (6) leads to

$$v_t = a^2 v_{xx} - f'(t)\mathrm{sgn}(x)$$

and then (4) leads to

$$u(t,x) = \int_0^t \frac{x}{\sqrt{4\pi a(t-s)^3}}\ e^{-x^2/(4a(t-s))} f(s)\,ds.$$