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NOTE

AN ASYMPTOTIC EQUIVALENT FOR THE NUMBER OF TOTAL PREORDERS ON A FINITE SET

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In this note, it is shown that, the number of total preorders on a finite set with n elements is equivalent, for n infinite, to $n!/2(\text{Log}\,2)^{n+1}$.

1. A proposition

Let X be a set with n elements and let $P(n)$ denote the number of total preorders on X.

Proposition. *For n infinite, $P(n)$ is equivalent to* $n!/2(\text{Log}\,2)^{n+1}$

2. The proof

The number of total preorders on X with exactly k classes is—obviously—$k!\,S(n,k)$, $S(n,k)$ being the Stirling number of second kind (number of X's partitions into exactly k classes). Let us compute the generating function $F(t)=\sum_{n\geqslant 0}[P(n)/n!]t^n$.

$$F(t)=\sum_{n\geqslant 0}\frac{1}{n!}\left(\sum_{k=0}^{n}k!\,S(n,k)\right)t^n$$
$$=\sum_{k\geqslant 0}k!\left(\sum_{n\geqslant k}\frac{S(n,k)}{n!}t^n\right).$$

It is well known that

$$\sum_{n\geqslant k}\frac{S(n,k)}{n!}t^n=\frac{(\mathrm{e}^t-1)^k}{k!}\quad\text{(see [2]).}$$

So

$$F(t)=\sum_{k\geqslant 0}(\mathrm{e}^t-1)^k=\frac{1}{2-\mathrm{e}^t}.$$

(convergence for $|t|<\text{Log}\,2$). $t=\text{Log}\,2$ is the only singularity of the function $1/(2-e^t)$ and it is a pole of order one. The "Darboux theorem" (see [1, 3]) gives us an asymptotic equivalent for $P(n)$:

$$P(n)=\frac{-n!}{(\text{Log}\,2)^{n+1}}c_{-1}+o((n-1)!)$$

where c_{-1} is the residue at $t=\text{Log}\,2$ of $F(t)$.

$$c_{-1}=\lim_{t\to\text{Log}\,2}\frac{t-\text{Log}\,2}{2-e^t}=-\frac{1}{2}.$$

Hence the result.

3. Some remarks

From the generating function

$$F(t)=\sum_{n\geq 0}\frac{P(n)}{n!}t^n=\frac{1}{2-e^t},$$

we can deduce that:

(1) The derivatives of $F(t)$ satisfying the recurrence:

$$2F^{(n)}(t)-\sum_{p=0}^{n}\binom{n}{p}e^tF^{(n-p)}(t)=0,$$

we get:

$$P(n)=\sum_{p=1}^{n}\binom{n}{p}P(n-p).$$

(2) The radius of convergence of $\sum_{n\geq 0}[P(n)/n!]t^n$ being Log 2,

$$\lim_{n\to\infty}\frac{nP(n-1)}{P(n)}=\text{Log}\,2.$$

$nP(n-1)/P(n)$ is the probability for a total preorder to have only one greatest element (when all total preorders are equiprobable).

(3) $$H(t)=\frac{t}{e^t-1}=\sum_{n\geq 0}\frac{B_n}{n!}t^n$$

being the generating function for Bernoullis numbers, we get

$$\begin{aligned}F(t)&=\frac{-1}{2(t-\text{Log}\,2)}H(t-\text{Log}\,2)\\&=\frac{-1}{2(t-\text{Log}\,2)}+\sum_{n=0}^{\infty}\frac{B_{n+1}}{(n+1)!}(t-\text{Log}\,2)^n\end{aligned}$$

(because $B_0=0$): this is the Laurent's expansion of $F(t)$ at $t=\operatorname{Log} 2$. From

$$\frac{-1}{2(t-\operatorname{Log} 2)}=\frac{1}{2\operatorname{Log} 2}\cdot\sum_{p\geq 0}\frac{t^p}{(\operatorname{Log} 2)}p$$

and

$$\sum_{n=0}^{\infty}\frac{B_{n+1}}{(n+1)!}(t-\operatorname{Log} 2)^n=$$

$$=\sum_{n=0}^{\infty}\frac{B_{n+1}}{(n+1)!}\sum_{p\leq n}\binom{n}{p}t^p\cdot(-1)^{n-p}(\operatorname{Log} 2)^{(n-p)}$$

$$=\sum_{p=0}^{\infty}t^p\left(\sum_{n>p}\frac{B_{n+1}}{(n+1)!}\binom{n}{p}(\operatorname{Log} 2)^{n-p}(-1)^{n-p}\right)$$

it follows that

$$P(n)=\frac{n!}{2(\operatorname{Log} 2)^{n+1}}+\sum_{p=0}^{\infty}(-1)^p\frac{B_{n+p+1}}{(n+p+1)p!}(\operatorname{Log} 2)^p.$$

$\sum_{p=0}^{\infty}(-1)^p[B_{n+p+1}/(n+p+1)p!](\operatorname{Log} 2)^p$ is the value of the o $((n-1)!)$ in the fomula

$$P(n)=\frac{n!}{2(\operatorname{Log} 2)^{n+1}}+\mathrm{o}((n-1)!).$$

References

[1] G. Darboux, Mémoire sur l'approximation des fonctions de très grands nombres, J.M. Pures Appl. 4 (1878) 5–56 and 377–416.

[2] J. Riordan, An Introduction to Combinatorial Analysis, (Wiley, New York, 1958).

[3] G. Szego, Orthogonal Polynomials, 3rd ed. (Amer. Math. Soc., Providence, RJ, 1967).