Fall 2018, MATH 407, Final Exam

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Name:

Instructions:

- No books, notes, or calculators.
- You have 120 minutes to complete the exam.
- **Show your work**.

Problem	Possible	Actual	Problem	Possible	Actual
1	20		6	20	
2	20		7	20	
3	20		8	20	
4	20		9	20	
5	20		10	20	
Total	100		Total	100	

Problem 1. A box contains 10 blue balls, 11 red balls, and 12 green balls [33 total, well mixed]. 15 balls are taken out of the box, all at once. What is the probability that at least one of those 15 balls is blue?

Problem 2. Imagine yourself making a salad by selecting one ingredient from each of the following groups:
Group 1: Potato, Yam;
Group 2: Cucumber, Carrot, Reddish;
Group 3: Garlic, Onion, Parsly.

Within every group, each ingredient is equally likely to be selected, and the selections are independent from group to group.

What is the probability that your salad will contain carrot or parsley, but not both?

Problem 3. A population contains twice as many females as males (that is, two females for each male). In this population, 5% of males and 0.5% of females are color-blind. A color-blind person is selected at random. Compute the probability that the person is female.

Problem 4. Let X be a standard Gaussian random variable. Define the random variable Y by

$$Y = \begin{cases} X, & \text{if } |X| > 10; \\ -X, & \text{if } |X| \leq 10. \end{cases}$$

Determine the probability density function f_Y of Y.

Problem 5. Let X be a random variable with uniform distribution on $[-1,1]$. Define the random variable Y by $Y = -\ln(X+1)$. Compute the probability density function of the random variable Y.

Problem 6. Let U be uniform random variable on $(-\pi, \pi)$ and let V be exponential random variable with mean 1. Assume that U and V are independent. Define the random variables $X = \sqrt{2V}\ \cos(U),\ Y = \sqrt{2V}\ \sin(U)$.

(a) Compute the joint pdf of X and Y.

(b) Are $X - Y$ and $X + Y$ independent? Explain your conclusion.

Problem 7. 100 (different) balls are dropped at random into 50 (different) boxes so that the balls are dropped independently of one another and each ball is equally likely to land in any of the boxes. In other words, there are 50^{100} different ways to put the balls into the boxes. Denote by X the number of non-empty boxes. Compute the expected number $E(X)$ of non-empty boxes. In the line below, circle the interval to which you think your answer belongs.

$$[0, 10] \qquad [11, 20] \qquad [21, 30] \qquad [31, 40] \qquad [41, 50]$$

Problem 8. At a certain location, there is, on average, one earthquake every two days. Assume that the earthquakes follow a Poisson process.

(a) Compute approximately the probability to have more than 70 earthquakes in 128 days. Use the Central Limit Theorem; depending on the way you set up your approximation, you may need to use continuity correction. Leave your answer in the form $P(Z < a)$ or $P(Z > b)$, where Z is a standard normal random variable.

(b) Is your answer in part (a) bigger than 0.1 or less than 0.1? Explain your conclusion.

Problem 9. Customers arrive at a bank at a Poisson rate λ. Suppose that three customers arrived during the first hour. Compute the probability that at least one arrived during the first 15 minutes.

Problem 10. Customers arrive at a bank at a Poisson rate λ. Suppose that five customers arrived between noon and 1pm. Let X be the arrival time of the first customer. Compute $E(X)$, the expected value of X. Make sure your answer is a time moment between noon and 1pm.