Spring 2017, MATH 407, Final Exam

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Instructor — S. Lototsky (KAP 248D; x0–2389; lototsky@math.usc.edu)

Name: ____________________

Instructions:

- No books, notes, or calculators.
- You have 120 minutes to complete the exam.
- **Show your work**.

Problem	Possible	Actual	Problem	Possible	Actual
1	20		6	20	
2	20		7	20	
3	20		8	20	
4	20		9	20	
5	20		10	20	
Total	100		Total	100	

Problem 1. A box contains 10 blue balls, 11 red balls, and 12 green balls [33 total, well mixed]. 15 balls are taken out of the box, all at once. What is the probability that at least one of those 15 balls is red?

Problem 2. Imagine yourself making a salad by selecting one ingredient from each of the following groups:
Group 1: Potato, Yam;
Group 2: Cucumber, Carrot, Reddish;
Group 3: Garlic, Onion, Parsly.

Within every group, each ingredient is equally likely to be selected, and the selections are independent from group to group.

What is the probability that your salad will contain potato or parsley, but not both?

Problem 3. A population contains twice as many males as females (that is, two males for each female). In this population, 5% of males and 0.5% of females are color-blind. A color-blind person is selected at random. Compute the probability that the person is female.

Problem 4. Let X be a standard Gaussian random variable. Define the random variable Y by

$$Y = \begin{cases} X, & \text{if } |X| < 1; \\ -X, & \text{if } |X| \geq 1. \end{cases}$$

Determine the probability density f_Y function of Y. [Use the definition of Y to conclude that $P(Y \leq t) = P(X \leq t)$ for all t. Keep in mind that $P(X > -t) = P(X < t)$ for all t. Your final answer should be a probability density function.]

Problem 5. Let X be a random variable with uniform distribution on $[2, 5]$. Define the random variable Y by $Y = -\ln(X - 2)$. Compute the probability density function of the random variable Y.

Problem 6. Let U be uniform random variable on $(-\pi, \pi)$ and let V be exponential random variable with mean 1. Assume that U and V are independent. Define the random variables $X = \sqrt{2V}\,\cos(U),\ Y = \sqrt{2V}\,\sin(U)$.

(a) Compute the joint pdf of X and Y.

(b) Are $X - Y$ and $X + Y$ independent? Explain your conclusion?

Problem 7. Four balls are dropped at random into five boxes so that the balls are dropped independently of one another and each ball is equally likely to land in any of the boxes. Denote by X the number of empty boxes. Compute the average number $E(X)$ of empty boxes and make sure that your answer is bigger than one.

Problem 8. At a certain location, there is, on average, one earthquake per day. Assume that the earthquakes follow a Poisson process.

(a) Compute approximately the probability to have fewer than 66 earthquakes in 64 days. Use the Central Limit Theorem; depending on the way you set up your approximation, you may need to use continuity correction. Leave your answer in the form $P(Z < a)$ or $P(Z > b)$, where Z is a standard normal random variable.

(b) Is the answer you got in part (a) bigger than 1/2 or less than 1/2? Explain your conclusion.

Problem 9. Customers arrive at a bank at a Poisson rate λ. Suppose that three customers arrived during the first hour. Compute the probability that at least one arrived during the last 15 minutes.

Problem 10. Customers arrive at a bank at a Poisson rate λ. Suppose that five customers arrived between noon and 1pm. Let X be the arrival time of the last customer. Compute $E(X)$, the expected value of X. [If you have no idea what to do, compute $E \max(U_1, U_2, U_3, U_4, U_5)$ where U_1, U_2, U_3, U_4, U_5 are iiid uniform on $(0, 1)$. Start by computing $P(\max(U_1, U_2, U_3, U_4, U_5) \leq x)$. For full credit, you final answer should be expressed as a time moment between noon and 1pm.]