Math 606, Summer 2022[1]: Gaussian Processes; version of June 25, 2022

HOMEWORK PROBLEMS

(1) Let Z be a standard Gaussian random variable. Determine the values of the real number r for which $\mathbb{E}|Z|^r$ exits and compute the expectation for those r. Express your answer in terms of the Gamma function

$$\Gamma(x) = \int_0^{+\infty} t^{x-1}e^{-t}dt,$$

and simplify the answer when possible (for example, when r is a positive even number). In particular, confirm that $\mathbb{E}Z^4 = 3$.

(2) Confirm, numerically or otherwise, that the function

$$F(x) = 2^{-22^{1-41^{x/10}}},\ x > 0,$$

can be a good approximation of the standard normal cdf. In what range of values x would you use such an approximation?

Reference: A. Soranzo and E. Epure, Very Simply Explicitly Invertible Approximations of Normal Cumulative and Normal Quantile Function, Applied Mathematical Sciences, volume 8, number 87, 4323–4341, 2014, http://dx.doi.org/10.12988/ams.2014.45338.

(3) Let Z be a standard normal random variable. Prove that, for every $x > 1$,

$$\frac{1}{\sqrt{2\pi}}\,\frac{x}{1+x^2}e^{-x^2/2} \le \mathbb{P}(Z \ge x) \le \frac{1}{\sqrt{2\pi}\,x}\,e^{-x^2/2}.$$

[This is all about integration by parts. For the lower bound, start by computing the derivative of $x^{-1}e^{-x^2/2}$; then note that the function $f(x) = x^2/(1+x^2)$ is increasing for $x > 0$.]

(4) Let X, Y be standard normal such that the joint distribution of X and Y is also normal and $\mathbb{E}XY = \rho$. Compute (a) $\mathbb{E}\big(|X|Y\big)$; (b) $\mathbb{E}(X^2Y^2)$ (c) the correlation coefficient between X^2 and Y^2. [Possible answers: 0, $2\rho^2+1$, ρ^2]

(5) Below, $\mathfrak{i} = \sqrt{-1}$, $(\cdot,\cdot)$ is inner product in Euclidean space; C^{-1} means inverse of the matrix C; C^T means the transpose of C. Vectors are thought of as matrices with one column.

(A) Confirm that the following three definitions of a Gaussian vector $X = (X_1,\dots,X_n)$ are equivalent:

(a) $\mathbb{E}e^{\mathfrak{i}(X,\lambda)} = e^{\mathfrak{i}(\lambda,\mu)-(1/2)(C\lambda,\lambda)}$ for some vector μ and a symmetric non-negative definite matrix C; with this characterization, also confirm that $\mu = \mathbb{E}X$ and $C = C_{XX}$ is the covariance matrix of X;

(b) (a, X) is a Gaussian random variable for every $a \in \mathbb{R}^n$

(c) $X = \mu + \mathcal{L}(Z)$, where Z is a vector with iid standard normal components and $\mathcal{L}: \mathbb{R}^n \to \mathbb{R}^n$ is a linear mapping.

(B) [The multi-dimensional normal correlation theorem] Let X be a Gaussian vector in $\mathbb{R}^n$, let Y be a Gaussian vector in $\mathbb{R}^m$ and assume that the combined vector X, Y is Gaussian in $\mathbb{R}^{m+n}$ and the covariance matrix C_{YY} of Y is invertible. Confirm that

$$\mathbb{E}(X|Y) = \mathbb{E}X + C_{XY}C_{YY}^{-1}(Y - \mathbb{E}Y),\ \mathbb{E}\Big(X - \mathbb{E}(X|Y)\Big)\Big(X - \mathbb{E}(X|Y)\Big)^T = C_{XX} - C_{XY}C_{YY}^{-1}C_{YX}.$$

Note that $C_{YX} = C_{XY}^T$.

Start by finding a matrix A such that the vector

$$X - \mathbb{E}X - A(Y - \mathbb{E}Y)$$

and the vector $Y - \mathbb{E}Y$ are uncorrelated. [Hint: $A = C_{XY}C_{YY}^{-1}$].

Confirm that if $m = n = 1$, then the conditional expectation is the equation of the regression line of Xon Y.

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What if the matrix C_{YY} is not invertible?

(6) Let $(X_1, \dots, X_n)$ be a Gaussian vector with non-singular covariance matrix $C = (C_{ij},\ i,j = 1, \dots, n)$.
(a) What can we say about the random variables X_1 and X_2 if $C_{12} = 0$? [Independent?]
(b) What can we say about the random variables X_1 and X_2 if $(C^{-1})_{12} = 0$? [Conditionally independent?]
(c) Construct two examples of the matric C such that (i) $C_{12} = 0$, $(C^{-1})_{12} \neq 0$; (ii) $C_{12} \neq 0$, $(C^{-1})_{12} = 0$.

(7) Let X be a standard Gaussian random variable. Given a real number $a > 0$, define the random variable Y_a by
$$Y_a = \begin{cases} X, & |X| > a \\ -X & |X| < a. \end{cases}$$
(a) Confirm that Y_a is standard Gaussian.
(b) Confirm that $\mathbb{E}XY_a = 0$ for some $a > 0$. [Use Intermediate Value Theorem.]
(c) Are there any values of a such that the vector (X, Y_a) is Gaussian? [No: $X + Y_a = 0$ with positive probability].

(8) Let X_1, X_2, X_3 be iid standard normal and define
$$Y = \frac{X_1 + X_2X_3}{\sqrt{1 + X_3^2}}.$$
(a) Confirm that Y is a standard Gaussian random variable [condition on X_3; do we need X_3 to be Gaussian for the computations to work?].
(b) For what i will the vector (X_i, Y) be Gaussian? [Looks like only for $i = 3$]

(9) Let X and Y be iid standard normal and define
$$Z = \begin{cases} X, & \text{if } XY > 0 \\ -X, & \text{if } XY \leq 0. \end{cases}$$
Confirm that Z is normal but the vector (Z, Y) is not jointly normal. [Note that $P(Z > 0|Y < 0) = 0$.]

(10) Let $(X_1, \dots, X_n)$ be a Gaussian vector with mean zero, $\mathbb{E}X_k^2 = 1$, and $\mathbb{E}X_kX_m = r$, $k \neq m$. What is the possible range of values for r? [$r \in [0, 1]$ is always OK, for $r < 0$, it depends on n. For example, $r \geq -1/2$ for $n = 3$].

(11) Let Z be a random vector in $\mathbb{R}^n$ with iid standard Gaussian components, and denote by $|Z|$ the Euclidean norm of Z. Confirm that the vector $Z/|Z|$ is uniformly distributed on the unit sphere in $\mathbb{R}^n$.

(12) Let X be standard normal, let $\xi_1, \xi_2, \dots$ be independent exponential random variables with mean 1, and let N be a Poisson random variable with mean $r^2/2$, $r > 0$. Assume that all the random variables are independent. Confirm that $(X + r)^2$ and $X^2 + 2\sum_{k=1}^{N} \xi_k$ have the same distribution. [Compare the moment generating functions.]

(13) Let X and Y be iid random variables with finite second moment. Confirm that if the random variables $X+Y$ and $X-Y$ are independent, then both X and Y are Gaussian. [Hint: Feller, Vol. 2, Theorem III. 4]

(14) Let H be an infinite-dimensional separable Hilbert space. Confirm that there is no (positive countably additive) measure on $\mathcal{B}(H)$ that is rotation-invariant and is finite on bounded open sets. [Look at balls of radius 1/2 centered at the elements of an orthonormal basis; they do not intersect, fit inside the ball of radius 2 centered at the origin, and must all have the same nonzero measure.]

(15) Let $K = K(x, y)$ be a continuous positive-definite kernel for $(x, y) \in [0, 1] \times [0, 1]$.
(a) Confirm that $K(x, x) \geq 0$ [argue by contradiction, by assuming that $K(a, a) < 0$ for some $a \in (0, 1)$ and then constructing a suitable function f, supported near a such that $\iint f(x)f(y)K(x, y)dxdy < 0$];

(b) Give an example of K such that $K(x,y)<0$ for some x,y. [$K(x,y)=\sin(2\pi x)\sin(2\pi y)$ might work. Now, how about $K(x,y)=F(x-y)$ for some function F?]

(16) Confirm that if $W=W(t)$ is a standard Brownian motions, then the process $X(t)=(1-t)W\big(t/(1-t)\big)$, $t\in(0,1)$, with $X(0)=X(1)=0$, is a Brownian bridge. [Note that the function $t\mapsto t/(1-t)$ is increasing on $(0,1)$.]

(17) Confirm that if $W=W(t),\ t\geq 0$, is a standard Brownian motions, then

$$X(t)=e^{-t}W\big(e^{2t}-1\big),\ t\geq 0,$$

is an Ornstein-Uhlenbeck process.

(18) Derive/verify the KL expansions for the standard Brownian motion and Brownian bridge on the interval $[0,L]$. Use the results (together with the appropriate wave equation) to estimate the lowest frequency of the clarinet [taking $L=0.6$ meters] and the flute [taking $L=0.7$ meters].

(19) If $W=W(t)$ is the standard Brownian motion on $(\Omega,\mathcal{F},\mathbb{P})$, then, by the Cameron-Martin formula,

$$\mathbb{E}\exp\left(-\frac{1}{2}\int_0^T W^2(t)\,dt\right)=\frac{1}{\sqrt{\cosh(T)}}.$$

Using this result confirm that, for every $p>0$,

$$\mathbb{E}\exp\left(-p\int_0^T W^2(t)\,dt\right)=\frac{1}{\sqrt{\cosh(T\sqrt{2p})}},$$

or, equivalently, for $\lambda>0$,

$$\mathbb{E}\exp\left(-\frac{\lambda^2}{2}\int_0^T W^2(t)\,dt\right)=\frac{1}{\sqrt{\cosh(\lambda T)}}.$$

[Use that $\sqrt{\lambda}W(t/\lambda)$ is a standard Brownian motion.] The original result of Cameron and Martin also includes $p<0$. Can you recover it?

(20) Let $W=W(t), t\geq 0$, be a standard Brownian motion in $\mathbb{R}^{\mathrm{d}}$, let X_0 be a Gaussian random vector in $\mathbb{R}^{\mathrm{d}}$ independent of W, and let A and B be square d-by-d matrices. Investigate the Gaussian process $X=X(t)$ defined by

$$X(t)=X_0+\int_0^t AX(s)\,ds+BW(t).$$

(21) Let $W=W(t),\ t\geq 0$, be a standard Brownian motion. For $a,b>0$, compute

$$\mathbb{E}\sup_{t>0}\frac{W(t)}{a+bt}.$$

[Note that the probability that $\sup_{t>0}\frac{W(t)}{a+bt}$ is bigger than $x>0$ is the same as the probability that the first time $W(t)$ hits the line $ax+bxt$ is finite].

(22) Let a,b be real numbers and let $\dot{W}$ be Gaussian white noise. Construct and implement on the computer an *exact* time discretization of the equations

$$\dot{X}=aX+\dot{W}\quad\text{and}\quad\ddot{X}+a\dot{X}+bX=\dot{W},$$

with zero initial conditions. Can you extend the method to higher-order equations?

[For the first equation, the starting point is the equality

$$X(t_{k+1})=X(t_k)e^{a(t_{k+1}-t_k)}+\xi_{k+1}$$

where $\xi_{k+1}=\int_{t_k}^{t_{k+1}}e^{a(t_{k+1}-s)}dW(s)$ is a Gaussian random variable with zero mean and known variance, and the random variables ξ_k are independent for different k. A similar formula exists for all inhomogeneous linear equations with constant coefficients.]

(23) Let $K = K(t,s)$ be a continuous symmetric $[K(t,s) = K(s,t)]$ real-valued function defined on $[0,1] \times [0,1]$. Consider the following two conditions:

$$\sum_{m,n=1}^{N} K(t_m, t_n) a_i a_j \geq 0 \text{ for all } t_1, \ldots t_N \in [0,1] \text{ and } a_1, \ldots, a_N \in \mathbb{R}; \tag{1.1}$$

$$\int_0^1 \int_0^1 K(t,s) f(t) f(s)\, ds dt \geq 0 \text{ for all } f \in L_2((0,1)). \tag{1.2}$$

(a) True or false: (1.1) implies (1.2)?
(b) True or false: (1.2) implies (1.1)?
In each case, either give a proof or construct a counterexample. [In each case, continuity implies that the answer is "yes" so the fun part is to relax continuity assumption to L_2 or L_∞]
(c) Confirm that condition (1.1) always [i.e. regardless of continuity of K] implies $K(t,t) \geq 0$ and $|K(t,s)|^2 \leq K(t,t)K(s,s)$.

(24) Use a software package of your choice to plot a sample path (surface) for each of the following zero-mean Gaussian fields on $[0,1] \times [0,1]$, defined by the covariance function $R(\boldsymbol{x}, \boldsymbol{y})$, $\boldsymbol{x} = (x_1, x_2)$, $\boldsymbol{y} = (y_1, y_2)$:
(a) $\min(x_1, x_2) \cdot \min(y_1, y_2)$ [Brownian sheet];
(b) $\min(x_1, x_2) \cdot \Big(\min(y_1, y_2) - y_1 y_2 \Big)$ [Kiefer field];
(c) $\Big(\min(x_1, x_2) - x_1 x_2 \Big) \cdot \Big(\min(y_1, y_2) - y_1 y_2 \Big)$;
(d) $\frac{|\boldsymbol{x}| + |\boldsymbol{y}| - |\boldsymbol{x} - \boldsymbol{y}|}{2}$ [Lévy Brownian motion];
(e) Green's function of the Dirichlet Laplacian [Gaussian free field].
Are the fields in parts (c) and (e) the same? [Hint: No].

(25) Let $\dot{W}$ be Gaussian white noise over $L_2((0,1))$. Denote by B_1 the unit ball in $L_((0,1))$ and denote by B_2 the unit ball in $H_1^0((0,1))$.
(a) Confirm that $\mathbb{P}\big(\sup_{f \in B_1} \dot{W}[f] = +\infty\big) = 1$. [Take $f = h_k$, an element of an orthonormal basis in $L_2((0,1))$.]
(b) True or false: $\mathbb{P}\big(\sup_{f \in B_2} \dot{W}[f] < +\infty\big) = 1$? Justify your answer. [Section 10.4 of our book might contain the solution.]

(26) Let μ be the standard Gaussian measure in $\mathbb{R}^n$. Denote by B_1 the unit ball in $\mathbb{R}^n$ with respect to the usual Euclidean metric. For a measurable set A, define

$$\mu^+(A) = \liminf_{\varepsilon \to 0} \frac{\mu(A + \varepsilon B_1) - \mu(A)}{\varepsilon}.$$

Confirm that if $A = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 \leq a\}$ for some $a \in \mathbb{R}$, then

$$\mu(A) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-x^2/2}\, dx \quad \text{and} \quad \mu^+(A) = \frac{1}{\sqrt{2\pi}} e^{-a^2/2}.$$

(27) Let Φ be the standard normal cdf and let φ be the standard normal pdf. For $x \in (0,1)$, define the function $I(x) = \varphi\big(\Phi^{-1}(x)\big)$. Confirm that

$$\lim_{x \to 0+} \frac{I(x)}{x\sqrt{2\ln(1/x)}} = 1.$$

[The key computation is $\Phi^{-1}(t) \sim \sqrt{2\ln(1/t)},\ t \to 0+$.]

(28) Identify the Cameron-Martin space of a Gaussian measure on $\mathbb{R}^n$. Do not assume that the covariance matrix is non-singular.

(29) (a) Confirm that if the random variable Y stochastically dominates the random variable Y ($\mathbb{P}(X > r) \leq \mathbb{P}(Y > r)$ for all $t \in \mathbb{R}$) and $\mathbb{E}|X| < \infty$, then $\mathbb{E}X \leq \mathbb{E}Y$ [note that we are not assuming $X \geq 0$].

(b) Using Fernique-Sudakov, confirm that $\mathbb{E}\sup_{t\in\mathbb{T}} X(t) \geq 0$ for every zero-mean Gaussian process X. When is the equality achieved?

(30) Let W^H be fractional Brownian motion with Hurst parameter $H \in (0,1)$. Then $\rho_X(t,s) = |t-s|^H$, and the Fernique-Sudakov inequality immediately implies that the function $H \mapsto \mathbb{E}\sup_{0<t<1} W^H(t)$ is decreasing in H. What can you say about the function $H \mapsto \mathbb{E}\sup_{0<t<T} W^H(t)$ for an arbitrary fixed $T > 0$? [Self-similarity of W^H might allow the reduction to the case $T = 1$.]

Key words and phrases

(1) Cameron-Martin space
(2) Chaining
(3) Dudley's integral
(4) Fernique's Theorem
(5) Gaussian inequalities: comparison, correlation, isoperimetric, measure concavity, etc.
(6) KL expansion
(7) LIL
(8) Slepian's lemma

Basic ideas.

(1) Many representations of Gaussian processes have an analogy with various matrix decompositions in linear algebra.
(2) A suitable representation of the Gaussian process can lead to the complete *mathematical* solution of a particular problem [e.g. KL expansion solves the small ball problem in a suitable Hilbert space; with a right kernel, integral representation solves the large deviation problem, and also identifies the Cameron-Martin space].
(3) Mercer's theorem is still work in progress.
(4) RKHS (reproducing kernel Hilbert space) is all over the place.

Reflective questions for discussions.[2]

(1) Select a book on the topic and write a review, either in the spirit of Mathematical Reviews, or following a more comprehensive approach of the Book Review section of the Bulletin of the American Mathematical Society.
(2) Take one homework problem you have worked on this semester that you struggled to understand and solve, and explain how (or if...) the struggle itself was valuable.
(3) What mathematical ideas are you curious to know more about as a result of taking this class? Give one example of a question about the material that you would like to explore further, and explain why you consider this question interesting.
(4) What three theorems did you most enjoy from the course, and why?
(5) Formulate a research question related to the course material that you would like to answer.
(6) Reflect on your overall experience in this class by describing an interesting idea that you learned, why it was interesting, and what it tells you about doing or creating mathematics.
(7) Think of one particular proof [of a result related to the topic of this class] and share your ideas about the ways you think the proof should be improved.
(8) If you were to write a textbook on the subject, what topics would you include, what topics would you exclude, and why? How about a research monograph?

[2]Most are not mine, including the wording. Suggestions for improvement will be part of the discussion.