62

HEAT FLOW IN A SEMI-INFINITE ROD

In Chapter 57 we solved the problem of heat flow in a semi-infinite rod initially at a uniform temperature θ_0 whose end is held at a fixed temperature 0. Less picturesquely but more precisely we proved the following lemma

Lemma 62.1. *Let* $\theta_0 \in \mathbb{R}$ *and*

$$\theta(y,t)=\frac{\theta_0}{2\sqrt{(\pi Kt)}}\int_0^\infty\left\{\exp\left(-\frac{(y-w)^2}{4Kt}\right)-\exp\left(-\frac{(y+w)^2}{4Kt}\right)\right\}dw$$

for all $y \geqslant 0$, $t>0$. *Then* $\theta:[0,\infty)\times(0,\infty)\to\mathbb{R}$ *is an infinitely differentiable function with*

(i) $(\partial\theta/\partial t)(y,t)=K(\partial^2\theta/\partial y^2)(y,t)$,
(ii) $\theta(y,t)\to\theta_0$ *as* $t\to 0+$ *for all* $y>0$,
(iii) $\theta(0,t)=0$ *for all* $t>0$.

Proof. This is Lemma 57.1. ■

By adding a constant we see that a solution to the problem in which the rod is initially at a uniform temperature 0 and the end is held at temperature θ_1 is

$$\theta(y,t)=\theta_1-\frac{\theta_1}{2\sqrt{(\pi Kt)}}\int_0^\infty\left\{\exp\left(-\frac{(y-w)^2}{4Kt}\right)-\exp\left(-\frac{(y+w)^2}{4Kt}\right)\right\}dw.$$

In this chapter we shall attack the more general problem in which the rod is initially at a uniform temperature 0 and the end is held at a varying temperature $f(t)$. More explicitly we shall try to solve the equations

(i) $(\partial\theta/\partial t)(y,t)=K(\partial^2\theta/\partial y^2)(y,t)[y,t>0]$,
(ii) $\theta(y,t)\to\theta_0$ as $t\to 0+$ for $y>0$,
(iii) $\theta(y,t)\to f(t)$ as $y\to 0+$ for $t>0$.

The discussion that follows is heuristic, though there is only one major gap in the reasoning. (I shall point this gap out when we come to it.)

Write $H_s(t)=0$ for $t \leqslant s$, $H_s(t)=1$ for $s<t$. We have solved the problem above for $f(t)=\theta_1 H_0(t)$ and this can be made the basis for a tentative solution when $f(t)=\lambda H_s(t)[s>0]$. For, if we set

$$\theta_s(y,t)=0 \quad \text{for } t \leqslant s,$$
$$\theta_s(y,t)=1-\frac{1}{2\sqrt{(\pi K(t-s))}}\int_0^\infty \left\{\exp\left(-\frac{(y-w)^2}{4K(t-s)}\right) - \exp\left(-\frac{(y+w)^2}{4K(t-s)}\right)\right\} dw \quad \text{for } t>s$$

then, certainly, $\lambda\theta_s$ satisfies conditions (i) and (iii) (with $f=\lambda H_s$) except possibly when $t=s$. (In fact it also satisfies them when $t=s$ but since the argument is heuristic, we shall not pause to verify this.)

The heat equation is linear and so, in particular, if

$$0<s(1)<s(2)<\cdots<s(n) \text{ and } \theta=\sum_{j=1}^{n}\lambda_j\theta_{s(j)},$$

(i) $(\partial\theta/\partial t)(y,t)=K(\partial^2\theta/\partial y^2)(y,t)$ for $y>0$, $t>0$, $t \neq s(1),\ldots,s(n)$,
(ii) $\theta(y,t)=0$ for $0 \leqslant t < s(1)$,
(iii) $\theta(y,t)\to\sum_{j=1}^{k}\lambda_j$ as $y\to 0+$ for $s(k)<t<s(k+1)$,
$\theta(y,t)\to\sum_{j=1}^{n}\lambda_j$ as $y\to 0+$ for $s(n)<t$.

Ignoring the (not very difficult) convergence problems involved, we may suspect that for reasonable choices of a sequence $0<s(1)<s(2)<\cdots$ and $\lambda_j\in\mathbb{C}$ the infinite sum $\theta=\sum_{j=1}^{\infty}\lambda_j\theta_{s(j)}$ will also satisfy

(i) $(\partial\theta/\partial t)(y,t)=K(\partial^2\theta/\partial y^2)(y,t)$ for $y>0, t>0, t\neq s(j)$,
(ii) $\theta(y,t)=0$ for $0\leqslant t<s(1)$,
(iii) $\theta(y,t)\to\sum_{j=1}^{k}\lambda_j$ as $y\to 0+$ for $s(k)<t<s(k+1)$.

If we now take a well behaved function f and some small $\delta>0$ then, putting $s(j)=j\delta$, $\lambda_1=f(\delta)$ and $\lambda_j=f((j+1)\delta)-f(j\delta)$, we conjecture that if

$$(*) \quad \phi_\delta(y,t)=\sum_{j=1}^{\infty}(f((j+1)\delta)-f(j\delta))\theta_{j\delta}(y,t),$$

then

(i) $(\partial\phi_\delta/\partial t)(y,t)=K(\partial^2\phi_\delta/\partial y^2)$ for $y>0, t>0, t\neq s(j)$,
(ii) $\phi_\theta(y,t)=0$ for $0<t<\delta$,
(iii) $\phi_\theta(y,t)\to f(j\delta)$ as $y\to 0+$ for $j\delta<t<(j+1)\delta$.

We now let $\delta\to 0$, and it is here that there is the widest gap in our argument. With luck we expect that

$$\phi_\delta(y,t)=\sum_{j=1}^{\infty}\frac{f((j+1)\delta)-f(j\delta)}{\delta}\theta_{j\delta}(y,t)\delta\to\int_0^\infty f'(s)\theta_s(y,t)\,ds=\phi(y,t), \text{ say}$$

and that

(i) $(\partial\phi/\partial t)(y,t) = K(\partial^2\phi/\partial y^2)(y,t)$ for $y>0, t>0$,
(ii) $\phi(y,t)\to 0$ as $t\to 0+$ for $y>0$,
(iii) $\phi(y,t)\to f(t)$ as $y\to 0+$ for $t>0$.

To get an expression for ϕ involving f rather that f' we integrate by parts to obtain

$$\phi(y,t) = \int_0^\infty f'(s)\theta_s(y,t)\,ds = [f(s)\theta_s(y,t)]_0^\infty - \int_0^\infty f(s)\left(\frac{\partial}{\partial s}\theta_s(y,t)\right)ds$$
$$= -\int_0^\infty f(s)\left(\frac{\partial}{\partial s}\theta_s(y,t)\right)ds = -\int_0^t f(s)\left(\frac{\partial}{\partial s}\theta_s(y,t)\right)ds,$$

since $\theta_s(y,t)=0$ for $s>t$.

The reader might like to pause to consider what we have done so far, but a further simplification is possible. Writing

$$\psi_0(x,t) = \frac{1}{2(\pi K t)^{\frac{1}{2}}}\int_0^\infty \exp\left(-\frac{(x-w)^2}{4Kt}\right)dw,$$

we see that

$$\theta_s(y,t) = \theta_0(y,t-s) = 1-\psi_0(y,t-s)+\psi_0(-y,t-s),$$

and so, by direct computation, or an appeal to Lemma 55.7 (ii), we have

$$\frac{\partial\theta_s}{\partial s}(y,t) = -D_2\theta_0(y,t-s) = D_2\psi_0(y,t-s)-D_2\psi_0(-y,t-s)$$
$$= \frac{-y}{4\pi^{\frac{1}{2}}K^{\frac{1}{2}}(t-s)^{3/2}}\exp\left(\frac{-y^2}{4K(t-s)}\right) - \frac{y}{4\pi^{\frac{1}{2}}K^{\frac{1}{2}}(t-s)^{3/2}}\exp\left(-\frac{(-y)^2}{4K(t-s)}\right)$$
$$= \frac{-y}{2\pi^{\frac{1}{2}}K^{\frac{1}{2}}(t-s)^{3/2}}\exp\left(\frac{-y^2}{4K(t-s)}\right) \quad \text{for } t>s.$$

In this way we have arrived at a putative solution

$$\phi(y,t) = \int_0^t f(s)\frac{y}{2\pi^{\frac{1}{2}}K^{\frac{1}{2}}(t-s)^{3/2}}\exp\left(-\frac{y^2}{4K(t-s)}\right)ds$$

to our problem. It remains to check our guess.

Theorem 62.2. *Suppose that $f:(0,\infty)\to\mathbb{C}$ is continuous and bounded. Then if we define $\phi:(0,\infty)\times(0,\infty)\to\mathbb{C}$ by*

$$\phi(x,t) = \int_0^t f(s)\frac{x}{2\pi^{\frac{1}{2}}K^{\frac{1}{2}}(t-s)^{3/2}}\exp\left(-\frac{x^2}{4K(t-s)}\right)ds,$$

if follows that ϕ is an infinitely differentiable function on $(0,\infty)\times(0,\infty)$ with

(i) $(\partial\phi/\partial t)(x,t) = K(\partial^2\phi/\partial x^2)(x,t)$ *for* $x, t > 0$,
(ii) $\phi(x,t) \to 0$ *as* $t \to 0+$ *for* $x > 0$,
(iii) $\phi(x,t) \to f(t)$ *as* $x \to 0+$ *for* $t > 0$.

Proof. The proof follows the same pattern as Theorem 55.4. It is not very interesting and I suggest that the reader just skims through it.
Step 1. (We show that we can differentiate under the integral sign.) Set

$$G(s,x,t) = f(s)\frac{x}{2\pi^{\frac{1}{2}}K(t-s)^{3/2}}\exp\left(-\frac{x^2}{4K(t-s)}\right)$$

if $t > s > 0$, $G(s,x,t) = 0$ otherwise. We claim that $G:\mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{C}$ is infinitely differentiable in x and t. This the reader will readily allow except possibly when $t = s$. To deal with the case $t = s$ we observe that the same proof as in Example 4.2 shows that, if

$$\begin{aligned} g(u) &= u^{-3/2}\exp(-1/u) \quad &&\text{for } u > 0,\\ g(u) &= 0 &&\text{for } u \leqslant 0, \end{aligned}$$

then g is infinitely differentiable everywhere including 0.

Since $G(s,x,t) = 0$ for $s \notin [0,t]$, it follows that $D_2^j D_3^k G(s,x,t) = 0$ for $s \notin [0,t]$ so that $D_2^j D_3^k G(\ ,x,t) \in L^1 \cap C$ and, if $t \in [T_1, T_2]$,

$$\int_{|s| \geqslant R} |D_2^j D_3^k G(s,x,t)|\,ds = 0 \quad \text{for} \quad R > \max(|T_1|, |T_2|).$$

Thus using the appropriate modification to Theorem 53.4 repeatedly we see that ϕ is infinitely differentiable with

$$\frac{\partial^j}{\partial x^j}\frac{\partial^k}{\partial t^k}\phi(x,t) = \int_{-\infty}^{\infty} D_2^j D_3^k G(s,x,t)\,ds.$$

Step 2. If we show that $D_2 G(x,s,t) = K D_3^2 G(x,s,t)$,
it will follow from Step 1 that ϕ satisfies the heat equation. We can verify this directly but it is simpler to observe that writing

$$E_{1/\surd(2Kt)}(x) = \frac{1}{2\pi^{\frac{1}{2}}(Kt)^{\frac{1}{2}}}\exp\left(-\frac{x^2}{4Kt}\right),$$

we have $$G(x,s,t) = f(s)\frac{\partial}{\partial x}E_{1/\surd(2K(t-s))}(x) \quad \text{for} \quad t > s > 0.$$

But we know that $E_{1/\surd(2Kt)}(x)$ satisfies the heat equation and so

$$\begin{aligned} \frac{\partial}{\partial t}\frac{\partial}{\partial x}E_{1/\surd(2Kt)}(x) &= \frac{\partial}{\partial x}\frac{\partial}{\partial t}E_{1/\surd(2Kt)}(x) = K\frac{\partial}{\partial x}\frac{\partial}{\partial x^2}E_{1/\surd(2Kt)}(x)\\ &= K\frac{\partial}{\partial x^2}\frac{\partial}{\partial x}E_{1/\surd(2Kt)}(x) \end{aligned}$$

and the required result now follows.

Step 3. Let

$$P(u) = \frac{1}{(2\pi)^{\frac{1}{2}} u^{3/2}} \exp\left(-\frac{1}{2u}\right) \quad u > 0,$$
$$P(u) = 0 \qquad\qquad u \leqslant 0,$$

(i.e. let $P(u) = 2\pi^{-\frac{1}{2}} g(2u)$). Then $P:\mathbb{R}\to\mathbb{R}$ is continuous and has the following properties

(i) $P(u) \geqslant 0$ for all u,
(ii) $u(P(u)) \to 0$ as $|u| \to \infty$,
(iii) $$\int_{-\infty}^{\infty} P(u)\,du = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_0^{\infty} \frac{1}{u^{\frac{3}{2}}} \exp\left(-\frac{1}{2u}\right) du = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{\infty}^{0} -2\exp\left(-\frac{v^2}{2}\right) dv$$
$$= \frac{2}{(2\pi)^{\frac{1}{2}}} \int_0^{\infty} \exp\left(-\frac{v^2}{2}\right) dv = 1,$$

making the substitution $v^2 = 1/u$ and remembering (for example from Lemma 48.10) that $\int_{-\infty}^{\infty} \exp(-t^2/2)\,dt = (2\pi)^{\frac{1}{2}}$.

Writing $\tilde{f}(t) = f(t)$ for $t \geqslant 0$, $\tilde{f}(t) = 0$ for $t < 0$ we see that with the notation of Theorem 51.5 we have

$$\phi(x,t) = \int_0^t \tilde{f}(s) P_{2K/x^2}(t-s)\,ds = \int_{-\infty}^{\infty} \tilde{f}(s) P_{2K/x^2}(t-s)\,ds = \tilde{f} * P_{2K/x^2}(t)$$
$$\to \tilde{f}(t) = f(t) \quad \text{as } x \to 0+ \text{ for all } t > 0.$$

All that remains is to verify that $\phi(x,t) \to 0$ as $t \to 0+$ for all $x > 0$. But

$$\phi(x,t) = \tilde{f} * P_{2K/x^2}(t) = \int_{-\infty}^{\infty} \tilde{f}(t-s) P_{2K/x^2}(s)\,ds = \int_0^t f(t-s) P_{2K/x^2}(s)\,ds,$$

so $$|\phi(x,t)| \leqslant \sup_{u\in\mathbb{T}} |f(u)| \int_0^t P_{2K/x^2}(x)\,ds \to 0 \quad \text{as } t \to 0+,$$

and we are done. ■

Suppose that we change the temperature at the end of the rod for a short time and then return it to its initial value. How will the resulting 'heat pulse' travel down the rod? Ignoring as usual the problem of uniqueness, we shall consider the question in the following form.

Problem. *Suppose $f:(0,\infty)\to\mathbb{R}$ is a continuous function with $f(t) = 0$ for $t \geqslant a$. What can we say about the behaviour of $\phi(x,t)$?*

We can make two remarks at once.

(1) If $f(u) > 0$ for all $0 < u < a$ then

$$\phi(x,t)=\int_0^t f(s)P_{2K/x^2}(t-s)\,ds>0 \quad \text{for all } t>0,\ x>0,$$

i.e. there is an instantaneous effect at all points.

(2) However, if $Kt \ll x^2$ (i.e. if Kt is small compared with x^2), then

$$|\phi(x,t)| \leqslant \sup_{u\in\mathbb{R}}|f(u)|\int_0^t P_{2K/x^2}(s)\,ds = \sup_{u\in\mathbb{R}}|f(u)|\int_0^{2Kt/x^2} P(v)\,dv$$

$$\leqslant \sup_{u\in\mathbb{R}}|f(u)|\sup_{v\in\mathbb{R}}|P(v)|2Kt/x^2 \ll 1,$$

so the effect is negligible at distances x which are large compared with $\sqrt{(Kt)}$.

If t/a is of the order of 1 (or less) then the behaviour of $\phi(x,t)$ depends on the precise behaviour of f. But if $t \gg a$ then

$$\phi(x,t)=\int_0^t f(s)P_{2K/x^2}(t-s)\,ds=\int_0^a f(s)P_{2K/x^2}(t-s)\,ds \approx \int_0^a f(s)P_{2K/x^2}(t)\,ds$$

(since
$$P_{2K/x^2}(t-s)=\frac{x}{2\pi^{\frac{1}{2}}(K(t-s))^{\frac{3}{2}}}\exp\left(-\frac{x^2}{K(t-s)}\right)$$
$$\approx \frac{x}{2\pi^{\frac{1}{2}}(Kt)^{\frac{3}{2}}}\exp\left(-\frac{x^2}{Kt}\right) \quad \text{for all } 0\leqslant s\leqslant a).$$

Hence
$$\phi(x,t)\approx \int_0^a f(s)\,ds\,P_{2K/x^2}(t).$$

Thus we have the following remark.

(3) If $t \gg a$ the 'shape' of the pulse $\phi(x,t)$ is independent of the 'shape' of f and its magnitude depends only on $\int_0^a f(s)\,ds$ (and, of course, x and t).

Using Remark (2) we obtain the following version of (3).

(4) If $x^2 \gg Ka$ then the shape of the pulse $\phi(x,\ )$ depends only on x and its magnitude depends only on $\int_0^a f(s)\,ds$ and x.

Thus the three pulses in Figure 62.1 produce much the same pulse $\phi(x,\ )\approx AP_{2K/x^2}(\ )$ (with $A=\int_0^a f(s)\,ds$) for large x.

To describe the pulse shape in more detail we make some preliminary observations.

Lemma 62.3. (i) *Let*

$$P(u)=(2\pi)^{-\frac{1}{2}}u^{-\frac{3}{2}}\exp(-1/2u) \quad [u>0].$$

Then P increases from 0 to $M=(2\pi)^{-\frac{1}{2}}3^{\frac{3}{2}}\exp(-3/2)$ as u runs from 0 to $T_0=1/3$ and then decreases to 0 as u runs from T_0 to ∞. In particular if $0<\delta<1$ there exist unique $T_1(\delta)$ and $T_2(\delta)$ with $0<T_1(\delta)<1/3<T_2(\delta)$ and $P(u)\geqslant \delta M$ for $u\in[T_1(\delta),T_2(\delta)]$, $P(u)<\delta M$ otherwise.

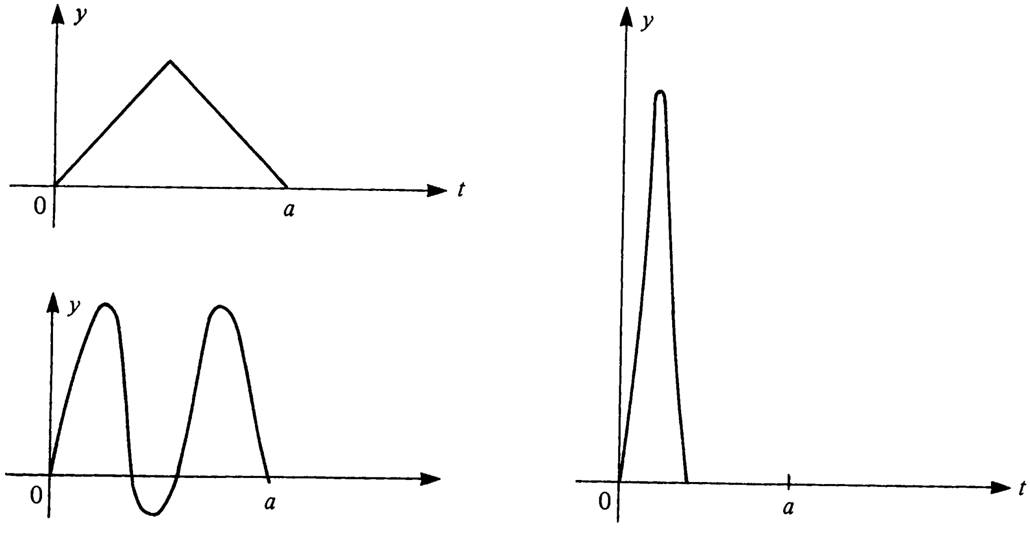

Fig. 62.1. Possible initial pulses.

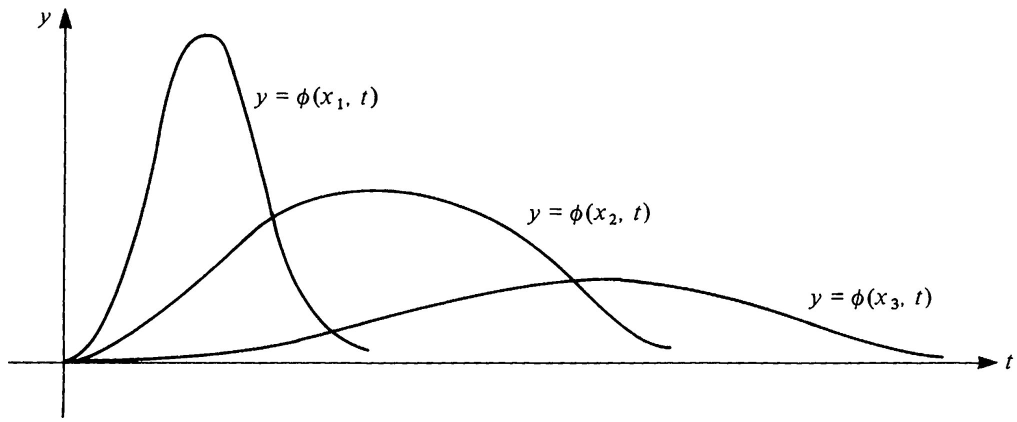

Fig. 62.2. Pulse shapes at various distances.

(ii) *Let* $Q(t) = AP_{2K/x^2}(t)[t > 0]$ *where* A, K, $x > 0$. *Then with the notation of* (i), Q *increases from* 0 *to* $(2K/x^2)MA$ *as* u *runs from* 0 *to* $(x^2/2K)T_0$ *and then decreases to* 0 *as* u *runs to* ∞. *If* $0 < \delta < 1$ *then*

$$Q(t) \geqslant 2K\delta MA/x^2 \text{ for } t \in [(x^2/2K)T_1(\delta), (x^2/2K)T_2(\delta)],\ Q(t) < \delta MA/x^2 \text{ otherwise.}$$

Proof. (i) Observe that

$$P'(u) = \frac{1}{(2\pi)^{\frac{1}{2}}}\left(-\frac{3}{2u^{\frac{5}{2}}} + \frac{1}{2u^{\frac{7}{2}}}\right)\exp\left(-\frac{1}{2u}\right) = \frac{1}{(2\pi)^{\frac{1}{2}}}\frac{1-3u}{2u^{\frac{7}{2}}}\exp\left(-\frac{1}{2u}\right).$$

(ii) Since $Q(t) = (A2K/x^2)P(2Kt/x^2)$ this follows directly from (i). ∎

Rewriting Lemma 62.3 in vaguer terms we obtain our concluding remark.

(5) If $x^2 \gg Ka$ then the pulse rises for a time proportional to x^2 (and inversely proportional to K) to a maximum which is inversely proportional to x^2 (and proportional to K) and then declines to 0 (Figure 62.2). The length of time during which the pulse is greater than a given fraction of its maximum is proportional to x^2 (and inversely proportional to K).