Calculus I: Homework problems (Fall 2020)

General guideline. Use any help you want, including people, computer algebra systems, Internet, and solution manuals, but make sure you are ready for quizzes and exams, when no help is allowed and you are on your own.

Some notations:

$$\sum_{k=1}^{n} a_k = a_1+a_2+\ldots+a_n,\ \prod_{k=1}^{n} a_k = a_1\cdot a_2\cdot\ldots\cdot a_n,\ n! = \prod_{k=1}^{n} k,\ 0! = 1,\ \exp(x) = e^x,\ \ln x = \log_e x,$$

$f(x)|_{x=x_0} = f(x_0)$, DNE = does not exist.

To keep in mind:

$$+\infty = \infty$$

Homework 1.

Compute the following limits:

$$\sqrt{2},\ \sqrt{2\sqrt{2}},\ \sqrt{2\sqrt{2\sqrt{2}}},\ldots; \qquad \sqrt{2},\ \sqrt{2+\sqrt{2}},\ \sqrt{2+\sqrt{2+\sqrt{2}}},\ldots;$$

$$\lim_{x\to 7}\frac{2-\sqrt{x-3}}{x^2-49}; \qquad \lim_{x\to 8}\frac{x-8}{x^{1/3}-2};$$

$$\lim_{x\to 0}\frac{x^2+x+\sin 3x}{x^3+\tan 2x}; \qquad \lim_{x\to 0}\frac{1-\cos x}{x^2}; \qquad \lim_{x\to 0} x^{1/3}\sin\frac{1}{x};$$

$$\lim_{x\to 0-}\frac{|\sin x|}{x}; \qquad \lim_{x\to 1}\frac{\sqrt{x^2+x-1}-1}{\sqrt{3x^2+2x-4}-1}$$

"Solutions"

$$x^2 = 2x \ \Rightarrow\ x = 2; \qquad x^2 = 2+x \Rightarrow\ x = 2;$$

$$-\left.\frac{1}{(x+7)(2+\sqrt{x-3})}\right|_{x=7}; \qquad (x^{2/3}+2x^{1/3}+4)|_{x=8} = 12;$$

$$\frac{0+1+3}{0+2} = \frac{1}{2}; \qquad \frac{2\sin^2(x/2)}{x^2} \to \frac{1}{2}; \qquad |\cdot| \le |x|^{1/3} \to 0;$$

$$-1; \qquad \left.\frac{(x+2)(\sqrt{3x^2+2x-4}+1)}{(3x+5)(\sqrt{x^2+x-1}+1)}\right|_{x=1}.$$

Homework 2.

Compute the following limits:

$$\lim_{x\to+\infty}\frac{(x+1)^2}{2x^2+1}; \qquad \lim_{x\to+\infty}\frac{\sqrt{x}}{\sqrt{x+\sqrt{x+\sqrt{x}}}};$$

$$\lim_{x\to+\infty} x(\sqrt{x^2+1}-x); \qquad \lim_{x\to+\infty} x\sin\frac{1}{x};$$

$$\lim_{x\to+\infty}\left(\frac{x+1}{2x+1}\right)^x; \qquad \lim_{x\to+\infty}\left(\frac{2x+3}{2x-2}\right)^x;$$

$$\lim_{x\to 1+}\exp\left(\frac{1}{1-x^2}\right); \qquad \lim_{x\to 0} e^{1/x}.$$

"Solutions"

$$\frac{x^2}{2x^2}=\frac{1}{2};\qquad \frac{\sqrt{x}}{\sqrt{x}}=1;$$
$$\frac{x}{\sqrt{x^2+1}+x}\to\frac{1}{2};\qquad \lim_{t\to 0+}\frac{\sin t}{t}=1;$$
$$\frac{1}{2^x}\to 0;\ 2x+3=2x-2+5,\qquad \left(1+\frac{5}{2x}\right)^x\to e^{5/2};$$
$$e^{-\infty}=0;\qquad 0+: e^{+\infty}=+\infty,\ 0-: e^{-\infty}=0\ \Rightarrow \text{DNE}.$$

Homework 3.

Problem 1. Compute the following limits:

$$\lim_{x\to 32}\frac{x^{2/5}-4}{x-32};\qquad \lim_{x\to 0}\frac{e^{\sin x}-1}{x};\qquad \lim_{x\to 2}\frac{x^{2x}-16}{x-2};$$
$$\lim_{x\to\pi}\frac{1-\sin(x/2)}{\pi-x};\qquad \lim_{x\to 1}\frac{1-x^2}{\sin(\pi x)};\qquad \lim_{x\to 1}(1-x)\tan(\pi x/2).$$

"Solutions" In each case, the answer involves $f'(a)$ for suitable function $f=f(x)$ and the point a:

$$f(x)=x^{2/5},\ a=32;\ f(x)=e^{\sin x},\ a=0;\ f(x)=x^{2x},\ a=2;$$
$$f(x)=\sin(x/2),\ a=\pi;\ f(x)=\sin(\pi x),\ a=1;\ f(x)=\cos(\pi x/2),\ a=1.$$

Problem 2. In each case, compute f'; simplify the answer as much as possible:

$$f(x)=\sin(\cos(x));\qquad f(x)=\big(\sin x\big)^{\cos x};\qquad f(x)=\frac{x}{\sqrt{1+x^2}};$$
$$f(x)=\ln\big(\sec(x)+\tan(x)\big);\qquad f(x)=2^{(\cos x)^{\ln x}};\qquad f(x)=\sec\Big(\cos\big(\tan(x)\big)\Big);$$
$$f(x)=x\ln x-x;\qquad f(x)=\prod_{k=1}^{212}\Big((x-k)^{\frac{10-k}{k}}e^{-k^2x}\Big).$$

Problem 3. Compute the first and second derivatives of the functions

$$f(x)=\frac{x}{1+x^2};\qquad f(x)=x^2e^{-\frac{x^2}{2}}.$$

Problem 4. Compute 10-th derivative of the functions

$$f(x)=\sin x,\quad f(x)=x\sin x,\ f(x)=x^{15}\sin(2x).$$

Here you can use the *Leibnitz formula* for the n-th derivative of the product of two functions:

$$\big(f(x)g(x)\big)^{(n)}=\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}f^{(n-k)}(x)g^{(k)}(x).$$

Problem 5. Let $y=y(x)$ be defined implicitly by $3x^2-2xy+y^3=3$. Compute $y'(x)$ and $y''(x)$.

Homework 4.

Problem 1. Find the dimensions of the rectangle with the greatest area that can be inscribed in the circle of radius R. Explain why your answer is indeed the maximum. Then answer the same question for a rectangle inscribed in an ellipse with semi-axis a and b, when the sides of the rectangle are parallel to the axis of the ellipse.

Problem 2. What dimensions should a closed cylinder have to minimize the overall surface area given the volume? Then answer the same question when cylinder is open at the top. In each case, explain why your answer is indeed the minimum.

Problem 3. Find the dimensions of a right circular cone with the greatest volume that can be inscribed in a sphere of radius R. Explain why your answer is indeed the maximum.

Problem 4. Sand falls from a conveyor belt into a conical pile at a rate of 10 cubic meters per minute. The radius of the base of the pile is always half the height of the pile. How fast are the height and the radius of the pile changing when the pile is 5 meters high?

Problem 5. Consider the curve $3x^2 - 2xy + y^2 = 3$.
(a) Write the equation of the tangent line to the curve at the point $(1, 0)$.
(b) The point $(1.01, y)$ is on the curve. Compute approximately the values of y.
(c) Identify the vertical and horizontal tangents to the curve.
(d) What kind of curve is this?

Homework 5.

Problem 1. F11-3 (Final exam Fall 2011 number 3).

Problem 2. S02-8 (Final exam Spring 2002 number 8).

Problem 3. F11-9.

Problem 4. Find the dimensions of a right circular cone with the smallest volume that can be circumscribed about a sphere of radius R. Explain why your answer is indeed the minimum.

Problem 5. An open container has the shape of a cylinder with a hemisphere at the bottom; the radius of the cylinder is the same as the radius of the hemisphere. Find the dimensions of the container to minimize the surface area given the volume. Explain why your answer is indeed the minimum.

Homework 6.

Problem 1. Compute the following integrals:

$$\int (\sqrt{x}+1)(x-\sqrt{x}+1)\,dx; \qquad \int 3^x e^x\,dx; \qquad \int x^2(2x^3-5)^{7/5}\,dx;$$

$$\int \tan^2 x\,dx \ [(\sec x)' = \sec x \tan x]; \qquad \int \frac{e^x}{e^x-1}\,dx \qquad \int (x+a)^r x^2\,dx;$$

$$\int \frac{2ax+b}{ax^2+bx+c}\,dx; \qquad \int \tan x\,dx; \qquad \int \frac{1+x}{1+\sqrt{x}}\,dx.$$

Problem 2. Compute the following integrals:

$$\int_0^4 \frac{dx}{1+\sqrt{x}}; \qquad \int_1^2 x^2(x-1)^{1/3}\,dx; \qquad \int_{-1}^1 \left(\sqrt{1+x+x^2}-\sqrt{1-x+x^2}\right)dx;$$

$$\int_0^1 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{1-x}}\,dx; \qquad \int_0^4 \sqrt{16-x^2}\,dx; \qquad \int_0^1 \sqrt{2x-x^2}\,dx.$$

Homework 7.

Problem 1. Compute the following limits

$$\lim_{n\to+\infty} \frac{1}{n^{r+1}} \sum_{k=1}^{n} k^r, \ r>0; \qquad \lim_{n\to+\infty} \frac{1}{n^2} \sum_{k=1}^{n} \sqrt{n^2-k^2};$$

$$\lim_{n\to+\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \ln n \right); \qquad \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k+n}; \qquad \text{Final Spring 2002 number 10.}$$

Problem 2. Prove the following inequalities

$$\ln n < \sum_{k=1}^{n} \frac{1}{k} < \ln(n+1); \qquad e^x \geq 1+x; \qquad \ln(1+x) \leq x;$$

$$(1+x)^r \leq 1+rx, \ r\in(0,1), \ x>0; \qquad (1+x)^r \geq 1+rx, \ r>1, \ x>0.$$

Homework 8.

Problem 1. Sketch the solution curves for the following equations:

$$y'(x) = (y+1)^3 y^2 (y-1); \quad y'(x) = y(1-y^2).$$

Problem 2. Identify the inflection points of the solution curves for the following equations:

$$y'(x) = y(1-y); \qquad y'(x) = y(1-y^2).$$

Problem 3. Determine the general solution of the following equations:

$$y'(x) = ay+b; \qquad y'(x) = -y^2; \qquad y'(x) = -xy.$$

Problem 4. F09-8; F10-12.

Problem 5. S15-8 [The problems on exponential growth/decay are straightforward, so there is only one in the homework].

Homework 9. This is a review of many of the things we studied.

Problem 1. Compute the following limits:

$$\lim_{x\to0} \frac{\tan x - \sin x}{x^3}; \qquad \lim_{x\to1} \left(\frac{1}{1-x} - \frac{3}{1-x^3} \right); \qquad \lim_{x\to2} \frac{\sqrt{x^2+4x-3}-3}{\sqrt{2x^2+x-6}-2};$$

$$\lim_{x\to\infty} \left(x + (1-x^3)^{1/3} \right); \qquad \lim_{x\to\pi/4} \frac{\sin x - \cos x}{1-\tan x}; \qquad \lim_{x\to1-} \frac{1}{1+\exp\left(-\frac{1}{1-x}\right)};$$

$$\lim_{n\to\infty} \frac{1}{n^{10}} \sum_{k=1}^{n} k^9; \qquad \lim_{n\to\infty} \frac{1}{n^{5/2}} \sum_{k=1}^{n} k\sqrt{n+k}.$$

Problem 2. Consider the function

$$f(x) = \begin{cases} \dfrac{\sin 2x}{x}, & x>0; \\ ax+b, & x<0. \end{cases}$$

(a) Determine the values of a and b for which the function f is continuous at $x=0$;
(b) Determine the values of a and b for which the function f is differentiable at $x=0$;
(c) Are there any values of a, b for which the function f is twice differentiable at $x=0$? Explain your conclusion.

Problem 3. The objective is to confirm that, for every $p, q > 0$,

$$\lim_{x\to 0+} x^p |\ln x|^q = 0, \quad \lim_{x\to+\infty} \frac{(\ln x)^q}{x^p} = 0. \tag{1}$$

For example, $\lim\limits_{x\to+\infty} x^{-0.0001}(\ln x)^{10000} = 0.$

(a) Argue that it is enough to confirm one of the limits; the other will follow after replacing x with $1/x$.

(b) Let us say we want to confirm the second limit. Argue that it is enough to show that

$$\lim_{x\to+\infty} \frac{\ln x}{x} = 0;$$

the rest will follow by using that $(\ln x)^q x^{-p} = \left(x^{-p/q}\ln x\right)^q$ and $\ln x^r = r\ln x$.

(c) Now complete the argument using the squeeze theorem: for $x > 1$, we have $x^{-1/2} > x^{-1} > 0$, so (after integration) $2(x^{1/2} - 1) > \ln x > 0$, and it remains to divide everything by x.

(d) Finally, confirm that the second equality in (1) implies $\lim\limits_{x\to+\infty} x^q e^{-px} = 0$ for all $p, q > 0$. For example, $\lim\limits_{x\to+\infty} x^{10000}e^{-0.0001x} = 0.$

Problem 4. (a) Compute the first derivative of the function

$$f(x) = \ln(1 + \tan x) + (1 + x^2)^{x\sqrt{\sin x}}.$$

Do not simplify the answer.

(b) Compute the derivative of the functions

$$f(x) = \ln\left(x + \sqrt{1 + x^2}\right), \quad g(x) = \frac{x\sqrt{1+x^2}}{2} + \frac{1}{2}\ln\left(x + \sqrt{1+x^2}\right).$$

Simplify the answers as much as possible.

Problem 5. At noon, car C1 is 100 miles east of point A and is moving west (toward point A) at 30 miles per hour, and car C2 is 50 miles north of point A and is moving north (away from point A) at 40 miles per hour.

(a) Compute the rate of change of the distance between the cars at 12:06pm and at 1pm.

(b) When is the distance between the cars the shortest?

Problem 6. Consider the function

$$F(x) = \int_{x^2}^{x^3} \frac{dt}{1 + \sqrt{t}}, \quad x > 0.$$

(a) Compute $F'(x)$.

(b) Confirm that $F'(a) = 0$ for exactly one value of $a \in (0, 1)$.

(c) Confirm that the F is one-to-one on $(a, +\infty)$.

(d) Let $G = G(x)$ be the inverse function of F defined for $x > F(a)$. Compute $G'(0)$.

Note that you can integrate

$$\int \frac{dt}{1 + \sqrt{t}}$$

using substitution $u = \sqrt{t}$. Evaluate the integral, derive the expression for F and confirm that your computations of the derivative were correct.

Finally, sketch the graph of F.

Problem 7. Sketch the graphs of the functions

$$f(x) = x^2 + \frac{1}{x}, \quad g(x) = x + \frac{1}{x^2}, \quad h(x) = \frac{x}{1 + x^2}.$$

Problem 8. Compute the following integrals:

$$\int_0^1 \left(2x + x^2\right)^2 dx; \qquad \int_0^{\pi} (3\sin(x/2) - \cos(x/3))\, dx;$$

$$\int \frac{\sin(x)}{\sqrt{1+\cos(x)}}\, dx; \qquad \int_1^{2^{1/4}} x^{11}\,\sqrt{x^4-1}\, dx;$$

$$\int \frac{2-x^2}{6x-x^3}\, dx; \qquad \int_{-5}^{5} \left(\frac{x^3}{2^x+2^{-x}} + \frac{3^x-3^{-x}}{1+x^2} + x^{22}\ln\frac{20+x}{20-x}\right) dx.$$

Problem 9. Let $f = f(x),\ x \in \mathbb{R}$, be a differentiable function.

(a) Assume that the equation $f(x) = 0$ has exactly 4 distinct real solutions. How many distinct real solutions can equation $f'(x) = 0$ have?

(b) Assume that the equation $f'(x)$ has exactly 4 distinct real solutions. How many distinct real solutions can equation $f(x) = 0$ have?

(c) Food for thought: in each of the two questions, are there any major differences between the answers depending on whether the number of distinct solutions is even or odd? For example, how will your answer in (a) change if there are exactly three distinct solutions?

Problem 10. Consider the curve $x^2y + xy^5 = 2$.

(a) Write the equation of the tangent line to the curve at the point $(1, 1)$.

(b) The point $(0.95, y)$ is on the curve. Compute approximately the values of y.

(c) Identify the vertical and horizontal tangents to the curve.

Homework 10.

Create your own version of the final exam for this class, and then solve it. Be creative, but also realistic.