Discrete Distributions: A Round-up

Bernoulli trials

Binomial $X \sim \mathcal{B}(n,p)$: number of successes in n trials

$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k},\ k=0,\ldots,n.$$

Geometric $X \sim G(p)$: number of trials TO GET the first success

$$P(X=k) = (1-p)^{k-1}p,\ k \geq 1.$$

Negative Binomial $X \sim \mathcal{NB}(p,m)$: number of trials TO GET m successes; $X = m, m+1, \ldots$

$$P(X=k) = \binom{k-1}{m-1} p^{m-1}(1-p)^{(k-1)-(m-1)}p = \binom{k-1}{m-1} p^m (1-p)^{(k-m)}$$

($m-1$ places to place the successes in the first $k-1$ trials; k-th trial has to be a success)

Runs: in 64 tosses of a fair coin, 6 consecutive heads or 6 consecutive tails are likely.

Jacob (Jacques) Bernoulli (1654 – 1705): Swiss.
Some contributions:

- First law of large numbers: $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}X_k = p$, if X_k are independent $\mathcal{B}(1,p)$;
- Bernoulli differential equation $y' = p(x)y + q(x)y^n$;
- polar coordinates;
- Brother of Johann Bernoulli and uncle of Daniel Bernoulli; his own two children were not mathematicians.

Poisson Distribution

$$X \sim \mathcal{P}(\lambda): \ P(X=k) = \frac{\lambda^k}{k!} e^{-\lambda}, \ \ k = 0, 1, 2, \ldots; E(X) = Var(X) = \lambda.$$

Poisson approximation of binomial distribution:

$$\lim_{n\to\infty,\, p\to 0,\, np\to\lambda} \binom{n}{k} p^k (1-p)^{n-k} = \frac{\lambda^k}{k!} e^{-\lambda}.$$

Quality of Approximation:

$$\sum_{k=0}^{\infty} \left| \binom{n}{k} p^k (1-p)^{n-k} - \frac{\lambda^k}{k!} e^{-\lambda} \right| \leq \frac{2\lambda}{n} \min(2, \lambda), \ \ \lambda = np,$$

a theorem of Prokhorov (for ideas, see Chapter 8 of the book)

Poisson and Prokhorov

Siméon Denis Poisson (1781–1840): French (distribution: 1837).
Yuri Vasilevich Prokhorov (1929–2013): Russian.

Hyper-Geometric Distribution: Definition

The main thing to remember: 23 balls; 11 red, 12 black; take 9;

$$P(5\ \mathrm{red}) = \frac{\binom{11}{5}\binom{12}{4}}{\binom{23}{9}}$$

General description: $X \sim \mathcal{H}(N, m; n)$
N objects of two types; m of type I; n is sample size; k is the number of Type I in the sample.

$$P(X = k) = \frac{\binom{m}{k}\binom{N-m}{n-k}}{\binom{N}{n}},\ k = 0, 1, \ldots, \min(m, n).$$

Key words: without replacement.

From Hyper-Geometric to Binomial

Theorem

$$\lim_{N\to\infty\, m\to\infty,\ m/N\to p} \frac{\binom{m}{k}\binom{N-m}{n-k}}{\binom{N}{n}} = \binom{n}{k} p^k (1-p)^{n-k}$$

Then can follow up with a Poisson approximation.
Example: a raffle; n^2 tickets, n winning. You buy $2n$ tickets.

$$P(3\,\mathrm{win}) = P(\mathcal{H}(n^2, n; 2n) = 3) \approx P(\mathcal{B}(2n, n^{-1}) = 3) \approx P(\mathcal{P}(2) = 3).$$

does not depend on n (exact in the limit $n \to \infty$).
Bottom line: $\mathcal{H}(N, m; n) \approx \mathcal{B}(n, m/N) \approx \mathcal{P}(nm/N)$

What else?

zeta distribution:

$$P(X=k) = \frac{1}{\zeta(\alpha+1)k^{1+\alpha}},\ \alpha>0,\ k=1,2,\ldots$$

Choosing a suitable α can fit many models...

Riemann zeta function: $\zeta(t) = \sum_{k=1}^{\infty} \frac{1}{k^t},\ t>1.$

Zipf distribution: $P(X=k) = C(N)k^{-1},\ k=1,\ldots,N.$
Frequency of N ranked objects.
Erlang's distribution:

$$P(X=k) = C(N)\frac{\lambda^k}{k!},\ k=0,\ldots,N.$$

(truncated Poisson distribution)

Who are those people?

Georg Friedrich Bernhard Riemann (1826–1866): German (1859)
George Kingsley Zipf (1902–1950): American (1932)
Agner Krarup Erlang (1878–1929): Danish (1906)