MATH 445

Implicit scheme for the wave equation

Consider the one-dimensional wave equation

$$u_{tt}(x,t) = c^2 u_{xx}(x,t); 0 < x < L,\ \ 0 < t \le T; u(0,t) = u(L,t) = 0;\ \ u(x,0) = f(x),\ \ u_t(x,0) = g(x).$$

Denote by $\Delta t = \tau$ the step size in time, and by $\Delta x = h$, in space; set $m^2 = \tau^2 c^2/h^2$, $x_i = (i-1)h, i = 1,\dots,M+1$; $t_j = (j-1)\tau,\ j = 1,\dots,N+1$. Note that $L = Mh = x_{M+1}, T = N\tau = t_{N+1}$. Write $u_{i,j}$ for the approximation of $u(x_i,t_j)$,

We approximate $u_{tt}(x_i,t_j)$ by the central difference at x_i:

$$u_{tt}(x_i,t_j) \approx \frac{1}{\tau^2}\left(u_{i,j+1} - 2u_{i,j} + u_{i,j-1}\right)$$

and $u_{xx}(i\Delta x, j\Delta t)$, by the average of the corresponding central differences at t_{j+1} and t_{j-1}:

$$u_{xx}(i\Delta x, j\Delta t) \approx \frac{1}{2}\left(\frac{1}{h^2}\left(u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}\right) + \frac{1}{h^2}\left(u_{i+1,j-1} - 2u_{i,j-1} + u_{i-1,j-1}\right)\right)$$

The result is

$$u_{i,j+1} - 2u_{i,j} + u_{i,j-1} = \frac{1}{2}m^2\Big(\left(u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}\right) + \left(u_{i+1,j-1} - 2u_{i,j-1} + u_{i-1,j-1}\right)\Big)$$

or

$$-m^2 u_{i+1,j+1} + 2(1+m^2)u_{i,j+1} - m^2 u_{i-1,j+1} = 4u_{i,j} + m^2 u_{i+1,j-1} - 2(1+m^2)u_{i,j-1} + m^2 u_{i-1,j-1}. \quad (1)$$

With zero boundary conditions, you get $u_{1,j} = u_{M+1,j} = 0$. For $i = 2,\dots,M$, use the initial conditions to get $u_{i,1} = u(x_i,0) = f((i-1)h)$, $u_{i,2} \approx u_{i,1} + \tau g((i-1)h) + \frac{\tau^2}{2}u_{tt}(x_i,0)$; from the equation, $u_{tt}(x_i,0) = c^2 u_{xx}(x_i,0) = c^2 f''(x_i) \approx c^2(f(ih) - 2f((i-1)h) + f((i-2)h))/h^2$.

Then, for each $j+1 = 3,\dots,N$, (1) is a linear system for the unknown vector $(u_{2,j+1},\dots,u_{M,j+1})$, and all you need is to solve this system. The matrix A of this system is of size $(M-1)\times(M-1)$, with $2(1+m^2)$ on the main diagonal, $-m^2$ just above and below the main diagonal, and 0 elsewhere. If you write $U(j)$ for the vector-column $(u_{2,j},\dots,u_{M-1,j})$, then the system to solve is $AU(j+1) = 4U(j) - AU(j-1)$.