Fall 2021, MATH 407, Final Exam

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Instructions:

- No books, notes, calculators, or help from other people.
- Turn off cell phones.
- Show your work/explain your answers.
- You have 120 minutes to complete the exam.
- There are 10 problems; 10 points per problem.
- Upload the solutions to GradeScope.

Standard normal pdf: $(2\pi)^{-1/2}e^{-x^2/2}$; Gamma(a,b) pdf: $b^a\big(\Gamma(a)\big)^{-1}x^{a-1}e^{-bx}$; Exponential with mean θ is Gamma$(1,1/\theta)$; Beta(a,b) pdf: $\big(B(a,b)\big)^{-1}x^{a-1}(1-x)^{b-1}$; Poisson, mean μ, pmf: $e^{-\mu}k^{\mu}/k!$.

Problem 1. A box contains 12 blue balls, 17 green balls and 21 red balls, [50 total, well mixed]. 15 balls are taken out of the box, all at once.

Compute the probability that 6 of those balls are blue, 5 are green, and 4 are red. DO NOT SIMPLIFY/EVALUATE BINOMIAL COEFFICIENTS.

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(12 choose 6)(17 choose 5)(21 choose 4)/(50 choose 15)
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Problem 2. In a certain town, there are twice as many cars as trucks, and 3% of cars and 1% of trucks are of yellow color. A vehicle is selected at random, and has yellow color. Compute the probability that the vehicle is a car.

$0.03*(2/3)/(0.03*(2/3)+0.01*(1/3)) = 6/7$

Problem 3. A charitable lottery has 20,000 tickets, of which 400 win prizes and the rest win nothing. You buy 50 tickets.

(a) Compute the number of the prize-winning tickets you expect to find.

(b) Which of the following numbers is closest to the probability that, out of 50 tickets, at least one is prize-winning:

$$\frac{1}{100},\quad \frac{1}{10},\quad \frac{1}{5},\quad \frac{1}{3},\quad \frac{1}{2},\quad \frac{2}{3},\quad \frac{3}{4}.$$

Explain your conclusion.

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Poisson(1) > 0:  1-(1/e) = 0.632, closest to 2/3.
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Problem 4. Let X be a random variable with uniform distribution on $[-2,2]$. Define the random variable Y by $Y=\ln(2-X)$. Compute the probability density function of the random variable Y. MAKE SURE TO INDICATE THE RANGE OF POSSIBLE VALUES FOR Y.

$P(Y<x)=P(X>2-e^x)=e^x/4, x<\ln 4; f_Y(x)=e^x/4I(x<\ln 4)$

Problem 5. For a randomly selected group of 60 people, compute the expected number of distinct birthdays (that is, the expected number of the days of the year that are a birthday of at least one person in the group). Assume 365 days in a year.

365 * P(somebody is born on Jan 1)
= 365 (1- P(nobody born Jan 1)) = 365 (1- $(364/365)^{60}$)

Problem 6. The joint probability density function of two random variables X and Y is

$$f_{X,Y}(x,y) = \begin{cases} Cxy, & \text{if } x^2+y^2 \le 1, \quad x \ge 0, \ y \ge 0, \\ 0, & \text{otherwise.} \end{cases}$$

Compute the conditional expectation $\mathbb{E}(X|Y)$. Note: there is no need to know C.

pdf of Y is $(Cy/2)(1-y^2); E(X|Y) = (2/3)(1-Y^2)^{1/2}$

Problem 7. At a particular location, there is, on average, one earthquake every 4 days. Assuming that the earthquakes follow Poisson process, compute, approximately, the probability that there are at least 90 earthquakes in 320 days. Leave your answer in the form $\mathbb{P}(\mathcal{N} < r)$ or $\mathbb{P}(\mathcal{N} > r)$, where $\mathcal{N}$ is a standard normal random variable and r is a real number. Then circle the interval that contains your answer:

$$(0, 0.1) \qquad [0.1, 0.3) \qquad [0.3, 0.5) \qquad [0.5, 1)$$

$Poisson(80) > 89.5$ [use continuity correction] or Gamma(90, 1/4) < 320;
either way, approximately $P(\mathcal{N} > 1.055) \in [0.1, 0.3)$
Remember that the variance of Poisson random variable is equal to the mean (in this case, 80),
and Gamma(90, 1/4) is the sum of 90 iid exponential random variables, each having the mean and standard deviation equal to 4 (days)

Problem 8. Let X, Y be independent exponential random variables, both with mean value equal to 1/3. Compute the probability density functions of the random variables $U = X + Y$ and $V = X/(X+Y)$. DO NOT FORGET TO INDICATE THE RANGE OF THE RANDOM VARIABLES U AND V.

pdfs of X and Y are $3e^{-3x}$, so $f_U(x) = 9xe^{-3x}, x > 0$; V is uniform on $(0,1)$

Problem 9. Customers arrive at a bank at a Poisson rate λ. Suppose that five customers arrive during the first hour. Compute the probability that at least one of the customers arrived during the last 15 minutes. THE FINAL ANSWER SHOULD BE IN THE FORM a/b FOR SOME POSITIVE NUMBERS $a < b$.

$1 - P(all\ five\ are\ in\ the\ first\ 3/4\ of\ the\ hour) = 1 - (3/4)^5 = 781/1024$

Problem 10. Let $X_1, X_2, \ldots$ be iid standard normal random variables.

(1) Identify the distribution of $n^{-1/2}\sum_{k=1}^{n} X_k$. Normal mean 0, variance 1, i.e. also standard normal

(2) Compute the limit

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} X_k^2 \Big(= EX_1^2 = 1 \Big) \quad (LLN)$$

(3) By completing the square, compute the limit

$$\lim_{n\to\infty} \frac{2}{n}\sum_{j=2}^{n}\sum_{k=1}^{j-1} X_k X_j \Big(= \Big(n^{-1/2}\sum_{k=1}^{n} X_k \Big)^2 - \frac{1}{n}\sum_{k=1}^{n} X_k^2 \to \chi_1^2 - 1 \Big)$$

Here we use the results of (1) and (2), together with the Slutsky theorem: convergence in (2) is with probability one. Recall that χ_1^2 is the square of the standard normal random variable.