Some Famous Problems in Probability

Agenda

- About Thomas Bayes
- Monty Hall problem
- The duel problem
- The gambler's ruin problem
- Second-order finite difference equations
- The probabilistic method

Thomas Bayes

- Lived 1702–1761
- Nonconformist minister in Tunbridge Wells, England, with a gambling problem.
- Essay *Towards Solving a Problem in the Doctrine of Chances*, published posthumously in 1763

Monty Hall Problem

1. **Monte Halperin** (1921–2017): Canadian.
2. *Let's Make a Deal* aired on NBC daytime from December 30, 1963 to December 27, 1968 and on ABC daytime from December 30, 1968 to July 9, 1976, along with two primetime runs. It also aired in syndication from 1971 to 1977, from 1980 to 1981, from 1984 to 1986, and again on NBC briefly from 1990 to 1991.
3. What's the big deal? (you `should` switch)

The duel problem

Input: A and B shooting at each other.
p_A: probability that A hits B; p_B: probability that B hits A;
$q = (1 - p_A)(1 - p_B)$: probability both miss.

What to compute:
P_A: probability A wins; P_B: probability B wins;
P_{AB}: probability of a draw; Q: probability the duel never ends.

The Solution

$$P_A = p_A(1-p_B) + qp_A(1-p_B) + q^2 p_A(1-p_B) + \cdots$$
$$= p_A(1-p_B)\sum_{n=0}^{\infty} q^n = \frac{p_A(1-p_B)}{1-q}$$
$$P_B = p_B(1-p_A) + qp_B(1-p_A) + q^2 p_B(1-p_A) + \cdots$$
$$= p_B(1-p_A)\sum_{n=0}^{\infty} q^n = \frac{p_B(1-p_A)}{1-q}$$
$$P_{AB} = p_A p_B \sum_{n=0}^{\infty} q^n = \frac{p_A p_B}{1-q},\ Q = \lim_{n\to\infty} q^n = 0,$$
$$\text{check}:\ P_A + P_B + P_{AB} = 1.$$

Gambler's Ruin Problem

Input: Two players (you and the casino) with the total capital $\$N$
on each round, you win or lose $\$1$
p: probability you win on one round.

What to compute: p_n, probability you win all the money without going broke first, if you start with $\$n$.

Solution: Denote by A_n the event in question, W: you win on round one, L: you loose on round one. Then $p_n = P(A_n) = P(A_n \bigcap W) + P(A_n \bigcap L) = P(A_n|W)P(W) + P(A_n|L)P(L)$. Then note that $P(A_n|W) = p_{n+1}$, $P(A_n|L) = p_{n-1}$ or

$$p_n = pp_{n+1} + (1-p)p_{n-1}$$

or

$$pp_{n+1} - p_n + (1-p) = 0,\ p_0 = 0,\ p_N = 1.$$

Second-Order Finite Difference Equations

$ay_{n+1} + by_n + cy_{n-1} = 0$, given two values of y_n; $a \neq 0,\ c \neq 0$.

Look for solution in the form $y_n = r^n$, then

$$ar^2 + br + c = 0 \ \Rightarrow \ r_1, r_2$$

Then the general solution is

$$y_n = \begin{cases} c_1 r_1^n + c_2 r_2^n, & r_1 \neq r_2 \\ (c_1 + nc_2) r^n, & r_1 = r_2 = r. \end{cases}$$

Gambler's ruin

p, probability of win on one round; p_n, probability of eventual win starting with $\$n$.

$pr^2 - r + (1-p) = 0,$

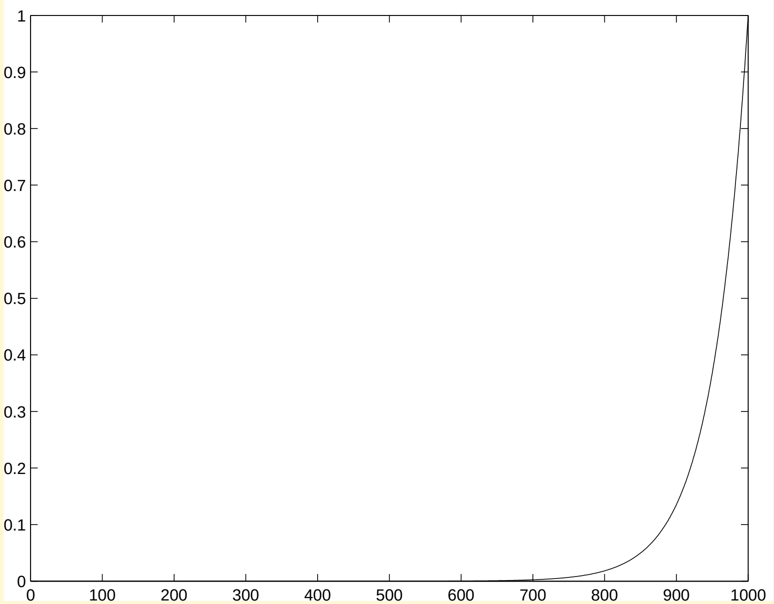
$$r = \frac{1 \pm \sqrt{1-4p(1-p)}}{2p} = \frac{1 \pm |2p-1|}{2p} = \begin{cases} 1, (1-p)/p, & p \neq 1/2; \\ 1, 1, & p = 1/2. \end{cases}$$

Final answer: (with $q = 1-p$)

$$p_n = \begin{cases} \frac{1-(q/p)^n}{1-(q/p)^N}, & p \neq 1/2 \\ n/N, & p = 1/2. \end{cases}$$

You have no chance!

$N = 1000$, $p = 0.495$, $n = 50$: $p_n \approx 3 \cdot 10^{-9}$



Probabilistic method

General idea: applying probabilistic tools to deterministic problems.
Some applications: (non-constructive) existence proofs; randomized algorithms
Basic example: graph edge coloring in two colors; n vertices, take k of them
$2 \cdot 2^{-k(k-1)/2}$ is the probability that one set of k vertices has all edges of one color. If

$$2\binom{n}{k} < 2^{k(k-1)/2}$$

then, with positive probability, all sets will have edges of both colors.
A related term: Ramsey's theory

Paul Erdős

- Lived 1913–1996
- Hungarian mathematician
- Ph.D. at 21
- Close to 1500 published papers with over 511 collaborators

What to remember

- Items on the agenda
- How to solve a second-order finite-difference equation
- Be careful if gambling: stop before it is too late!