Applications of Fourier Analysis to the Study of Analog, Discrete, and Digital Signals[1]

Main notations used below

- $\mathbb{N} = \{1, 2, \ldots\},\ \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\},\ \mathbb{Z}_+ = \{0, 1, 2, \ldots\},\ \mathbb{R} = (-\infty, +\infty)$;
- $\mathfrak{i} = \sqrt{-1}$, Re is the real part of a complex number, Im is the imaginary part of a complex number;
- 1_A: indicator function of the set or condition A, so that $1_A = 1$ if you are in the set A or condition A holds and $1_A = 0$ otherwise.

Analog signal $f = f(t),\ f \in (A, B) \subseteq \mathbb{R},\ t \in (a, b) \subseteq \mathbb{R}$. In other words, we have a "continuum" of possible values for both the domain and the range; note that f does not have to be continuous as a function of t. As we know, such signals are represented by Fourier series (if the interval (a, b) is finite) or Fourier transform (if at least one end of the interval (a, b) is infinite).

An application to ODEs: `Forced oscillations`. Let $f = f(t)$ be periodic with period $T = 2\pi/\omega$ and consider the equation

$$y''(t) + \gamma y'(t) + \omega_0^2\, y(t) = f(t),\ \omega_0 > 0,\ \gamma \geq 0.$$

Recall that this equation models basic mechanical and electrical systems exhibiting (damped) oscillations; γ describes damping.

Writing

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{\mathfrak{i}k\omega t},\ y(t) = \sum_{k=-\infty}^{\infty} y_k e^{\mathfrak{i}k\omega t}$$

and substituting into the equation, we find $y_k(-k^2\omega^2 + \mathfrak{i}\gamma k\omega + \omega_0^2) = f_k$ or

$$y_k = \frac{f_k}{(\omega_0^2 - k^2\omega^2) + \mathfrak{i}\gamma k}.$$

In other words, we found a particular solution of the equation:

$$y(t) = \sum_{k=-\infty}^{+\infty} \frac{f_k}{(\omega_0^2 - k^2\omega^2) + \mathfrak{i}\gamma k}\ e^{\mathfrak{i}k\omega t}. \tag{1}$$

This is great as long as we are not dividing by zero, that is

$$(\omega_0^2 - k^2\omega^2) + \mathfrak{i}\gamma k \neq 0,\ k \in \mathbb{Z}. \tag{2}$$

If $\gamma \neq 0$, then (2) always holds. `Resonance` happens exactly when (2) fails, that is, if

$$\gamma = 0 \text{ AND } \omega_0 = k_0\omega$$

for some $k_0 \in \mathbb{N}$. In that case, the equation does not have a particular solution in the form (1); instead, the particular solution becomes unbounded as $t \to +\infty$.

If $\gamma > 0$ but small, the solution, while staying bounded, can still be [arbitrarily] large. For example, take the input a 2π-periodic rectangular wave

$$f(t) = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\sin(2k+1)t}{2k+1},$$

that is, $f(t) = \pi$ on $(0, \pi)$ and $f(t) = -1$ on $(\pi, 2\pi)$. Then $\omega = 1$ and, with $\sin kt = \mathrm{Im}\Big(e^{\mathfrak{i}kt}\Big)$, equality (1) becomes

$$y(t) = \sum_{k=0}^{\infty} Y_k(t),$$

where

$$Y_k(t) = \mathrm{Im}\left(\frac{4e^{\mathfrak{i}(2k+1)t}}{\pi\big((\omega_0^2 - (2k+1)^2) + \mathfrak{i}\gamma(2k+1)\big)}\right). \tag{3}$$

If $\omega_0 = 2\ell + 1$ for some $\ell \in \mathbb{Z}_+$, then [by (3), with $\omega = 1$, $k = 2\ell + 1 = \omega_0$, and $f_k = 4/(\pi(2\ell + 1))$]

$$Y_{2\ell+1}(t) = \mathrm{Im}\left(\frac{e^{\mathfrak{i}(2\ell+1)t}}{\mathfrak{i}\gamma(2\ell+1)^2}\right) = -\frac{\cos(2\ell+1)t}{\gamma(2\ell+1)^2}.$$

As a numerical example, take $\omega_0 = 3$ and $\gamma = 0.01$. Then $Y_3(t) \approx -14\cos(3t)$. Roughly speaking, the input gets amplified by a factor of 10, so the system might still break down.

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Discrete signal f_k, $k \in \mathbb{K} \subseteq \mathbb{Z}$, each $f_k \in (A, B) \subseteq \mathbb{R}$: we discretize the domain of f, but not the range. In other words, we have at most countable many values of the signal, but each value still has a continuum of options. A typical scenario is $f_k = f(k\triangle t)$: samples of a continuous signal f with constant step $\triangle t$.

The *canonical discretization* of a signal $f = f(t)$ is

$$f_d(t) = \triangle t \sum_{k=-\infty}^{+\infty} f(k\triangle t)\delta(t - k\triangle t); \tag{4}$$

as usual δ is the Dirac delta function. Note that (4) can be written as

$$f_d(t) = \triangle t\Big(f * \sqcup\!\sqcup_{\triangle t}\Big)(t), \tag{5}$$

and the (generalized) function $\sqcup\!\sqcup_{\triangle t}(t) = \sum_{k=-\infty}^{\infty} \delta(t - k\triangle t)$ is called the `Dirac comb`. The extra factor $\triangle t$ in (4) and (5) is necessary to make the units agree: if we think of t as a quantity having unites of time, then the (physical) definition the Dirac delta function implies that δ has the units of inverse time. In this connection, note that, assuming the original signal f has no units, the Fourier coefficients $c_k(f)$ have no units either [in particular, if the period is 2π, then π has to come with units of time]; the Fourier transform has units of time: check it out, in particular, in the context of Parseval and Plancherel equalities.

The (discrete) signal f_d is represented by the Fourier transform as follows.

Theorem 1. Under suitable conditions,

$$\hat{f}_d(\omega) = \sum_{k=-\infty}^{+\infty} \hat{f}\left(\omega - \frac{2\pi}{\triangle t}\, k\right). \tag{6}$$

The proof relies on another useful result.

Theorem 2. [`Poisson Summation Formula`] Under suitable conditions on the function h,

$$\sum_{k=-\infty}^{+\infty} h(2kL) = \frac{\sqrt{2\pi}}{2L} \sum_{n=-\infty}^{+\infty} \hat{h}\left(\frac{\pi}{L}\, n\right) \tag{7}$$

for every $L > 0$.

Proof of Theorem 2. Consider the function

$$g_L(x) = \sum_{k=-\infty}^{+\infty} h(x + 2kL).$$

Then $g_L(x) = g_L(x + 2kL)$ for all $k \in \mathbb{Z}$ (that is, g_L is periodic with period $2L$), and, under suitable conditions, $S_{g_L}(0) = g_L(0)$. On the other hand, direct computations show that

$$c_n(g_L) = \frac{\sqrt{2\pi}}{2L} \hat{h}\left(\frac{\pi}{L}\, n\right),$$

and the result follows. □

The following are immediate consequences of (7):

- With $\theta > 0$, $L = \pi\sqrt{\theta}$ and $h(t) = e^{-t^2}$, (7) becomes the θ `formula`

$$\sum_{k=-\infty}^{+\infty} e^{-4\theta\pi^2 n^2} = \frac{1}{\sqrt{2\pi\theta}} \sum_{n=-\infty}^{+\infty} e^{-n^2/(4\theta)}.$$

  This formula helps with approximations of the sum $\sum_{k=-\infty}^{+\infty} e^{-ak^2}$ for various values of $a > 0$, which is helpful, for example, in statistics, for example, because of the `Kolmogorov-Smirnov test`.
- With $s \in \mathbb{R}$, $h(t) = \delta(t - s)$, and $L = \triangle t/2$, we get an alternative expression for Dirac's comb:

$$\sum_{k=-\infty}^{+\infty} \delta(s - k\triangle t) = \frac{1}{\triangle t} \sum_{n=-\infty}^{+\infty} \exp\Big(-\mathrm{i}s(2\pi/\triangle t)n\Big).$$

Proof of Theorem 1 using Theorem 2. Apply $\mathcal{F}$ to both sides of (4) fixing $\omega = \omega_0$. The result is

$$\hat{f}_d(\omega_0) = \frac{\triangle t}{\sqrt{2\pi}} \sum_{k=-\infty}^{+\infty} f(k\triangle t) e^{-\mathrm{i}\omega_0 k\triangle t} \tag{8}$$

Now apply (7) with $h(t) = f(t)e^{-\mathrm{i}\omega_0 t}$ and $2L = \triangle t$. The result is

$$\hat{f}_d(\omega_0) = \sum_{k=-\infty}^{+\infty} \hat{f}\left(\omega_0 + \frac{2\pi}{\triangle t}\, k\right),$$

which, upon a brief inspection, is the same as (6). $\square$

Theorem 2 has various ramifications, of various degree of importance. Here are some, in no particular order:

(1) The function $\hat{f}_d$ [or some multiple of it] is called the `Discrete Time Fourier Transform` (DTFT) of the (discretized) signal f.

(2) The right-hand side of (6) illustrates overlapping (or `aliasing`) of the spectrum as a result of discretization, and, consequently, impossibility of a perfect recovery of the signal from the samples unless $|\hat{f}(\omega)| = 0$ for $|\omega| > \omega_0$ and we take $\triangle t \leq \frac{\pi}{\omega_0}$ (*sampling at least at twice the rate of the highest frequency*) which takes us back to the sampling theorem.

(3) The function $\hat{f}_d$ is periodic with period $2\pi/\triangle t$. Representing $\hat{f}_d$ as a Fourier series, we can get some other interesting identities, including [under the right conditions] the sampling theorem.

(4) No physical signal f can have $|\hat{f}(\omega)| = 0$ for $|\omega| > \omega_0$. More precisely, if a signal f has `compact support`: there exists a t_0 such that $|f(t)| = 0$ for all $|t| > t_0$, then the support of $\hat{f}$ is not compact: for every $\omega_0 > 0$ there exists an ω_1 with $|\omega_1| > \omega_0$ and $|\hat{f}(\omega_1)| > 0$. Conversely, if $\hat{f}$ has compact support (as required in the sampling theorem), then f does not. An intuitive way to see it is to notice that if $\hat{f}$ has compact support, then $\hat{f}(\omega) = \hat{f}(\omega)1_{|\omega|<\omega_0}$. Taking the inverse Fourier transform we conclude that $f(t) = (f * g)(t)$, where $g(t) \sim \sin(\omega_0 t)/t$ [give or take a constant factor], and g [clearly] does not have compact support. At deeper level, this observation takes you to the `Heisenberg uncertainty principle` [the more concentrate the signal, the more spread out the Fourier transform] and the `Paley-Wiener theorem` [what kind of (generalized) functions have compactly supported Fourier transform].

(5) If we put $z = e^{\mathrm{i}\omega_0 \triangle t}$ in (8), then the right-hand side becomes

$$F(z) = \frac{\triangle t}{\sqrt{2\pi}} \sum_{k=-\infty}^{+\infty} f(k\triangle t) z^{-k},$$

which, up to the $\sqrt{2\pi}$ factor, is called the z `transform` of the discrete signal f_d. If the signal is `causal`, that is, $f(t) = 0$ for $t < 0$ [equivalently, $f(k\triangle t) = 0,\ k < 0$], then we get a more familiar expression

$$F(z) = \frac{\triangle t}{\sqrt{2\pi}} \sum_{k=0}^{+\infty} f(k\triangle t) z^{-k}.$$

Digital signal $X_k \in \{x_1, \ldots, x_M\}$, with $k \in \{k_1, \ldots, k_N\}$. In other words, we discretize both the domain and the range of f, and get at most MN numbers. Note that in many applications, you can choose M ["number of bits"] and N [number of samples].

With no loss of generality, we assume that $k \in \{0, 1, \ldots, N-1\}$. For the discussion that follows, it does not really matter whether the set of values of X_k is finite or not.

The main relations are as follows:

$$\sum_{n=0}^{N-1} e^{2\pi \mathrm{i} n(m-k)/N} = \begin{cases} N, & m = k + \ell N,\ \ell \in \mathbb{Z} \\ 0, & \text{otherwise}; \end{cases} \tag{9}$$

$$X_k = \sum_{n=0}^{N-1} F_n e^{2\pi \mathrm{i} kn/N}, \text{ where} \tag{10}$$

$$F_n = \sum_{k=0}^{N-1} X_k e^{-2\pi \mathrm{i} kn/N}. \tag{11}$$

All three are verified by direct computation.

The collection of numbers $F_n,\ n = 0, 1, \ldots, N-1$ is called the `discrete Fourier transform` [or DFT] of the signal $X = X_k,\ k = 0, \ldots, N-1$. The main point is that, when the number N has sufficiently many divisors [for example, $N = 2^r$ for some integer r] then special algebraic properties of the complex exponentials make it possible to organize the computations of F_n in such as way that the total number of operations required to compute all N numbers $F_0, \ldots, F_{N-1}$ is (much) smaller than the N^2 required by the straightforward computation. The "optimal" bound of *order* $N \log N$ is achieved exactly when $N = 2^r$. This is the idea behind the `fast Fourier transform`

[FFT], which is an algorithm [probably more like a family of algorithms] for computing DFT. Even without the details [which are rather long and eventually turn into a trade secret], there are several points to keep in mind:

- The first known appearance of FFT goes back to the early 1800's, when Gauss used it to simplify computations related to his tracking of a particular comet [while at that, he also introduced many other fundamental concepts, such as the normal distribution and the method of least squares; as always, fascinating history]. The result [about FFT] was eventually published in 1865 [10 years after Gauss died], but remained unnoticed for another 100 years, until AFTER Cooley and Tukey re-discovered it on their own.
- An operation in this context is one addition AND one multiplication.
- The word *order* in the above discussion means that the number of operations required by FFT is bounded from above by $CN \log N$ for some "absolute" number C. In other words, C stays bounded as N goes to infinity, but the actual value of C might depend on the "details" [such as whether you have $N = 2^r$ or $N = 4^r$]. Because of that, the base of the log does not matter. These considerations also lead to whole new concept of `computational complexity` of an algorithm.
- The ideas of FFT are used in the `Strassen algorithm` (and many similar algorithms) for multiplying $N \times N$ matrices using fewer than the "direct" N^3 operations [here, "fewer" means order of N^r with $r < 3$], as well as in the `Karatsuba algorithm` for multiplying two N-digit numbers using fewer than the "direct" N^2 single-digit multiplications [here, "fewer" means order of N^r with $r < 2$].
- The bottom line: because DFT of convolution is the product of DFT-s, and the convolution is a fundamental operation in engineering, FFT makes it possible to compute convolutions using (order of) $N \log N$ operations by computing DFT-s of the signals using FFT, multiplying the results [multiplication only takes N operations] and then using FFT again to compute the inverse DFT. As always, full details can be a trade secret.

Linear systems and stability

Recall that, for a linear system of ODEs with constant coefficients,

$$X'(t) = AX(t),\ t \geq 0,$$

[X is a vector, A is a (square) matrix] the "elementary building blocks" of the solution are $he^{\lambda t}$ for some vector h and some (complex) number λ. Substitution in the equation show that the equality

$$\lambda h = Ah$$

must hold, that is, h is an eigenvector of A and λ is the corresponding eigenvalue. The system is (asymptotically) stable if (and only if) all eigenvalues of A are in the left half-plane [strictly to the left of the imaginary axis in the complex plane]. Recall that neutrally stable system [when at least one eigenvalue has zero real part] can blow up with suitable input: think $y''(t) + y(t) = \sin t$; this is why we indeed want asymptotic stability [and this is why, for engineering and other applications, stability usually means `bounded output for every bounded input`]. When done via the Laplace transform, the analysis leads to the concept of the `transfer function`, and the fundamental result in this connection is that *a system is stable if and only if the poles of the transfer function have negative real parts.*

Similarly, in the case of discrete-time [and digital] systems, the starting point is the system of `finite difference equations` $X_{k+1} = AX_k,\ k \geq 0$, with vectors X_k and a square matrix A. Now we look for solutions in the form $r^k h$, with a vector h and number r. Substitution shows that the equality $rh = Ah$ must hold, that is, r has to be an eigenvalue of the matrix A. Just as in the case of continuous time, the number $r = a + \mathrm{i}b = |r|e^{\mathrm{i}\theta}$ can be complex. Now, to have (asymptotic) stability, we need $\lim_{k\to\infty} |r|^k = 0$, that is, $|r| < 1$, that is, *all eigenvalues of A must be strictly inside the unit disk in the complex plane.* When done via the z transform, the analysis leads to the concept of the [discrete time] `transfer function`, and the fundamental result in this connection is that *a system is stable if and only if all poles of the transfer function are strictly inside the unit disk in the complex plane.*

Keep in mind that some authors, in an attempt to further confuse the reader [or because of the conventions in that particular field] write the z transform as $\sum_{k\geq 0} a_k z^k$ [as opposed to $\sum_{k\geq 0} a_k z^{-k}$], and then stability means having the poles strictly *outside* of the unit disk.