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BEHAVIOUR AT POINTS OF DISCONTINUITY II

We have seen in Chapter 8 how Kelvin invented machines which could compute periodic functions from their Fourier series and conversely obtain the Fourier series of a given periodic function. One such machine was constructed by Michelson to work to a higher accuracy and to involve many more terms than previous models. (Michelson's ability to build and operate equipment to new standards of accuracy was legendary. Of his interferometer which he invented and used in the Michelson Morley experiments it was said that it was a remarkable instrument – provided you had Michelson to operate it. His experiments to measure the diameter of the nearest stars using an interferometer were not reproduced for 30 years.)

Michelson tested his machine by feeding in the first 80 Fourier coefficients of the sawtooth function h defined in Chapter 16. To his surprise the machine did not produce an exact sawtooth but instead added two little blips on either side of the discontinuity as shown in Figure 17.1. Even after making every effort to remove any mechanical defects which could account for them, the blips still remained. Finally hand calculation confirmed the existence of blips in $S_n(h, \cdot)$ close to the discontinuity. The effect of increasing n was to move the blips closer and closer to the discontinuity but they remained and their height (in absolute value) remained 17 or 18% above the correct absolute value. How could this be reconciled with Theorem 16.4 (or indeed Lemma 16.1)?

Gibbs in two letters to *Nature* (the second a correction of the first) clarified and resolved the issue. The difficulty is due to a confusion between '*the limit of the graphs* and ... *the graph of the limit* of the sum. A misunderstanding on this point is a natural consequence of the usage which allows us to omit the word *limit* in certain connections as when we speak of the sum of an infinite series.'

In other words $S_n(h, t) \rightarrow h(t)$ pointwise (after all the blips move towards the discontinuity), but pointwise convergence of f_n to f does not imply that the graph of f_n starts to look like f for large n . The reader has already met more extreme examples of this when the notion of uniform convergence of a function g_n to g (which does imply that the graph of g_n starts to look like g) was introduced. For

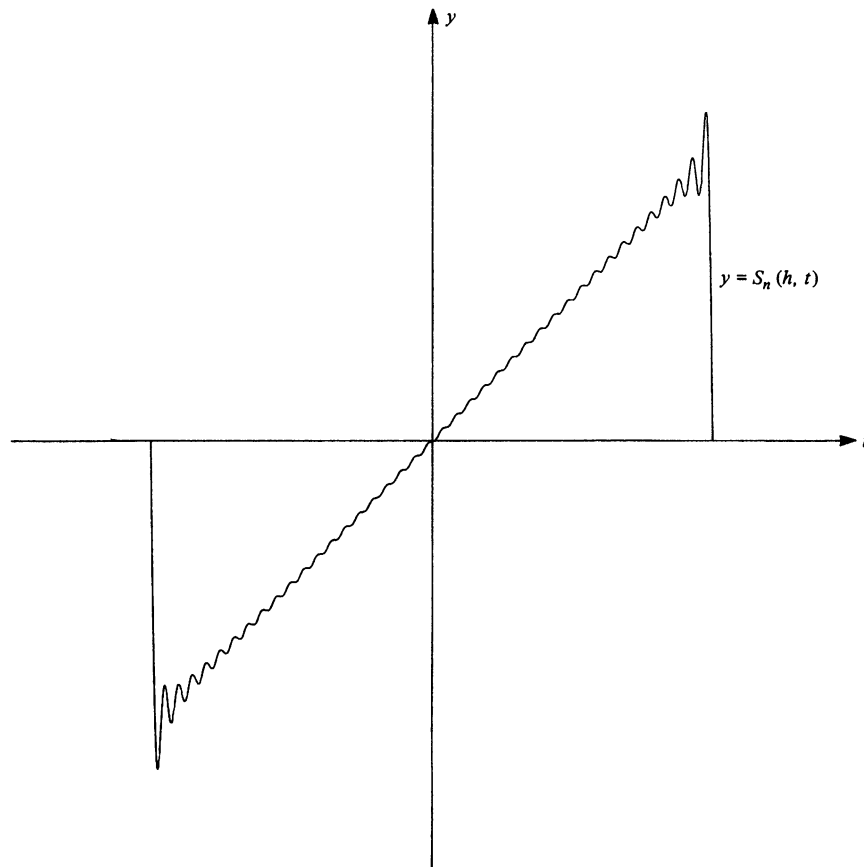


Fig. 17.1. A partial Fourier sum for the sawtooth function.

example, consider the 'witch's hats'

$$\begin{aligned} f_n(x) &= n^2(1 - n|x - n^{-1}|) & \text{for } 0 \leq x \leq 2n^{-1}, \\ f_n(x) &= 0 & \text{otherwise.} \end{aligned}$$

Then $f_n \rightarrow 0$ pointwise but the graph of f_n does not resemble 0. (see Figure 17.2).

In conclusion we may repeat once more that delicate and sophisticated results like Theorem 16.4 require much more care in use and interpretation than crude and unsubtle results like Theorem 9.1, and that the reader must always be careful to understand the limitations of any particular mode of convergence under discussion.

A full investigation of the 'Gibbs phenomenon' is not very difficult but neither is it very interesting. We shall therefore limit ourselves to demonstrating its reality.

Theorem 17.1. *If h is the sawtooth function defined by $h(x) = x[x \neq \pi]$, $h(\pi) = 0$ then*

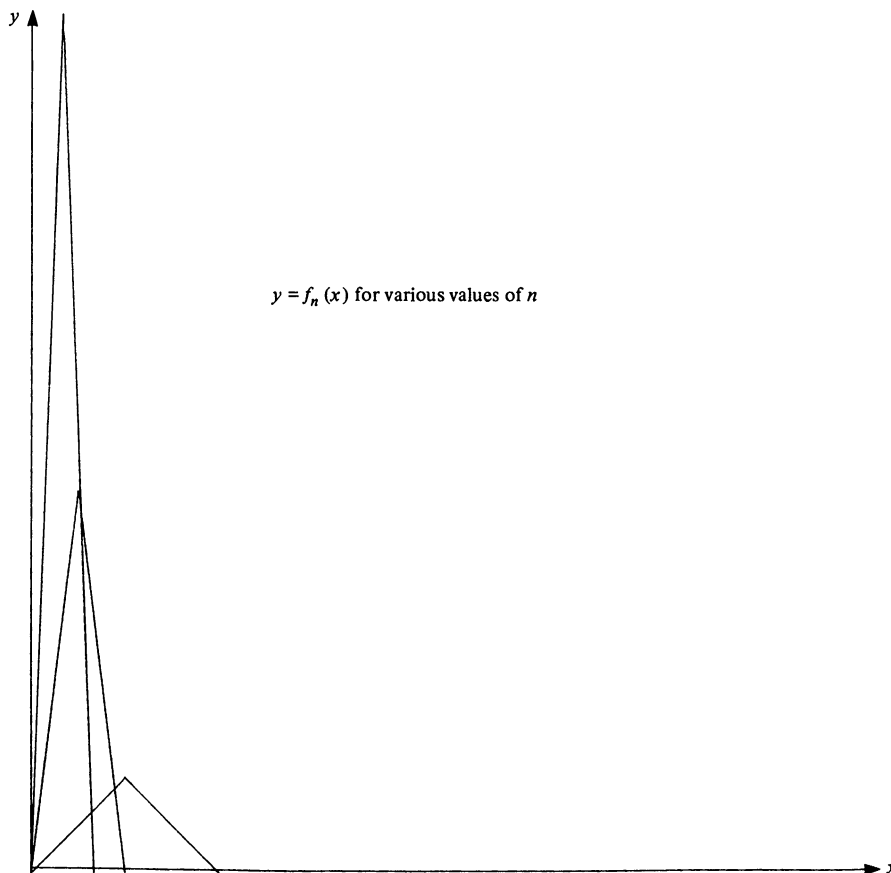


Fig. 17.2. The 'witch's hat' counter example.

$$S_n(h, \pi - \pi/n) \rightarrow A\pi,$$

$$S_n(h, -\pi + \pi/n) \rightarrow -A\pi \quad \text{as } n \rightarrow \infty,$$

where $A = 2/\pi \int_0^\pi (\sin x/x) dx > 1.17$.

Proof. At the beginning of Chapter 16 we saw that

$$S_n(h, x) = \sum_{r=1}^n (-1)^{r+1} \frac{2}{r} \sin rx$$

$$\text{Thus } S_n(h, \pi - \pi/n) = \sum_{r=1}^n \frac{2}{r} \sin \frac{r\pi}{n} = 2 \sum_{r=1}^n \frac{\pi}{n} \left(\frac{n}{r\pi} \sin \frac{r\pi}{n} \right) \rightarrow 2 \int_0^\pi \frac{\sin x}{x} dx$$

(using the standard results on the approximation of integrals by sums together with the observation that $\sin x/x$ is defined, continuous and bounded on $[0, \pi]$).

Similarly,

$$S_n(h, -\pi + \pi/n) \rightarrow -2 \int_0^\pi (\sin x/x) dx,$$

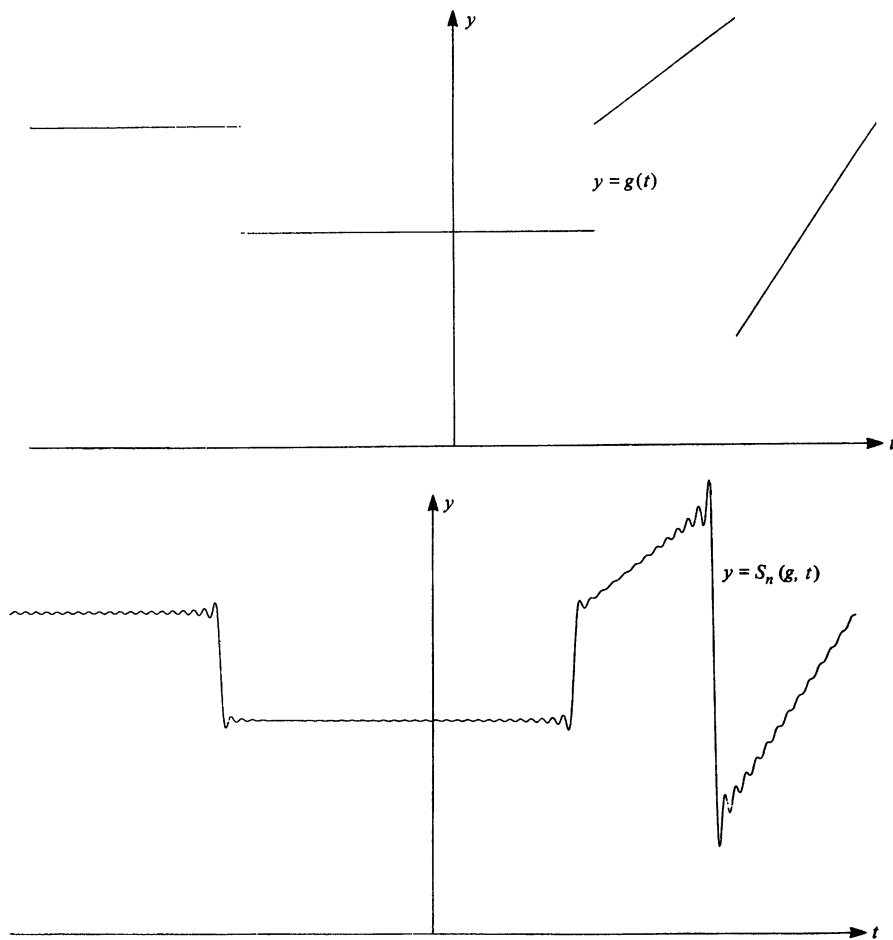


Fig. 17.3. The Gibbs phenomenon.

so all that remains to be done is to show that

$$\frac{2}{\pi} \int_0^{\pi} \frac{\sin x}{x} dx > 1.17.$$

This we do by direct numerical calculation. Since

$$\frac{\sin x}{x} = \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r}}{(2r+1)!}$$

with the power series having infinite radius of convergence, we may integrate term by term to get

$$\int_0^{\pi} \frac{\sin x}{x} dx = \pi \left(1 - \frac{\pi^2}{3!3} + \frac{\pi^4}{5!5} - \frac{\pi^6}{7!7} + \cdots \right).$$

The series on the right is an oscillating decreasing series, so the error due to truncation is less, in absolute value, than the first term neglected. In other words

$$\left| \frac{2}{\pi} \int_0^\pi \frac{\sin x}{x} dx - 2 \sum_{r=0}^n \frac{\pi^{2r} (-1)^r}{(2r+1)^2 (2r)!} \right| \leq \frac{2\pi^{2n+2}}{(2n+3)^2 (2n+2)!}.$$

Taking $n = 4$ and performing the calculations on a hand calculator we obtain

$$\frac{2}{\pi} \int_0^\pi \frac{\sin x}{x} dx > 1.17,$$

as required. ■

Suppose now we have a function $f: \mathbb{T} \rightarrow \mathbb{C}$ with $\hat{f}(r) = O(|r|^{-1})$ which has only a finite number of discontinuities, at x_1, x_2, \dots, x_N say, and suppose further that f is continuous on the left and on the right at each of these. Then the reader will easily verify (if she is interested) that we can write

$$f(t) = g(t) + \sum_{j=1}^N \lambda_j h(t - x_j) \quad [t \in \mathbb{T}],$$

where $\lambda_j \in \mathbb{C} [1 \leq j \leq n]$ and $g: \mathbb{T} \rightarrow \mathbb{C}$ is a continuous function with $\hat{g}(r) = O(|r|^{-1})$ and so, by Theorem 15.3 (i), with $S_n(g, \cdot) \rightarrow g$ uniformly on \mathbb{T} . Thus the same phenomenon which we described for h (of a blip overshooting by $8\frac{1}{2}$ to 9% of the total jump) will occur at each of the discontinuities (see Figure 17.3).

The phenomenon described in this chapter is called the ‘Gibbs phenomenon’ but could perhaps more fittingly be described as the ‘Gibbs–Wilbraham phenomenon’ since it had already been discovered and explained by an English mathematician called Wilbraham 60 years before. However this first discovery must have appeared as an isolated curiosity of no practical relevance and was soon forgotten.

During the early British development of radar it was decided to use the sawtooth function h to give the x coordinate on the oscilloscopes. The engineers produced h in the obvious way as a Fourier sum and the Gibbs–Wilbraham phenomenon was rediscovered yet again.