Elementary Fluid Mechanics[1]

Equation of continuity

$$\rho_t + \nabla \cdot (\mathbf{u}\rho) = 0,$$

where ρ is the density of the "continuum" moving with velocity $\mathbf{u}$. The main particular case is the `incompressibility condition`, when the density ρ is not changing, either in time or in space:

$$\nabla \cdot \mathbf{u} = 0. \tag{1}$$

The convection term $(\mathbf{u}\cdot\nabla)\mathbf{u}$. Let $\mathbf{r}$ denote the position vector of a particle (in $\mathbb{R}^n$, with $n = 2$ or $n = 3$ in most applications).

First, recall that, if $\mathbf{r} = \mathbf{r}(t)$ and f is a scalar field in $\mathbb{R}^n$, then, by the chain rule,

$$\frac{d\,f(\mathbf{r}(t))}{dt} = \dot{\mathbf{r}}(t) \cdot (\nabla f)(\mathbf{r}(t))$$

Now, let us assume that $f = f(t, \mathbf{r})$ and

$$\dot{\mathbf{r}}(t) = \mathbf{u}(t, \mathbf{r}(t)). \tag{2}$$

Then the chain rule implies

$$\frac{d\,f(t,\mathbf{r}(t))}{dt} = \frac{\partial\,f(t,\mathbf{r}(t))}{\partial t} + \mathbf{u}(t,\mathbf{r}(t)) \cdot (\nabla f)(t,\mathbf{r}(t)). \tag{3}$$

The right-hand side of (3) is sometimes called the `material derivative` of f.

Now, instead of a scalar field f consider a vector field $\mathbf{G} \in \mathbb{R}^n$. Then (3) applies to each component of $\mathbf{G}$, and the result is written as

$$\frac{d\,\mathbf{G}(t,\mathbf{r}(t))}{dt} = \frac{\partial\,\mathbf{G}(t,\mathbf{r}(t))}{\partial t} + (\mathbf{u}(t,\mathbf{r}(t)) \cdot \nabla)\mathbf{G}(t,\mathbf{r}(t)). \tag{4}$$

Finally, let us assume that $\mathbf{r} = \mathbf{r}(t)$ is, in fact, a trajectory of a particle, with velocity $\mathbf{u} = \dot{\mathbf{r}}$. Then, by the second law of Newton,

$$m\ddot{\mathbf{r}}(t) = \mathbf{F},$$

where $\mathbf{F}$ are the forces acting on the particle. On the other hand, by (4),

$$\ddot{\mathbf{r}}(t) = \frac{d\mathbf{u}(t,\mathbf{r}(t))}{dt} = \frac{\partial\,\mathbf{u}(t,\mathbf{r}(t))}{\partial t} + (\mathbf{u}(t,\mathbf{r}(t)) \cdot \nabla)\mathbf{u}(t,\mathbf{r}(t)),$$

leading to the relation

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = \mathbf{F}/m, \tag{5}$$

which is a collection of n equations and is the starting point in the derivation of many of the equations related to fluids. Here are some of them.

Euler equations are the combination of (5) and (1). The key is that the term $\mathbf{F}/m$ contains one more unknown, the pressure p, which is necessary to ensure incompressibility and makes the number of unknowns match the number of equations. Assuming there are no other forces, the system of equations becomes

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = 0, \quad \nabla \cdot \mathbf{u} = 0. \tag{6}$$

Navier-Stokes equations (NSE) bring another "internal" force into the picture, related to viscosity ν of the fluid. The result is

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \nu\boldsymbol{\Delta}\mathbf{u}, \quad \nabla \cdot \mathbf{u} = 0. \tag{7}$$

When $n = 3$ and $\mathbf{u} = \langle u, v, w\rangle$, the system (6) becomes

$$\begin{aligned}
u_t + uu_x + vu_y + wu_z + p_x &= \nu\boldsymbol{\Delta}u\\
v_t + uv_x + vv_y + wv_z + p_y &= \nu\boldsymbol{\Delta}v\\
w_t + uw_x + vw_y + ww_z + p_x &= \nu\boldsymbol{\Delta}w.
\end{aligned}$$

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The reason why the internal friction in the fluid due to viscosity is indeed $\nu\boldsymbol{\Delta}\mathbf{u}$ is completely, and highly, non-trivial.

Burgers equations are NSE without the incompressibility condition (and without pressure term):

$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} = \nu\boldsymbol{\Delta}\mathbf{u}. \tag{8}$$

Unlike Euler and Navier-Stokes, a one-dimensional version of Burgers is non-trivial:

$$u_t + uu_x = \nu u_{xx}. \tag{9}$$

Equation (9) is connected to the heat equation via the `Hopf-Cole transformation`: if $v = v(t,x)$ satisfies

$$v_t = \nu v_{xx}, \tag{10}$$

then $u(t,x) = -\dfrac{2\nu v_x(t,x)}{v(t,x)} \equiv -2\nu\Big(\ln|v(t,x)|\Big)_x$ satisfies (9). Indeed, taking an arbitrary $x_0 \in \mathbb{R}$ and writing

$$U(t,x) = \int_{x_0}^{x} u(t,y)\,dy,$$

we get

$$v(t,x) = e^{-U(t,x)/(2\nu)}$$

so that

$$v_t = -\frac{U_t v}{2\nu},\ v_x = -\frac{U_x v}{2\nu} = -\frac{uv}{2\nu},\ v_{xx} = \left(\frac{u^2}{4\nu^2} - \frac{u_x}{2\nu}\right)v,$$

and then (10) implies

$$U_t = -\frac{u^2}{2} + \nu u_x. \tag{11}$$

After differentiation with respect to x, equation (11) becomes (10).

One can also consider the one-dimensional version of the Euler equations without incompressibility; the result is an example of a `hyperbolic conservation law`

$$u_t + uu_x = 0.$$

In the case of NSE, there is a big difference between 2 and 3 space dimensions. The (literally) million-dollar question is whether the solution remains a smooth function of (x,y,z) for all $t > 0$ if the initial condition $\mathbf{u}(0,x,y,z)$ is a smooth function.