Spring 2022, MATH 408, Final Exam

Monday, May 9; 8–10am

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Instructions:

- You should have access to a calculator or some other computing device, and to the distribution tables (normal, t, χ^2, and F). Instead of the tables, you are welcome to use the statistical functions available on your computing device.
- Answer all questions and clearly indicate your answers; upload the solutions to GradeScope.
- **There are ten problems; each problem is worth 10 points.**

Problem 1. Let $X_1, \dots, X_{25}$ be an independent random sample from a normal population with unknown mean μ and unknown variance σ^2. It is known that

$\bar{X}_n = 10 \quad S_n = 5$

$n = 25$

$\chi^2_{24,\,0.025} = 39.364$

$\chi^2_{24,\,0.975} = 12.401$

$$\sum_{k=1}^{25} X_k = 250, \quad \sum_{k=1}^{25} X_k^2 = 3100.$$

$S_n^2 \cdot (n-1) = \underbrace{\sum X_k^2 - n(\bar{x}_n)^2}_{= 600 = 25 \cdot 24} \sim \chi^2_{n-1}$

Construct the 95% confidence interval for σ. $\rightarrow \left[\sqrt{\frac{600}{39.364}}, \sqrt{\frac{600}{12.401}}\right] \approx [4, 7]$.

To get full credit, indicate the values of the sample mean, sample standard deviation, and the quantile of the corresponding distribution you need to construct the confidence interval.

Problem 2. Let $X_1, \dots, X_n$ be an independent random sample from the $Gamma(3, 1/\theta)$ distribution; in particular, the pdf of each X_k is

$L_n(\vec{X}_n; \theta) \propto \exp(-n\theta\bar{X}_n + 3n \ln\theta)$

$$f(x;\theta) = \frac{\theta^3}{2} x^2 e^{-x\theta} 1(x > 0).$$

$\frac{\partial}{\partial\theta} \ln L_n(\vec{X}_n; \theta) = -n\bar{X}_n + \frac{3n}{\theta}$ $\quad + \ \underset{\hat{\theta}_n}{|} \ -$

$\hat{\theta}_n = \frac{3}{\bar{X}_n}$

Construct the MLE of θ. **Make sure to check that you indeed maximized the likelihood function.**

$H_0: \mu_{Pb} = \mu_{Pv} \quad H_1: \mu_{Pb} \neq \mu_{Pv}$. Not due to chance

Problem 3. A study reports that freshmen at public universities work 11.1 hours a week for pay, on average, and the s_n is 8.4 hours; at private universities, the average is 9.3 hours and the s_n is 7.2 hours. Assume these data are based on two independent simple random samples, each of size $n =$ 900. Is the difference between the averages due to chance? Explain your conclusion by stating the corresponding null and alternative hypotheses and computing the p-value.

Large $\sqrt{n} = 30$ $\rightarrow$ z-test. $z^* = (11.1 - 9.3)\cdot 30 / (8.4^2 + 7.2^2)^{1/2} = 4.88 \Rightarrow$ p-value $= \mathbb{P}(|N(0,1)| > 4.88) < 0.01$

Problems 4. Let $X_1, \dots, X_n$ be an independent random sample from the $Gamma(3, 1/\theta)$ distribution; in particular, the pdf of each X_k is

N-P: reject H_0 if $\frac{L_n(\bar{X}_n, 1)}{L_n(\bar{X}_n, 2)}$ is small

$\frac{L_n(\bar{X}_n, 1)}{L_n(\bar{X}_n, 2)} \propto \exp(n\bar{X}_n)$ $\quad n\bar{X}_n \sim Gamma(3n, 1)$

$$f(x;\theta) = \frac{\theta^3}{2} x^2 e^{-x\theta} 1(x > 0).$$

Construct the most powerful test with Type-I error equal to 0.05 for testing $H_0 : \theta = 1$ against $H_1 : \theta = 2$. reject H_0 if $n\bar{X}_n < \underbrace{Gamma(3n, 1)_{0.95}}$

Problem 5. For the first-year students at a certain university, the correlation between SAT $= X$ scores and the first-year GPA $= Y$ was 0.6. Assume that the distribution of the scores is jointly normal. Predict the percentile rank on the first-year GPA for a student whose percentile rank on the SAT was 80%.

$z_{0.8} = 0.84$

$$\underbrace{\frac{Y - \mu_Y}{\sigma_Y}}_{\sim N(0,1)} = \overset{0.6}{\rho} \frac{X - \mu_X}{\sigma_X}$$

$\Rightarrow$ GPA rank is $\mathbb{P}(N(0,1) < \underbrace{0.6 \cdot 0.84}_{= 0.504}) \approx 0.7$

$\Rightarrow$ 70%

Problem 6. Fill in the rest of the following two-way ANOVA table.

Source	SS	df	MS	F	Prob $> F$
Columns	87	5	17.4	1	0.44
Rows	157	4	39.25	2.25	0.1
Error	348	20	17.4	X	X
Total	592	29	X	X	X

(> 0.1)

Problem 7. To test whether a die is fair, 78 rolls were made, and the corresponding outcomes were as follows:

H_0: die is fair
H_1: die is not fair

$exp. = \frac{78}{6} = 13$

Face value	Observed frequency		$(Obs - exp)^2$
1	10	13	9
2	11	13	4
3	18	13	25
4	17	13	16
5	12	13	1
6	10	13	9
			64

(a) Estimate the p-value if the χ^2 test is used. $\varphi^* = \frac{64}{13} \simeq 5$

(b) Based on your estimate, would you consider the die fair? Explain your conclusion.

p-value $= P(\chi_5^2 > 5) > P(\chi_5^2 > 9.236) = 0.1 \Rightarrow$ Yes

Problems 8. Assume the following is an independent sample from a population with a continuous cdf $F_X = F(x)$:

$$X_1 = 14 \quad X_2 = 6 \quad X_3 = 10.5 \quad X_4 = 11 \quad X_5 = 12 \quad X_6 = 9.5,$$

and assume that the following is an independent sample from a population with cdf $F_Y = F(x - \theta)$

$$Y_1 = 14.5 \quad Y_2 = 8 \quad Y_3 = 7 \quad Y_4 = 9 \quad Y_5 = 13 \quad Y_6 = 11.5.$$

$P(Y > X) > 1/2$

Compute the p-value of the sign test for the null hypothesis $\theta = 0$ against the alternative $\theta > 0$.
You will need the binomial coefficients $1, 6, 15, 20, 15, 6, 1$.

$1(X<Y)$ 1 1 0 0 1 1 $\to M = 4$

p-value $= P(B(6, 1/2) \geq 4) = 22/64 = 11/32$

Problem 9. A coin-making machine produces pennies with unknown probability p to turn up heads; this probability is equally likely to be any number between 0 and 1.
A coin pops out of the machine, flipped 25 times and lands heads 8 times. Compute the Bayesian estimate $\hat{p}$ of p.

$f(\text{prior}) = U(0,1) = Beta(1,1) \Rightarrow f^* = Beta(9, 18) \Rightarrow \hat{p} = \frac{9}{27} = 1/3.$

Problem 10. Let $X_1, \ldots, X_n$ be an independent random sample from a population with uniform distribution on the interval $(\theta, 2\theta)$, $\theta > 0$. Compute the method of moments estimator $\widehat{\theta}_n$ of θ and the bias and the MSE of $\widehat{\theta}_n$.

$EX = \frac{3}{2}\theta$; $Var X = \frac{\theta^2}{12}$

$\frac{X}{\theta} - 1 \sim U(0,1)$

$MSE(\hat{\theta}_n) = Var(\hat{\theta}_n) = \frac{4}{9} Var(\bar{X}_n) = \frac{\theta^2}{27n}$

$Var(X)/n$

$\hat{\theta}_n = \frac{2}{3}\bar{X}_n$, $E\hat{\theta}_n = \frac{2}{3} E\bar{X}_n = \frac{2}{3} EX = \theta \Rightarrow Bias = 0$