Computations related to Fourier transform[1]

Recall that, for "nice" functions $f = f(x)$,

(1) $$\hat{f}(\omega) \equiv \mathcal{F}[f](\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-\mathrm{i}\omega x}\, dx,\ I_f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{f}(\omega) e^{\mathrm{i}\omega x}\, d\omega, \quad \mathrm{i} = \sqrt{-1},$$

and

(2) $$I_f(x) = \frac{1}{2}\Big(f(x+) + f(x-)\Big); \quad f(x+) = \lim_{y\to x,\ y>x} f(y),\ \ f(x-) = \lim_{y\to x,\ y<x} f(y).$$

We also remember that

$$e^{\mathrm{i}\omega x} = \cos(\omega x) + \mathrm{i}\sin(\omega x)$$

(Euler) and that

- an even function $f = f(x)$ satisfies $f(x) = f(-x)$;
- an odd function $f = f(x)$ satisfies $f(x) = -f(-x)$;
- cos is an even function;
- sin is an odd function;
- product of two even or two odd functions is even;
- product of an even and an odd function is odd;
- integral of an odd function over a symmetric interval (of the form $(-a, a)$, $a > 0$, with a possibility $a = \infty$) is zero;
- integral of an even function over a symmetric interval (of the form $(-a, a)$, $a > 0$, with a possibility $a = \infty$) is twice the integral of the same function over $(0, a)$;
- Derivative of an even function is odd; derivative of an odd function is even.

With this in mind, we take a closer look at the Fourier transform of even and odd functions.

For an *even* function $f(x)$, the Fourier integral becomes the Fourier cosine integral:

$$I_f(x) = \int_0^{\infty} A(\omega)\cos(\omega x)\, d\omega \text{ where } A(\omega) = \frac{2}{\pi}\int_0^{\infty} f(v)\cos(\omega v)\, dv.$$

Accordingly, we define the **Fourier cosine transform** $\hat{f}_c(\omega)$ (or $\mathcal{F}_c[f]$) by

$$\hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x)\cos(\omega x)\, dx.$$

In this case,

$$I_f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \hat{f}_c(\omega)\cos(\omega x)\, d\omega,$$

and we call I_f the **inverse Fourier cosine transform** of $\hat{f}_c(\omega)$.

Similarly, for an *odd* function $f = f(x)$, the Fourier integral is the Fourier sine integral:

$$I_f(x) = \int_0^{\infty} B(\omega)\sin(\omega x)\, d\omega \text{ where } B(\omega) = \frac{2}{\pi}\int_0^{\infty} f(v)\sin(\omega v)\, dv.$$

Accordingly, we define the **Fourier sine transform** $\hat{f}_s(\omega)$ (or $\mathcal{F}_s[f]$) by

$$\hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x)\sin(\omega x)\, dx.$$

In this case,

$$I_f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \hat{f}_s(\omega)\sin(\omega x)\, dw,$$

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and we call I_f the **inverse Fourier sine transform** of $\hat{f}_s(\omega)$.

If a function is originally defined only on $(0,+\infty)$, then we have two "natural" ways to continue this function to $(-\infty,0)$: in an even way or in an odd way. These two choices will lead to two different representations (of the same function!), using, respectively, the Fourier cosine integral and the Fourier sine integral. The third option would be to set the function equal to zero for $x<0$, and then the corresponding (third!) representation of f is by the Fourier integral I_f. [Notice the analogy with the Fourier series!] **Fourth** representation of a function $f=f(x),\ x>0$ is given by the Laplace transform:

$$\mathcal{L}[f](s)=\int_0^{+\infty} f(x)e^{-sx}\,dx.$$

If we set $f(x)=0$ for $x<0$, then the Laplace transform becomes the Fourier transform evaluated on the imaginary axis [and the other way around]:

$$\mathcal{L}[f](s)=\sqrt{2\pi}\mathcal{F}[f](-\mathfrak{i}\,s),\quad \mathcal{F}[f](\omega)=\frac{1}{\sqrt{2\pi}}\mathcal{L}[f](\mathfrak{i}\,\omega). \tag{3}$$

If a function $f=f(x)$ (real-valued or complex-valued) is originally defined for all x, we can still compute $\hat{f}_c$ and $\hat{f}_s$ by considering only the values of f for $x>0$. The Fourier cosine transform is an even function and represents the even extension of f [that is, when $f(x)$ is re-defined to be equal to $f(-x)$ for $x<0$]; the Fourier sine transform is an odd function and represents t he odd extension of f [that is, when $f(x)$ is re-defined to be equal to $-f(-x)$ for $x<0$].

If $f=f(x)$ is even to begin with, then

$$\hat{f}(\omega)=\hat{f}_c(\omega);$$

if $f=f(x)$ is odd, then

$$\hat{f}(\omega)=-\mathfrak{i}\,\hat{f}_s(\omega),$$

One can play further with the formulas and get additional equalities of this type. For example, if f is real-valued, then

$$I_f(x)=\frac{1}{\pi}\int_0^{+\infty}\int_{-\infty}^{+\infty} f(y)\cos\big(\omega(x-y)\big)\,dy\,d\omega;$$

if f is even, then

$$I_f(x)=\mathcal{F}_c[\hat{f}_c](x).$$

In particular, the Fourier cosine transform is an *involution*:

$$\mathcal{F}_c=\mathcal{F}_c^{-1}. \tag{4}$$

More generally, an involution is any operation that is equal to its own inverse. Examples include

(1) taking the reciprocal of a non-zero number $x\mapsto 1/x$;
(2) complex conjugation $x+\mathfrak{i}y\mapsto x-\mathfrak{i}y$;
(3) transposition of a matrix $A\mapsto A^\top$;
(4) inversion of an invertible square matrix $A\mapsto A^{-1}$;
(5) various operations on functions, for example reflections $f(x)\mapsto f(-x)$, $f(x)\mapsto -f(x)$, etc

To get a better idea,

- Confirm that Fourier sine transform is also an involution.
- Confirm that if f is real-valued, then

  $$\hat{f}(-\omega)=\overline{\hat{f}(\omega)};$$

  as usual, the line on top means complex conjugation.
- Determine how I_f looks if f takes purely imaginary values [that is, if the real part of f is zero].

- Confirm that *every* function $f = f(x)$ can be written as a sum of even function and an odd function,

(5) $$f(x) = f_e(x) + f_o(x),\ f_e(x) = \frac{f(x) + f(-x)}{2},\ f_o(x) = \frac{f(x) - f(-x)}{2},$$

  and then investigate the corresponding formulas for $\hat{f}$ and I_f.
- Confirm that, in general,
$$\hat{f}(\omega) \neq \hat{f}_c(\omega) - \mathfrak{i}\,\hat{f}_s(\omega)$$
  (the only exception is $f \equiv 0$); instead, with f_e, f_o from (5), we have

(6) $$\hat{f}(\omega) = \mathcal{F}_c[f_e](\omega) - \mathfrak{i}\,\mathcal{F}_s[f_o](\omega).$$

- The Fourier transform is not an involution, but comes pretty close:

(7) $$\mathcal{F}[\hat{f}](x) = f(-x),$$

  which follows either by looking at (1) or by combining (5) and (6) to get
$$\mathcal{F}[\hat{f}] = f_e - f_o.$$

If you pay close attention, you note a difference between (2) and related equalities, such as (7). The way to reconcile the differences is to recall that the value of an integral does not change if the values of a function are changed in a few points. As a result, different functions can have identical Fourier transforms. The way to resolve this discrepancy is to consider *equivalence classes* of functions so that all functions from a given equivalence class have the same Fourier transform.

Example 1.
For $f(x)$ defined by
$$f(x) = \begin{cases} 1 & \text{if } 0 < x < a, \\ 0 & \text{if } x > a, \end{cases}$$
the Fourier cosine transform and Fourier sine transform are
$$\hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^a \cos(\omega x)\, dx = \sqrt{\frac{2}{\pi}} \frac{\sin(a\omega)}{\omega},$$
$$\hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^a \sin(\omega x)\, dx = \sqrt{\frac{2}{\pi}} \frac{1 - \cos(a\omega)}{\omega}.$$
The Fourier cosine integral gives the even extension of the function f:
$$\frac{2}{\pi} \int_0^\infty \frac{\sin(a\omega)}{\omega} \cos(\omega x)\, d\omega = \begin{cases} 1 & \text{if } |x| < a, \\ 0 & \text{if } |x| > a; \end{cases}$$
we can put $x = 0$ and recover the familiar result
$$\int_0^\infty \frac{\sin(a\omega)}{\omega}\, d\omega = \frac{\pi}{2},\ a > 0.$$
The Fourier sine integral gives the odd extension of f:
$$\frac{2}{\pi} \int_0^\infty \frac{1 - \cos(a\omega)}{\omega} \sin(\omega x)\, d\omega = \begin{cases} 1 & \text{if } 0 < x < a, \\ -1 & \text{if } -a < x < 0, \\ 0 & \text{if } |x| > a \text{ or } x = 0. \end{cases}$$
In particular, for $0 < x < a$, we get interesting equalities
$$\int_0^\infty \frac{\sin(a\omega)}{\omega} \cos(\omega x)\, d\omega = \int_0^\infty \frac{1 - \cos(a\omega)}{\omega} \sin(\omega x)\, d\omega = \frac{\pi}{2}.$$

Finally,

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_0^a e^{-\mathrm{i}x\omega}\, dx = \frac{1 - e^{-\mathrm{i}\omega a}}{\mathrm{i}\sqrt{2\pi}\,\omega},$$

so that

$$I_f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1 - e^{-\mathrm{i}\omega a}}{\mathrm{i}\omega}\, e^{\mathrm{i}\omega x}\, d\omega = \begin{cases} 1 & \text{if } 0 < x < a, \\ 1/2 & \text{if } x = 0 \text{ or } x = a, \\ 0 & \text{if } x > a \text{ or } x < 0, \end{cases} \tag{8}$$

which also looks like an interesting identity.

We also know that

$$\mathcal{L}[f](s) = \int_0^1 e^{-sx}\, dx = \frac{1 - e^{-s}}{s},$$

which, together with (8) suggests that we can invert the Laplace transform as follows:

$$\frac{1}{2\pi\mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{1 - e^{-sx}}{s}\, e^{sx}\, ds = 1,\ \ 0 < x < a. \tag{9}$$

□

Remark 1.

(1) If $f(x)$ is *absolutely integrable* (*i.e.*, $\int_{-\infty}^{\infty} |f(x)|\, dx$ is well-defined and *finite*) and piecewise continuous on every finite interval, then the Fourier cosine transform and Fourier sine transform of f exist.

(2) The Fourier cosine transform, Fourier sine transform, and the (original) Fourier transform are *linear operators*,

$$\begin{aligned} \mathcal{F}_c[af + bg] &= a\mathcal{F}_c[f] + b\mathcal{F}_c[g], \\ \mathcal{F}_s[af + bg] &= a\mathcal{F}_s[f] + b\mathcal{F}_s[g], \\ \mathcal{F}[af + bg] &= a\mathcal{F}[f] + b\mathcal{F}_[g], \end{aligned}$$

where a and b are constants (real or complex numbers) and f and g are absolutely integrable and piecewise continuous on every finite interval. Indeed, everything follows from linearity of integrals. For the cosine transform,

$$\begin{aligned} \mathcal{F}_c[af + bg] &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} [af(x) + bg(x)] \cos(\omega x)\, dx \\ &= a\sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos(\omega x)\, dx + b\sqrt{\frac{2}{\pi}} \int_0^{\infty} g(x) \cos(\omega x)\, dx. \end{aligned}$$

□

Theorem 1. (Fourier cosine transform, Fourier sine transform, and Fourier transform of derivatives) Let $f(x)$ be continuous and absolutely integrable, let $f'(x)$ be piecewise continuous on each finite interval, and $f(x) \to 0$ as $|x| \to \infty$. Then

$$\begin{aligned} &\mathcal{F}_c[f'](\omega) = \omega \hat{f}_s(\omega) - \sqrt{\frac{2}{\pi}} f(0+); \\ &\mathcal{F}_s[f'](\omega) = -\omega \hat{f}_c(\omega); \\ &\mathcal{F}[f'](\omega) = \mathrm{i}\,\omega \hat{f}(\omega). \end{aligned}$$

Proof. Integrate by parts. For the first one,

$$\begin{aligned}\mathcal{F}_c[f'](\omega) &= \sqrt{\frac{2}{\pi}}\int_0^\infty f'(x)\cos(\omega x)\,dx \\ &= \sqrt{\frac{2}{\pi}}\left(f(x)\cos(\omega x)\Big|_0^\infty + w\int_0^\infty f(x)\sin(\omega x)\,dx\right) \\ &= -\sqrt{\frac{2}{\pi}}f(0+) + \omega\mathcal{F}_s[f](\omega).\end{aligned}$$

The other two are the same. Note that, because we only consider the function f on $(0, +\infty)$, we have to write $f(0+)$ as opposed to $f(0)$. □

Corollary. Two consecutive applications of the above equalities result in the corresponding formula for the second derivatives [check it out!]:

$$\mathcal{F}_c[f''](\omega) = -\omega^2\hat{f}_c(\omega) - \sqrt{\frac{2}{\pi}}f'(0);$$
$$\mathcal{F}_s[f''](\omega) = -\omega^2\hat{f}_s(\omega) + \sqrt{\frac{2}{\pi}}\omega f(0);$$
$$\mathcal{F}[f''](\omega) = -\omega^2\hat{f}(\omega).$$

Example 2. Let us compute Fourier cosine transform, Fourier sine transform, and the Fourier transform of the function $f(x) = e^{-ax}, \ x \geq 0$, where $a > 0$.

Fourier cosine transform

Because $f''(x) = a^2 f(x), \ x > 0$, the corollary implies

$$a^2\mathcal{F}_c[f] = \mathcal{F}_c[a^2 f] = \mathcal{F}_c[f''] = -\omega^2\mathcal{F}_c[f] - \sqrt{\frac{2}{\pi}}f'(0) = -\omega^2\mathcal{F}_c[f] + a\sqrt{\frac{2}{\pi}},$$

i.e.,

$$(a^2 + \omega^2)\mathcal{F}_c(f) = a\sqrt{\frac{2}{\pi}}.$$

Thus,

$$\mathcal{F}_c(f) = \sqrt{\frac{2}{\pi}}\left(\frac{a}{a^2 + \omega^2}\right)$$

and

$$\frac{2}{\pi}\int_0^{+\infty}\frac{a\cos(\omega x)}{a^2 + \omega^2}\,d\omega = e^{-a|x|}.$$

With $a = x = 1$, we recover a familiar result:

$$\int_{-\infty}^{\infty}\frac{\cos\omega}{1 + \omega^2}\,d\omega = \frac{\pi}{e}.$$

Moreover, using (4), we conclude that if

$$h(x) = \frac{1}{x^2 + a^2},$$

then, because the function h is even,

$$\hat{h}(\omega) = \hat{h}_c(\omega) = \sqrt{\frac{\pi}{2}}\frac{e^{-a|\omega|}}{a}.$$

To make sure we got the constants right, note that, for every "nice" function $f = f(x)$, we have by (1)

$$\hat{f}(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\, dx. \tag{10}$$

Now we confirm that

$$\hat{h}(0) = \sqrt{\frac{\pi}{2}} \frac{1}{a} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{dx}{x^2 + a^2}.$$

Fourier sine transform

Similarly,

$$a^2 \mathcal{F}_s[f] = \mathcal{F}_s[a^2 f] = \mathcal{F}_s[f''] = -\omega^2 \mathcal{F}_s[f] + \sqrt{\frac{2}{\pi}}\, \omega f(0) = -\omega^2 \mathcal{F}_s[f] + \sqrt{\frac{2}{\pi}}\, \omega,$$

or $(a^2 + \omega^2)\mathcal{F}_s(f) = \omega\sqrt{\frac{2}{\pi}}$. Thus,

$$\mathcal{F}_s[f] = \sqrt{\frac{2}{\pi}} \left(\frac{\omega}{a^2 + \omega^2} \right)$$

so that

$$\frac{2}{\pi} \int_0^{+\infty} \frac{a\omega \sin(\omega x)}{a^2 + \omega^2}\, d\omega = \begin{cases} e^{-ax}, & x > 0 \\ 0, & x = 0, \\ -e^{-a|x|}, & x < 0. \end{cases}$$

With $a = x = 1$, we recover a familiar result:

$$\int_{-\infty}^{\infty} \frac{\omega \sin \omega}{1 + \omega^2}\, d\omega = \frac{\pi}{e}.$$

Fourier transform. We integrate directly:

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-x(a + \mathrm{i}\omega)}\, dx = \frac{1}{\sqrt{2\pi}} \frac{1}{a + \mathrm{i}\omega}.$$

In particular,

$$\sqrt{2\pi}\hat{f}(-\mathrm{i}s) = \frac{1}{a + s} = \mathcal{L}[f](s)$$

and

$$I_f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{\mathrm{i}\omega x}}{a + \mathrm{i}\omega}\, d\omega = \begin{cases} e^{-ax}, & x > 0, \\ 1/2, & x = 0, \\ 0, & x < 0. \end{cases}$$

By (3), $\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\mathcal{L}[f](\mathrm{i}\omega)$, and so we also get

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{L}[f](\mathrm{i}\omega) e^{\mathrm{i}\omega x}\, d\omega = \frac{1}{2\mathrm{i}\pi} \lim_{b \to +\infty} \int_{-\mathrm{i}b}^{\mathrm{i}b} \mathcal{L}[f](s) e^{sx}\, ds, \ x > 0.$$

[Similar to (9), we change the variables back to $s = \mathrm{i}\omega$ and write explicitly the (improper) integral as a limit, to emphasize that integration is now over the imaginary axis]. Could

$$f(x) = \frac{1}{2\pi\mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathcal{L}[f](s) e^{sx}\, ds$$

be a general formula for recovering a function from its Laplace transform? [No, but it is a good start...There is a general formula for inverting the Laplace transform, and it is remarkable that one needs complex numbers to derive it.] □

Convolutions and generalized functions.

By definition, the *convolution* $h = f * g$ of two (absolutely integrable) functions $f = f(x)$ and $g = g(x)$ is

$$h(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\,dy \equiv \int_{-\infty}^{\infty} g(x-y)f(y)\,dy. \tag{11}$$

It follows that

- The function h is also absolutely integrable:
$$\int_{-\infty}^{\infty} |h(x)|\,dx \le \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} |f(x-y)g(y)|\,dy\right)dx = \left(\int_{-\infty}^{\infty}|f(x)|\,dx\right)\left(\int_{-\infty}^{\infty}|g(y)|\,dy\right).$$
In fact, if $f \ge 0$ and $g \ge 0$, then the absolute values in the above computations can be removed and the inequality becomes an equality:
$$\int_{-\infty}^{\infty} h(x)\,dx \le \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(x-y)g(y)\,dy\right)dx = \left(\int_{-\infty}^{\infty}f(x)\,dx\right)\left(\int_{-\infty}^{\infty}g(y)\,dy\right).$$
In particular, if both f and g are *probability density functions*:
$$f(x) \ge 0,\ \int_{-\infty}^{\infty} f(x)\,dx,\quad g(x)\ge 0,\ \int_{-\infty}^{\infty} g(x)\,dx,$$
then so is h.
- If the functions f and g represent *causal* signals, that is, $f(x)=0,\ g(x)=0$ for $x<0$, then

$$h(x) = \int_0^x f(x-y)g(y)\,dy = \int_0^x g(x-y)f(y)\,dy. \tag{12}$$

Theorem 2. Up to a constant, the Fourier transform of the convolution is the product of the Fourier transforms:

$$\mathcal{F}[f*g](\omega) = \sqrt{2\pi}\hat{f}(\omega)\hat{g}(\omega). \tag{13}$$

Proof is by direct computation, even though changing the order of integration must be justified [and, in case you are wondering, is done best by the *Fubini theorem*]:

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\mathcal{F}[f*g](\omega) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}(f*g)(x)e^{-\mathrm{i}x\omega}\,dx = \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} f(x-y)g(y)\,dy\right)e^{-\mathrm{i}x\omega}\,dx\\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} f(x-y)e^{-\mathrm{i}\omega(x-y)}\,dx\right)e^{-\mathrm{i}y\omega}g(y)\,dy\\ &= \left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)e^{-\mathrm{i}\omega(x)}\,dx\right)\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\mathrm{i}y\omega}g(y)\,dy\right) = \hat{f}(\omega)\hat{g}(\omega).\end{aligned}$$

□

Corollary. The sum of two independent Gaussian random variables is a Gaussian random variable, and you add the corresponding means and variances: if

$$f(x) = \frac{1}{\sqrt{2\pi\sigma_1^2}}e^{-(x-\mu_1)^2/(2\sigma_1^2)},\quad g(x) = \frac{1}{\sqrt{2\pi\sigma_2^2}}e^{-(x-\mu_2)^2/(2\sigma_2^2)}$$

then

$$(f*g)(x) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/(2\sigma^2)},\ \mu=\mu_1+\mu_2,\ \sigma^2=\sigma_1^2+\sigma_2^2.$$

Proof. We know (or should know...) that

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}e^{-\mathrm{i}\omega\mu_1-(\sigma_1^2\omega^2/2)},\quad \hat{g}(\omega) = \frac{1}{\sqrt{2\pi}}e^{-\mathrm{i}\omega\mu_2-(\sigma_2^2\omega^2/2)},$$

and therefore
$$\mathcal{F}[f*g](\omega) = \frac{1}{\sqrt{2\pi}} e^{-\mathrm{i}\omega\mu-(\sigma^2\omega^2/2)}.$$
It remains to use the (very useful) fact that the pdf [*probability density function*] of a sum of two (absolutely continuous) independent random variables is the convolution of the corresponding pdf-s. Note that this example can also be considered a motivation for using Fourier transform methods [such as the Fast Fourier Transform (FFT)] to compute convolutions: direct computation of $(f*g)$ using the integral from (11) is rather unpleasant. □

Exercise. A Cauchy random variable with location parameter $\mu \in (-\infty, \infty)$ and scale parameter $\sigma > 0$ is a random variable with pdf
$$f(x) = \frac{1}{\pi\sigma\Big(1 + \big((x-\mu)/\sigma\big)^2\Big)}.$$
Confirm that f is indeed a pdf, and then use an earlier result (from Example 2)
$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} e^{\mathrm{i}\mu\omega - \sigma|\omega|}$$
to confirm that the sum of two independent Cauchy random variables is again a Cauchy random variable, and we add the corresponding location and scale parameters.

If you want to learn more: The Gaussian and Cauchy distributions are two examples of α-*stable distributions*, $0 < \alpha \le 2$, for which the characteristic function [a slightly modified version of the Fourier transform of the pdf] is
$$\varphi(t) = e^{\mathrm{i}\mu t - \sigma^\alpha |t|^\alpha}.$$
In fact, the general form is (slightly) more complicated, but the above is enough to see that Gaussian and Cauchy are two particular cases. corresponding to $\alpha = 2$ and $\alpha = 1$, respectively. □

A whole new direction of study is provided by the observation that there are no α-stable distributions with $\alpha > 2$: a function
$$\varphi(t) = e^{-|t|^\alpha}$$
is not a Fourier transform of anything if $\alpha > 2$. Indeed, how do we know? How can we determine if a given function is a Fourier transform of something? [If you want to know more, check out the *Bochner-Khnichin theorem.*]

Meanwhile, we will try to address the following three questions:

(1) Is there a function $g_0 = g_0(x)$ for which the Fourier transform is constant [does not depend on ω]?

(2) Is there a function $g = g(x)$ such that
$$(f * g_0)(x) = f(x) \tag{14}$$
for all f?

(3) The function
$$u_1(x) = \begin{cases} 0, & x < 0, \\ x, & x > 0, \end{cases}$$
is not differentiable at $x = 0$, but if we ignore the point $x = 0$ and stretch our definition of the derivative, then the function
$$u_0(x) = \begin{cases} 0, & x < 0, \\ 1, & x > 0, \end{cases} \tag{15}$$

can be considered a derivative of h_1: for example

$$u_1(x) = \int_0^x u_0(y)\, dy.$$

Can we keep going and construct a function $g = g(x)$ so that, in some sense,

$$u_0'(x) = g(x)\ ?$$

Consider the family of functions

$$g_\sigma(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-x^2/(2\sigma^2)}. \tag{16}$$

[Yes, these are Gaussian pdf-s with mean zero and variance σ^2.] We know that

$$\mathcal{F}[g_\sigma](\omega) = \frac{1}{\sqrt{2\pi}} e^{-\sigma^2\omega^2/2},$$

so that, by Theorem 2, for every (absolutely integrable) function $f = f(x)$,

$$\mathcal{F}[f * g_\sigma](\omega) = e^{-\sigma^2\omega^2/2}\hat{f}(\omega).$$

Because

$$\lim_{\sigma\to 0} \mathcal{F}[f * g_\sigma](\omega) = \hat{f}(\omega),$$

the function

$$g_0(x) = \lim_{\sigma\to 0} g_\sigma(x),$$

if exists, will satisfy (14) and

$$\mathcal{F}[g_0](\omega) = \frac{1}{\sqrt{2\pi}}. \tag{17}$$

But what happens if you try to pass to the limit $\sigma \to 0$ in (16)? Clearly,

$$\lim_{\sigma\to 0} g_\sigma(0) = \lim_{\sigma\to 0} \frac{1}{\sqrt{2\pi\sigma^2}} = +\infty.$$

On the other hand, for $x \neq 0$,

$$\lim_{\sigma\to 0} g_\sigma(x) = 0:$$

one application of the L'Hospital rule shows that [but first you replace σ with $1/t$ and pass to the limit $t \to +\infty$]. Moreover,

$$\int_{-\infty}^{+\infty} g_\sigma(x)\, dx = 1$$

for all $\sigma > 0$. As a result, our limit g_0 must satisfy the following three conditions:

$$g_0(0) = +\infty;\ g_0(x) = 0,\ x \neq 0;\ \int_{-\infty}^{+\infty} g_0(x)\, dx = 1.$$

In words, we are looking for an infinite peak at zero with total mass equal to one. While there is no usual function like that, these three properties are exactly the (physical) characterization of the *Dirac's delta function* $\delta = \delta(x)$ [Paul Adrien Maurice Dirac (1902–1984) was a British physicist, Nobel Prize 1933]. In mathematics, objects like this are called *generalized functions*. In particular, we have

$$(f * \delta)(x) = f(x), \tag{18}$$

$$\mathcal{F}[\delta](\omega) = \frac{1}{\sqrt{2\pi}}. \tag{19}$$

Putting $x = 0$ in (18), we get an important particular case of (18):

$$\int_{-\infty}^{+\infty} f(x)\delta(x)\,dx = f(0). \tag{20}$$

To see that the derivative of the unit step function u_0 from (15) is the Dirac delta function δ, we cannot use the Fourier transform directly because the function u_0 is not absolutely integrable. A direct application of the Laplace transform does not work either. Indeed,

$$\mathcal{L}[f'](s) = s\mathcal{L}[f](s) - f(0+), \quad \mathcal{L}[u_0](s) = \frac{1}{s}, \ u_0(0+) = 1,$$

so that $\mathcal{L}[u_0'](s) = \frac{s}{s} - 1 = 0$ [which does make sense because the Laplace transform only "sees" the function on $(0, +\infty)$, where u_0 is constant] whereas, by (3) and (18), $\mathcal{L}[\delta](s) = 1$.

An approach that works is integration by parts: if u_0 were differentiable in the usual sense, then we would have

$$\int_{-1}^{1} \varphi(x)u_0'(x)\,dx = -\int_{-1}^{1} \varphi'(x)u_0(x)\,dx$$

for every continuous function satisfying $\varphi(-1) = \varphi(1) = 0$. On the other hand, for each such function φ,

$$-\int_{-1}^{1} \varphi'(x)u_0(x)\,dx = -\int_{0}^{1} \varphi'(x)\,dx = \varphi(0),$$

that is, the "derivative" of u_0 acts as (20), that is, as a (bounded linear) functional $\varphi \mapsto \varphi(0)$, that is, is the delta function.

More generally we define $\delta_a(x) = \delta(x - a)$ as the corresponding "infinite peak at the point a with total mass equal to one". The resulting convention is $\delta_0 = \delta$. Then (18) becomes

$$\int_{-\infty}^{\infty} f(x)\delta_a(x)\,dx = f(a). \tag{21}$$

Using $\mathcal{F}[f(x - a)](\omega) = e^{-\mathrm{i}a\omega}\hat{f}(\omega)$ we conclude that

$$\mathcal{F}[\delta_a](\omega) = \frac{e^{-\mathrm{i}a\omega}}{\sqrt{2\pi}}$$

and

$$\mathcal{F}[\delta_a + \delta_{-a}](\omega) = \sqrt{\frac{2}{\pi}}\cos(a\omega).$$

Next, we use (7) and (19) to conclude that

$$\mathcal{F}[\cos(ax)](\omega) = \sqrt{\frac{\pi}{2}}\Big(\delta_a(\omega) + \delta_{-a}(\omega)\Big). \tag{22}$$

And you can keep going like that for a while, deriving numerous interesting identities.

In particular, we can now provide a Laplace-transform argument suggesting that $u_0' = \delta$. If $u_a(x) = u_0(x - a)$, $a > 0$, is the unit jump at the point a, then

$$\mathcal{L}[u_a](s) = \frac{e^{-sa}}{s}, \ \mathcal{L}[u_a'](s) = e^{-sa} = \mathcal{L}[\delta_a](s)$$

that is, $u_a' = \delta_a$ for all $a > 0$, and then, by passing to the limit $a \to 0$, we can at least imagine that $u_0' = \delta$ is a reasonable outcome.

The tables below summarize the main properties of the Fourier transform, and presents some of the related properties of the Fourier series. The properties of the Laplace transform are in a separate table.

Properties of the Fourier series and transform

Series	Name	Transform
$c_k(f) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(x)e^{-\mathrm{i}kx}\,dx$	Forward	$\hat{f}(\omega) = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int\limits_{-\infty}^{+\infty} f(x)e^{-\mathrm{i}x\omega}\,dx$
$S_f(x) = \displaystyle\sum_{k=-\infty}^{+\infty} c_k(f)e^{\mathrm{i}kx}$	Inverse	$I_f(x) = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int\limits_{-\infty}^{+\infty} \hat{f}(\omega)e^{-\mathrm{i}x\omega}\,d\omega$
$c_0(f) = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\pi}^{\pi} f(x)\,dx$	Obvious	$\hat{f}(0) = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{+\infty} f(x)\,dx$
$\displaystyle\sum_{k=-\infty}^{+\infty} c_k(f) = S_f(0) = \dfrac{\tilde{f}(0+) + \tilde{f}(0-)}{2}$	Obvious	$\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{+\infty} \hat{f}(\omega)\,d\omega = I_f(0)$ $= \frac{f(0+)+f(0-)}{2}$
$\displaystyle\lim_{\lvert k\rvert\to\infty} \lvert c_k(f)\rvert = 0$	Riemann-Lebesgue: $f \in L_1$	$\displaystyle\lim_{\lvert\omega\rvert\to\infty} \lvert\hat{f}(\omega)\rvert = 0,\ \hat{f}$ continuous
$\displaystyle\sum_{k=-\infty}^{+\infty} \lvert c_k(f)\rvert^2 = \dfrac{1}{2\pi}\int_{-\pi}^{\pi} \lvert f(x)\rvert^2\,dx$	Parseval/Plancherel: $f \in L_2$	$\displaystyle\int\limits_{-\infty}^{+\infty} \lvert\hat{f}(\omega)\rvert^2\,d\omega = \int\limits_{-\infty}^{+\infty} \lvert f(x)\rvert^2\,dx$

Further properties of the Fourier transform

Function	Fourier transform	Function	Fourier transform
$f(x)$	$\hat{f}(\omega) = \mathcal{F}[f](\omega)$	$\hat{f}(x)$	$f(-\omega)$
$f(x-a)$	$e^{-\mathrm{i}a\omega}\hat{f}(\omega)$	$e^{\mathrm{i}ax}f(x)$	$\hat{f}(\omega-a)$
$f(x/\sigma)$	$\sigma\hat{f}(\sigma\omega)$	$e^{-x^2/2}$	$e^{-\omega^2/2}$
$f'(x)$	$\mathrm{i}\omega\hat{f}(\omega)$	$xf(x)$	$\mathrm{i}\hat{f}'(\omega)$
$f''(x)$	$-\omega^2\hat{f}(\omega)$	$x^2f(x)$	$-\hat{f}''(\omega)$
$\int f(x)\,dx$	$\dfrac{\hat{f}(\omega)}{\mathrm{i}\omega}$	$\dfrac{f(x)}{x}$	$\dfrac{1}{\mathrm{i}}\int \hat{f}(\omega)\,d\omega$
$(f*g)(x)$	$\sqrt{2\pi}\hat{f}(\omega)\hat{g}(\omega)$	$f(x)g(x)$	$\dfrac{1}{\sqrt{2\pi}}(\hat{f}*\hat{g})(\omega)$
$e^{-\lvert x\rvert}$	$\sqrt{\dfrac{2}{\pi}}\,\dfrac{1}{1+\omega^2}$	$\dfrac{1}{1+x^2}$	$\sqrt{\dfrac{\pi}{2}}\,e^{-\lvert\omega\rvert}$
$1(\lvert x\rvert \le 1)$	$\sqrt{\dfrac{2}{\pi}}\,\dfrac{\sin\omega}{\omega}$	$\dfrac{\sin\omega}{\omega}$	$\sqrt{\dfrac{\pi}{2}}1(\lvert x\rvert \le 1)$
$\delta_a(x)$	$e^{-\mathrm{i}\omega a}/\sqrt{2\pi}$	$\cos(ax)$	$\sqrt{\pi/2}\Big(\delta_a(\omega)+\delta_{-a}(\omega)\Big)$

Properties of the Laplace transform

Function	Laplace transform	Function	Laplace transform
$f(t)$	$F(s) = \int_0^{+\infty} e^{-st} f(t) dt$	$1 = u_0(t)$	$\frac{1}{s}$
$af(t) + bg(t)$	$aF(s) + bG(s)$	$\delta(t)$	1
$e^{-ct} f(t)$	$F(s+c)$	e^{at}	$\frac{1}{s-a}$
$f(t-c) = f(t-c)u_c(t)$	$e^{-cs} F(s), \ (c > 0)$	$\sin(at)$	$\frac{a}{s^2+a^2}$
$f'(t)$	$sF(s) - f(0)$	$\cos(at)$	$\frac{s}{s^2+a^2}$
$f''(t)$	$s^2 F(s) - sf(0) - f'(0)$	$\frac{t}{2a} \sin(at)$	$\frac{s}{(s^2+a^2)^2}$
$\int_0^t f(\tau) d\tau$	$\frac{F(s)}{s}$	$\frac{1}{2a^3}\big(\sin(at) - at\cos(at)\big)$	$\frac{1}{(s^2+a^2)^2}$
$tf(t)$	$-F'(s)$	$t^r, \ r > -1$	$\frac{\Gamma(r+1)}{s^{r+1}}$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$	$t^n, \ n = 1, 2, 3, \dots$	$\frac{n!}{s^{n+1}}$
$\frac{f(t)}{t}$	$\int_s^{+\infty} F(z) dz$	$f(t+T) = f(t), \ T > 0$	$\frac{\int_0^T e^{-st} f(t) dt}{1 - e^{-sT}}$
$f(ct), \ c > 0$	$\frac{1}{c} F(s/c)$	$\int_0^t f(t-\tau) g(\tau) d\tau$	$F(s)G(s)$