Calculus I: Final summary

Computing Limits

(1) Algebraic manipulations:

Factoring

$$\lim_{x\to 2}\frac{x^2-4}{x^2-3x+2}=\lim_{x\to 2}\frac{(x-2)(x+2)}{(x-1)(x-2)}=4;$$

Rationalization (multiplying by a conjugate expression)

$$\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x}=\lim_{x\to 0}\frac{2x}{x(\sqrt{1+x}+\sqrt{1-x})}=1;$$

(2) Using the standard limits:

$$\lim_{x\to 0}\frac{\sin rx}{x}=r,\quad \lim_{x\to 0}(1+rx)^{1/x}=\lim_{x\to+\infty}\left(1+\frac{r}{x}\right)^{x}=e^r;$$

For example

$$\lim_{x\to 0}\frac{x-\sin 2x}{x+\sin 3x}=\lim_{x\to 0}\frac{1-(\sin 2x)/x}{1+(\sin 3x)/x}=\frac{1-2}{1+3}=-\frac{1}{4},$$

$$\lim_{x\to+\infty}\left(\frac{2x-1}{2x}\right)^{x}=\lim_{x\to+\infty}\left(\left(1-\frac{1}{2x}\right)^{2x}\right)^{x/(2x)}=e^{-1/2};$$

(3) Looking at the leading term:

$$\lim_{x\to+\infty}\frac{2\ln x+1}{3\ln x+5}=\lim_{x\to 0+}\frac{2\ln x+1}{3\ln x+5}=\frac{2}{3};$$

(4) Interpreting as a derivative:

$$\lim_{x\to 2}\frac{x^x-4}{x-2}=f'(2),\ f(x)=x^x;$$

(5) Interpreting as a Riemann sum

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}f(k/n)=\int_0^1 f(x)\,dx;$$

(6) Combinations of several ideas:

$$\lim_{x\to 0}\frac{\sqrt{1+\sin x}-\sqrt{1-\sin x}}{x}=\frac{2\sin x}{x(\sqrt{1+\sin x}+\sqrt{1-\sin x})}=1.$$

Computing Derivatives

(1) Direct (sum/product/chain);

(2) Logarithmic derivative: $f'(x)=f(x)\big(\ln f(x)\big)'$

Powers: $(x^x)'=(e^{x\ln x})'=x^x(x\ln x)'=x^x(\ln x+1)$;

Products: $f(x)=(x-1)^{-1/2}x^{-1/3}(x+1)^{3/2}(x+2)^{5/3}\ \Rightarrow$

$$f'(x)=f(x)\big(\ln f(x)\big)'=f(x)\left(-\frac{1}{2(x-1)}-\frac{1}{3x}+\frac{3}{2(x+1)}+\frac{5}{3(x+2)}\right).$$

(3) Inverse function: $f(f^{-1}(x))=x,\ f'(f^{-1}(x))(f^{-1}(x))'=1.$

(4) Implicit: $x^2+xy^2+y^3=1,\ y=y(x)\ \Rightarrow$

$$2x+y^2+2xyy'+3y^2y'=0.$$

(5) Higher order: $f(x)=x^{-1},\ f^{(n)}(x)=(-1)^n n! x^{-n-1}.$

(6) FTC:

$$F(x)=\int_{x^2}^{x^3}\sqrt{1+\sin^2 t}\,dt\ \Rightarrow\ F'(x)=3x^2\sqrt{1+\sin^2 x^3}-2x\sqrt{1+\sin^2 x^2}.$$

Integration

(1) Substitution, including the $\int x^n(1+x)^r\,dx$ forms (n positive integer, r real).

(2) Geometric argument: $\int_0^1 \sqrt{x-x^2}\,dx = \pi/8$: half the area of the circle with center at $(1/2,0)$ and radius $1/2$.

(3) Approximation:

$$\int_0^2 \sqrt{1+x^4}\,dx \approx \begin{cases} 1+\sqrt{2}, & \text{Two}-\text{step left point; underestimate} \\ \sqrt{2}+\sqrt{17}, & \text{Two}-\text{step right point; overestimate} \\ (1+2\sqrt{2}+\sqrt{17})/2, & \text{Two}-\text{step trapezoidal; overestimate} \end{cases}$$

Separable ordinary differential equations (sometimes, with solution given)

(1) Exponential growth/decay;

(2) Newton's Law of Cooling;

(3) Custom.

The "value" theorems

(1) Existence/number of solutions: if $f(x) = x(x+1)(x+2)(x+3)$, then $f'(x) = 0$ has three distinct real roots.

(2) Inequalities, including convexity arguments, for example,
$\sqrt{3+x^4} \geq 1+x,\ x \in \mathbb{R}$: the graph is above the tangent line at $(1,1)$
$\sqrt{1+x^3} \leq 1+x,\ x \in [0,2]$: the graph is below the secant line through $(0,1)$ and $(2,3)$.

Further applications

(1) Curve sketching: asymptotes via limits; $f'(x) \geq 0 \Rightarrow$ increasing; $f''(x) \geq 0 \Rightarrow$ concave down, etc.

(2) Tangent line/linear approximation: $f(x) \approx f(a) + f'(a)(x-a)$.

(3) Optimization: looking for global extremal values.

(4) Related rates, with the notation $\dot{v}(t) = dv(t)/dt$.

Explicit: if $y(t) = f(x(t))$, then $\dot{y} = f'(x)\dot{x}$;

Implicit: for example, if $x^2+y^2 = 1$ and $x = x(t),\ y = y(t)$, then $x\dot{x} + y\dot{y} = 0$.