Chapter 2

Real and complex Wigner matrices

2.1 Real Wigner matrices: traces, moments and combinatorics

Start with two independent families of independent and identically distributed (i.i.d.) zero mean, real-valued random variables $\{ Z_{i,j} \}_{1 \le i < j}$ and $\{ Y_i \}_{1 \le i}$, such that $EZ_{1,2}^2 = 1$ and for all integers $k \ge 1$,

$$r_k := \max(E|X_{1,2}|^k, E|Y_1|^k) < \infty. \tag{2.1.1}$$

Consider the (symmetric) $N \times N$ matrix X_N with entries

$$X_N(j,i) = X_N(i,j) = \begin{cases} Z_{i,j}/\sqrt{N}, & \text{if } i < j \\ Y_i/\sqrt{N}, & \text{if } i = j \end{cases} \tag{2.1.2}$$

We call such a matrix a *Wigner matrix*, and if the random variables $Z_{i,j}$ and Y_i are Gaussian, we use the term *Gaussian Wigner matrix.* The case of Gaussian Wigner matrices in which $EY_1^2 = 2$ is of particular importance, and for reasons that will become clearer in Chapter 3, such matrices (rescaled by $\sqrt{N}$) are referred to as Gaussian orthogonal ensemble (GOE) matrices.
Let λ_i^N denote the (real) eigenvalues of X_N, with $\lambda_1^N \le \lambda_2^N \le \cdots \le \lambda_N^N$, and define the *empirical distribution* of the eigenvalues as the (random) probability measure on $\mathbb{R}$ defined by

$$L_N = \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i^N}.$$

Define the *semicircle distribution* (or *law*) as the probability distribution $\sigma(x)\,dx$ on $\mathbb{R}$ with density

$$\sigma(x) = \frac{1}{2\pi} \sqrt{4 - x^2}\,\mathbf{1}_{|x| \le 2}. \tag{2.1.3}$$

The following theorem can be considered the starting point of random matrix theory (RMT).

Theorem 2.1.1 (Wigner) For a Wigner matrix, the empirical measure L_N converges weakly, in probability, to the semicircle distribution.

In greater detail, Theorem 2.1.1 asserts that for any $f \in C_b(\mathbb{R})$, and any $\epsilon > 0$,

$$\lim_{N \to \infty} P\left(|\langle L_N, f \rangle - \langle \sigma, f \rangle| > \epsilon\right) = 0.$$

Remark 2.1.2 The assumption (2.1.1) that $r_k < \infty$ for all k is not really needed. See Theorem 2.1.21 in Section 2.1.5.

2.1.1 The semicricle distribution, Catalan numbers, and Dyck paths

2.1.2 Proof #1 of Wigner's Theorem 2.1.1

2.1.3 Proof of Lemma 2.1.6: words and graphs

2.1.4 Proof of Lemma 2.1.7: sentences and graphs

2.1.5 Some useful approximations

Lemma 2.1.19 (Hoffman-Wielandt) Let A, B be $N \times N$ symmetric matrices, with eigenvalues $\lambda_1^A \leq \lambda_2^A \leq \cdots \leq \lambda_N^A$ and $\lambda_1^B \leq \lambda_2^B \leq \cdots \leq \lambda_N^B$. Then,

$$\sum_{i=1}^{N} \left|\lambda_i^A - \lambda_i^B\right|^2 \leq \operatorname{tr}(A-B)^2.$$

Remark 2.1.20 The statement and proof of Lemma 2.1.19 carry over to the case where A and B are both Hermitian matrices.

Lemma 2.1.19 allows one to perform all sorts of truncations when proving convergence of empirical measures. For example, let us prove the following variant of Wigner's Theorem 2.1.1.

Theorem 2.1.21 Assume X_N is as in (2.1.2), except that instead of (2.1.1), only $r_2 < \infty$ is assumed. Then, the conclusion of Theorem 2.1.1 still holds.
Proof. Fix a constant C and consider the symmetric matrix $\hat{X}_N$ whose elements satisfy, for $1 \leq i \leq j \leq N$,

$$\hat{X}_N(i,j) = X_N(i,j)\mathbf{1}_{\sqrt{N}|X_N(i,j)|\leq C} - E\left(X_N(i,j)\mathbf{1}_{\sqrt{N}|X_N(i,j)|\leq C}\right).$$

Then, with $\hat{\lambda}_i^N$ denoting the eigenvalues of $\hat{X}_N$, ordered, it follows from Lemma 2.1.19 that

$$\frac{1}{N}\sum_{i=1}^{N}\left|\lambda_i^N - \hat{\lambda}_i^N\right|^2 \leq \frac{1}{N}\operatorname{tr}\left(X_N - \hat{X}_N\right)^2.$$

But,

$$\begin{aligned} W_N &:= \frac{1}{N}\operatorname{tr}\left(X_N - \hat{X}_N\right)^2 \\ &\leq \frac{1}{N^2}\sum_{i,j}\left[\sqrt{N}X_N(i,j)\mathbf{1}_{\sqrt{N}|X_N(i,j)|\geq C} - E\left(\sqrt{N}X_N(i,j)\mathbf{1}_{\sqrt{N}|X_N(i,j)|\geq C}\right)\right]^2. \end{aligned}$$

Since $r_2 < \infty$, and the involved random variables are identical in law to either $Z_{1,2}$ or Y_1, it follows that $E\left[\left(\sqrt{N}X_N(i,j)\right)^2\mathbf{1}_{\sqrt{N}|X_N(i,j)|\geq C}\right]$ converge to 0 uniformly in N, i, j, when C converges to infinity. Hence, one may choose for each ϵ a large enough C such that $P(|W_N| > \epsilon) < \epsilon$. Further, let

$$\operatorname{Lip}(\mathbb{R}) = \left\{ f \in C_b(\mathbb{R}) : \sup_x |f(x)| \leq 1, \sup_{x\neq y}\frac{|f(x)-f(y)|}{|x-y|} \leq 1 \right\}.$$

Then, on the event $\{\,|W_N| < \epsilon\}$, it holds that for $f \in \operatorname{Lip}(\mathbb{R})$,

$$\left|\langle L_N, f\rangle - \langle \hat{L}_N, f\rangle\right| \leq \frac{1}{N}\sum_i \left|\lambda_i^N - \hat{\lambda}_i^N\right| \leq \sqrt{\epsilon},$$

where $\hat{L}_N$ denotes the empirical measure of the eigenvalues of $\hat{X}_N$, and Jensen's inequality was used in the second inequality. This, together with the weak convergence in probability of $\hat{L}_N$ toward the semicircle law assured by Theorem 2.1.1, and the fact that weak convergence is equivalent to convergence with respect to the Lipschitz bounded metric (Theorem C.8) complete the proof of Theorem 2.1.21. ∎

2.1.6 Maximal eigenvalues and Füredi-Komlós enumeration

Theorem 2.1.22 (Maximal eigenvalue) Consider a Wigner matrix X_N satisfying $r_k \leq k^{Ck}$ for some constant C and all positive integers k. Then, λ_N^N converges to 2 in probability.

2.1.7 Central limit theorems for moments

With X_N a Wigner matrix and L_N the associated empirical measure of its eigenvalues, set $W_{N,k} := N\left[\langle L_N, x_k\rangle - \langle \bar{L}_N, x_k\rangle\right]$. Let

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2}\, du$$

denote the Gaussian distribution. We set σ_k^2 as in (2.1.44) below, and prove the following.

Theorem 2.1.31 The law of the sequence of random variables $W_{N,k}/\sigma_k$ converges weakly to the standard Gaussian distribution. More precisely,

$$\lim_{N\to\infty} P\left(\frac{W_{N,k}}{\sigma_k} \leq x\right) = \Phi(x).$$

Most of the proof consists of a variance computation.

2.2 Complex Wigner matrices

In this section, we start with two independent familes of i.i.d. random variables $\{ Z_{i,j} \}_{1\leq i<j}$ (complex-valued) and $\{ Y_i \}_{1\leq i}$ (real-valued), zero mean, such that $EZ_{1,2}^2 = 0$, $E|Z_{1,2}|^2 = 1$ and, for all integers $k \geq 1$,

$$r_k := \max\left(E|Z_{1,2}|^k, E|Y_1|^k\right) < \infty. \tag{2.2.4}$$

Consider the (Hermitian) $N \times N$ matrix X_N with entries

$$X_N^*(j,i) = X_N(i,j) = \begin{cases} Z_{i,j}/\sqrt{N} & \text{if } i<j, \\ Y_i/\sqrt{N} & \text{if } i=j. \end{cases} \tag{2.2.5}$$

We call such a matrix a *Hermitian Wigner matrix*, and if the random variables $Z_{i,j}$ and Y_i are Gaussian, we use the term *Gaussian Hermitian Wigner matrix.* The case of Gaussian Hermitian Wigner matrices in which $EY_1^2 = 1$ is of particular importance, and for reasons that will become clearer in Chapter 3, such matrices (rescaled by $\sqrt{N}$) are referred to as Gaussian unitary ensemble (GUE) matrices.

As before, let λ_i^N denote the (real) eigenvalues of X_N, with $\lambda_1^N \leq \lambda_2^N \leq \cdots \leq \lambda_N^N$, and recall that the empirical distribution of the eigenvalues is the probability measure on $\mathbb{R}$ defined by

$$L_N = \frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i^N}.$$

The following is the analog of Theorem 2.1.1.

Theorem 2.2.1 (Wigner) For a Hermitian Wigner matrix, the empirical measure L_N converges weakly, in probability, to the semicircle distribution.

2.3 Concentration for functionals of random matrices and logarithmic Sobolev inequalities

In this short section, we digress slightly and prove that certain functionals of random matrices have the concentration property, namely, with high probability these functionals are close to their mean value. A more complete treatment of concentration inequalities and their application to random matrices is postponed to Section 4.4. The results of this section will be useful in Section 2.4, where they will play an important role in the proof of Wigner's theorem via the Stieltjes transform.

2.3.1 Smoothness properties of linear functions of the empirical measure

Let us recall that if X is a symmetric (Hermitian) matrix and f is a bounded measurable function, $f(X)$ is defined as the matrix with the same eigenvectors as X but with eigenvalues that are the image by f of those of X, namely, if e is an eigenvector of X with eigenvalue λ, $Xe = \lambda e$, $f(X)e = f(\lambda)e$. In terms of the spectral decomposition $X = UDU^*$ with U orthogonal (unitary) and D diagonal real, one has $f(X) = Uf(D)U^*$ with $f(D)_{ii} = f(D_{ii})$. For $M \in \mathbb{N}$, we denoted by $\langle \cdot, \cdot \rangle$ the Euclidean scalar product on $\mathbb{R}^M$ (or $\mathbb{C}^M$).

Throughout this section, we denote the Lipschitz constant of a function $G : \mathbb{R}^M \to \mathbb{R}$ by

$$|G|_{\mathcal{L}} := \sup_{x \neq y \in \mathbb{R}^M} \frac{|G(x) - G(y)|}{\|x - y\|_2},$$

and call G a *Lipschitz function* if $|G|_{\mathcal{L}} < \infty$. The following lemma is an immediate application of Lemma 2.1.19. In its statement, we identify $\mathbb{C}$ with $\mathbb{R}^2$.

Lemma 2.3.1 Let $g : \mathbb{R}^N \to \mathbb{R}$ be Lipschitz with Lipschitz constant $|g|_{\mathcal{L}}$. Then, with X denoting the Hermitian matrix with entries $X(i,j)$, the map

$$\{ X(i,j) \}_{1 \le i \le j \le N} \mapsto g(\lambda_1(X), \ldots, \lambda_N(X))$$

is a Lipschitz function on $\mathbb{R}^{N^2}$ with Lipschitz constant bounded by $\sqrt{2}|g|_{\mathcal{L}}$. In particular, if f is a Lipschitz function on $\mathbb{R}$,

$$\{ X(i,j) \}_{1 \le i \le j \le N} \mapsto \mathrm{tr}(f(X))$$

is a Lipschitz function on $\mathbb{R}^{N(N+1)}$ with Lipschitz constant bounded by $\sqrt{2N}|f|_{\mathcal{L}}$.

2.3.2 Concentration inequalities for independent variables satisfying logarithmic Sobolev inequalities

To begin with, recall that a probability measure P on $\mathbb{R}$ is said to satisfy the *logarithmic Sobolev inequality* (LSI) with constant c if, for any differentiable function f in $L^2(P)$,

$$\int f^2 \log \frac{f^2}{\int f^2\, dP}\, dP \le 2c \int |f'|^2\, dP.$$

It is not hard to check, by induction, that if P_i satisfy the LSI with constant c and if $P^{(M)} = \otimes_{i=1}^M P_i$ denotes the product measure on $\mathbb{R}^M$, then $P^{(M)}$ satisfies the LSI with constant c.

The interest in the logarithmic Sobolev inequality, in the context of concentration inequalities, lies in the following argument, that among other things, shows that LSI implies sub-Gaussian tails.

Lemma 2.3.3 (Herbst) Assume that P satisfies the LSI on $\mathbb{R}^M$ with constant c. Let G be a Lipschitz function on $\mathbb{R}^M$, with Lipschitz constant $|G|_{\mathcal{L}}$. Then, for all $\lambda \in \mathbb{R}$,

$$E_P \left[e^{\lambda (G - E_P(G))} \right] \le e^{c\lambda^2 |G|_{\mathcal{L}}^2 / 2},$$

and so, for all $\delta > 0$,

$$P(|G - E_P(G)| \ge \delta) \le 2e^{-\delta^2 / 2c|G|_{\mathcal{L}}^2}.$$

Note that part of the statement in Lemma 2.3.3 is that $E_P G$ is finite.

2.3.3 Concentration for Wigner-type matrices

We consider in this section (symmetric) matrices X_N with independent (and not necessarily identically distributed) entries $\{ X_N(i,j) \}_{1 \le i \le j \le N}$. The following is an immediate corollary of Lemmas 2.3.1 and 2.3.3.

Theorem 2.3.5 Suppose that the laws of the independent entries $\{ X_N(i,j) \}_{1 \le i \le j \le N}$ all satisfy the LSI with constant c/N. Then, for any Lipschitz function f on $\mathbb{R}$, for any $\delta > 0$,

$$P(|\mathrm{tr}(f(X_N)) - E[\mathrm{tr}(f(X_N))]| \ge \lambda N) \le 2e^{-\frac{1}{4c|f|_{\mathcal{L}}^2} N^2 \delta^2}.$$

Further, for any $k \in \{1, \ldots, N\}$,

$$P(|f(\lambda_k(X_N)) - Ef(\lambda_k(X_N))| \geq \lambda) \leq 2e^{-\frac{1}{4c|f|_{\mathcal{L}}^2} N\delta^2}.$$

We note that under the assumptions of Theorem 2.3.5, $E\lambda_N(X_N)$ is uniformly bounded.

2.4 Stieltjes transforms and recursions

We begin by recalling some classical results concerning the Stieltjes transfrom of a probability measure.

Definition 2.4.1 Let μ be a positive, finite measure on the real line. The *Stieltjes transform* of μ is the function

$$S_\mu(z) := \int_{\mathbb{R}} \frac{\mu(dx)}{x - z}, z \in \mathbb{C} \setminus \mathbb{R}.$$

Note that for $z \in \mathbb{C} \setminus \mathbb{R}$, both the real and imaginary parts of $1/(x - z)$ are continuous bounded functions of $x \in \mathbb{R}$ and, further, $|S_\mu(z)| \leq \mu(\mathbb{R})/|\Im z|$. These crucial observations are used repeatedly in what follows.

Stieltjes transforms can be inverted. In particular, one has

Theorem 2.4.3 For any open interval I with neither endpoint on an atom of μ,

$$\mu(I) = \lim_{\epsilon \to 0} \frac{1}{\pi} \int_I \frac{S_\mu(\lambda + i\epsilon) - S_\mu(\lambda - i\epsilon)}{2i} \, d\lambda = \lim_{\epsilon \to 0} \frac{1}{\pi} \int_I \Im S_\mu(\lambda + i\epsilon) \, d\lambda.$$

Theorem 2.4.3 allows for the reconstruction of a measure from its Stieltjes transform. Further, one has the following.

Theorem 2.4.4 Let $\mu_n \in M_1(\mathbb{R})$ be a sequence of probability measures.
(a) If μ_n converges weakly to a probability measure μ, then $S_{\mu_n}(z)$ converges to $S_\mu(z)$ for each $z \in \mathbb{C} \setminus \mathbb{R}$.
(b) If $S_{\mu_n}(z)$ converges for each $z \in \mathbb{C} \setminus \mathbb{R}$ to a limit $S(z)$, then $S(z)$ is the Stieltjes transform of a sub-probability measure μ, and μ_n converges vaguely to μ.
(c) If the probability measures μ_n are random and, for each $z \in \mathbb{C} \setminus \mathbb{R}$, $S_{\mu_n}(z)$ converges in probability to a deterministic limit $S(z)$ that is the Stieltjes transform of a probability measure μ, then μ_n converges weakly in probability to μ.

(We recall that μ_n *converges weakly* to μ if, for any continuous function f on $\mathbb{R}$ that decays to 0 at infinity, $\int f \, d\mu_n \to \int f \, d\mu$. Recall also that a positive measure μ on $\mathbb{R}$ is a *sub-probability measure* if it satisfies $\mu(\mathbb{R}) \leq 1$.)

For a matrix X, define $\boldsymbol{S}_X(z) := (X - zI)^{-1}$. Taking $A = X$ in the matrix inversion lemma (Lemma A.1), one gets

$$\boldsymbol{S}_X(z) = z^{-1}(X\boldsymbol{S}_X(z) - I), z \in \mathbb{C} \setminus \mathbb{R}.$$

Note that with L_N denoting the empirical measure of the eigenvalues of X_N,

$$S_{L_N}(z) = \frac{1}{N} \mathrm{tr} \boldsymbol{S}_{X_N}(z), \quad S_{\overline{L}_N}(z) = \frac{1}{N} E \mathrm{tr} \boldsymbol{S}_{X_N}(z).$$

Two proofs are given for Wigner's theorem using Stieltjes transforms.

2.5 Joint distribution of eigenvalues in the GOE and the GUE

We are going to calculate the join distribution of eigenvalues of a random symmetric or Hermitian matrix under a special type of probability law which displays a high degree of symmetry but still makes on-or-above-diagonal entries independent so that the theory of Wigner matrices applies.

2.5.1 Definition and preliminary discussion of the GOE and the GUE

Let $\{\, \xi_{i,j}, \eta_{i,j} \,\}_{i,j=1}^{\infty}$ be an i.i.d. family of real mean 0 variance 1 Gaussian random variables. We define

$$P_2^{(1)}, P_3^{(1)}, \ldots$$

to be the laws of the random matrices

$$\begin{bmatrix} \sqrt{2}\xi_{1,1} & \xi_{1,2} \\ \xi_{1,2} & \sqrt{2}\xi_{2,2} \end{bmatrix} \in \mathcal{H}_2^{(1)}, \begin{bmatrix} \sqrt{2}\xi_{1,1} & \xi_{1,2} & \xi_{1,3} \\ \xi_{1,2} & \sqrt{2}\xi_{2,2} & \xi_{2,3} \\ \xi_{1,3} & \xi_{2,3} & \sqrt{2}\xi_{3,3} \end{bmatrix} \in \mathcal{H}_3^{(1)}, \ldots,$$

respectively. We define

$$P_2^{(2)}, P_3^{(2)}, \ldots$$

to be the laws of the random matrices

$$\begin{bmatrix} \xi_{1,1} & \frac{\xi_{1,2}+i\eta_{1,2}}{\sqrt{2}} \\ \frac{\xi_{1,2}-i\eta_{1,2}}{\sqrt{2}} & \xi_{2,2} \end{bmatrix} \in \mathcal{H}_2^{(2)}, \begin{bmatrix} \xi_{1,1} & \frac{\xi_{1,2}+i\eta_{1,2}}{\sqrt{2}} & \frac{\xi_{1,3}+i\eta_{1,3}}{\sqrt{2}} \\ \frac{\xi_{1,2}-i\eta_{1,2}}{\sqrt{2}} & \xi_{2,2} & \frac{\xi_{2,3}+i\eta_{2,3}}{\sqrt{2}} \\ \frac{\xi_{1,3}-i\eta_{1,3}}{\sqrt{2}} & \frac{\xi_{2,3}-i\eta_{2,3}}{\sqrt{2}} & \xi_{3,3} \end{bmatrix} \in \mathcal{H}_3^{(2)}, \ldots,$$

respectively. A random matrix $X \in \mathcal{H}_N^{(\beta)}$ with law $P_N^{(\beta)}$ is said to belong to the *Gaussian orthogonal ensemble* (GOE) or the *Gaussian unitary ensemble* (GUE) according as $\beta = 1$ or $\beta = 2$, respectively. The theory of Wigner matrices developed in previous sections applies here. In particular, for fixed β, given for each N a random matrix $X(N) \in \mathcal{H}_N^{(\beta)}$ with law $P_N^{(\beta)}$, the empirical distribution of the eigenvalues of $X_N := X(N)/\sqrt{N}$ tends to the semicircle law of mean 0 and variance 1.

Definition 2.5.1 Let $x = (x_1, \ldots, x_N) \in \mathbb{C}^N$. The *Vandermonde determinant* associated with x is

$$\Delta(x) = \det\left(\left\{ x_i^{j-1} \right\}_{i,j=1}^{n} \right) = \prod_{i<j} (x_j - x_i).$$

The main result in this section is the following.

Theorem 2.5.2 (Joint distribution of eigenvalues: GOE and GUE) Let $X \in \mathcal{H}_N^{(\beta)}$ be random with law $P_N^{(\beta)}$, $\beta = 1, 2$. The joint distribution of eigenvalues $\lambda_1(X) \leq \cdots \leq \lambda_N(X)$ has density with respect to Lebesgue measure which equals

$$N! \overline{C}_N^{(\beta)} \mathbf{1}_{x_1 \leq \cdots \leq x_N} \, |\Delta(x)|^{\beta} \prod_{i=1}^{N} e^{-\beta x_i^2/4},$$

where

$$\begin{aligned} N! \overline{C}_N^{(\beta)} &= N! \left(\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |\Delta(x)|^{\beta} \prod_{i=1}^{N} e^{-\beta x_i^2/4} \, dx_i \right)^{-1} \\ &= (2\pi)^{-N/2} \left(\frac{\beta}{2} \right)^{\beta N(N-1)/4 + N/2} \prod_{j=1}^{N} \frac{\Gamma(\beta/2)}{\Gamma(j\beta/2)}. \end{aligned}$$

A consequence of Theorem 2.5.2 is that almost surely, the eigenvalues of the GOE and GUE are all distinct. Let $\nu_1, \ldots, \nu_N$ denote the eigenvectors corresponding to the eigenvalues $(\lambda_1^N, \ldots, \lambda_N^N)$ of a matrix X from GOE(N) or GUE(N), with their first nonzero entry positive real. Recall that $O(N)$ (the group of orthogonal matrices) and $U(N)$ (the group of unitary matrices) admit a unique Haar probability measure (Theorem F.13). The invariance of the law of X under arbitrary orthogonal (unitary) transformations implies then the following.

Corollary 2.5.4 The collection $(\nu_1, \ldots, \nu_N)$ is independent of the eigenvalues $(\lambda_1^N, \ldots, \lambda_N^N)$. Each of the eigenvectors $\nu_1, \ldots, \nu_N$ is distributed uniformly on

$$S_+^{N-1} = \{\, \boldsymbol{x} = (x_1, \ldots, x_N) : x_i \in \mathbb{R}, \|\boldsymbol{x}\|_2 = 1, x_1 > 0 \,\}$$

(for the GOE), or on

$$S_{\mathbb{C},+}^{N-1} = \{\, \boldsymbol{x} = (x_1, \ldots, x_N) : x_1 \in \mathbb{R}, x_i \in \mathbb{C} \text{ for } i \geq 2, \|\boldsymbol{x}\|_2 = 1, x_1 > 0 \,\}$$

(for the GUE). Further, $(\nu_1, \ldots, \nu_N)$ is distributed like a sample of Haar measure on $O(N)$ (for the GOE) or $U(N)$ (for the GUE), with each column multiplied by a norm one scalar so that the columns all belong to S_+^{N-1} (for the GOE) and $S_{\mathbb{C},+}^{N-1}$ (for the GUE).

2.5.2 Proof of the joint distribution of eigenvalues

2.5.3 Selberg's integral formula and proof of (2.5.4)

To complete the description of the joint distribution of eigenvalues of the GOE, GUE and GSE, we derive in this section an expression for the normalization constant in (2.5.4).
We begin by stating Selberg's integral formula. We then describe in Corollary 2.5.9 a couple of limiting cases of Selberg's formula. The evaluation of the normalization constant in (2.5.4) is immediate from Corollary 2.5.9. Recall, that $\Delta(x)$ denotes the Vandermonde determinant of x.

Theorem 2.5.8 (Selberg's integral formula) For all positive numbers a, b and c, we have

$$\frac{1}{n!}\int_0^1 \cdots \int_0^1 |\Delta(x)|^{2c} \prod_{i=1}^n x_i^{a-1}(1-x_i)^{b-1}\, dx_i = \prod_{j=0}^{n-1} \frac{\Gamma(a+jc)\Gamma(b+jc)\Gamma((j+1)c)}{\Gamma(a+b+(n+j-1)c)\Gamma(c)}.$$

Corollary 2.5.9 For all positive numbers a and c, we have

$$\frac{1}{n!}\int_0^\infty \cdots \int_0^\infty |\Delta(x)|^{2c} \prod_{i=1}^n x_i^{a-1} e^{-x_i}\, dx_i = \prod_{j=1}^{n-1} \frac{\Gamma(a+jc)\Gamma((j+1)c)}{\Gamma(c)},$$

and

$$\frac{1}{n!}\int_{-\infty}^\infty \cdots \int_{-\infty}^\infty |\Delta(x)|^{2c} \prod_{i=1}^n e^{-x_i^2/2}\, dx_i = (2\pi)^{n/2} \prod_{j=0}^{n-1} \frac{\Gamma((j+1)c)}{\Gamma(c)}.$$

Remark 2.5.10 The identities in Theorem 2.5.8 and Corollary 2.5.9 hold under rather less stringent conditions on the parameters a, b, and c. For example, one can allow a, b, and c to be complex with positive real parts. We note also that the second part of Corollary 2.5.9 is directly relevant to the study of normalization constants for the GOE and GUE. The usefulness of the other more complicated formulas will become apparent in Section 4.1.

2.5.4 Joint distribution of eigenvalues: alternative formulation