MATH 445

THE CRANK-NICOLSON SCHEME FOR THE HEAT EQUATION

Consider the one-dimensional heat equation

(1) $$u_t(x,t) = au_{xx}(x,t); 0 < x < L,\ 0 < t \le T; u(0,t) = u(L,t) = 0;\ u(x,0) = f(x),$$

The idea is to reduce this PDE to a system of ODEs by discretizing the equation in space, and then apply a suitable numerical method to the resulting system of ODEs.

Denote by $\Delta x = L/N$ the step size in space and approximate $u_{xx}(n\Delta x, t)$ for every $t \in [0,T]$ using central differences:

(2) $$u_{xx}(n\Delta x, t) \approx \frac{1}{(\Delta x)^2}\left(u((n+1)\Delta x, t) - 2u(n\Delta x, t) + u((n-1)\Delta x, t)\right).$$

Define the column vector $U(t) = (U_1(t), \ldots, U_{N-1}(t))^T$ as the solution of the system of equations

(3) $$\frac{dU_n(t)}{dt} = \frac{a}{(\Delta x)^2}\big(U_{n+1}(t) - 2U_n(t) + U_{n-1}(t)\big),\ n = 1, \ldots, N-1,\ 0 < t \le T,$$

with initial condition $U_n(0) = f(n\Delta x)$, and set $U_0(t) = U_N(t) = 0$ for all t. By (2), it is natural to consider $U_n(t)$ as an approximation of $u(n\Delta x, t)$. Note that, from the definition of U and the boundary conditions for u, we have $u(0,t) = 0 = U_0(t)$ and $u(N\Delta x, t) = u(L,t) = 0 = U_N(t)$ for all t.

In the matrix form, (3) becomes

(4) $$\frac{dU(t)}{dt} = \frac{a}{(\Delta x)^2} A_{N-1}^{[-2,1]} U(t),\ 0 < t \le T,$$

where $A_{N-1}^{[-2,1]}$ is a tri-diagonal square matrix of the size $(N-1)\times(N-1)$:

(5) $$A_{N-1}^{[-2,1]} = \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0\cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0\ 1 & -2 \end{pmatrix}.$$

Next, we discretize time by introducing a uniform grid with step $\Delta t = T/M$. By (4),

(6) $$U((m+1)\Delta t) = U(m\Delta t) + \frac{a}{(\Delta x)^2}\int_{m\Delta t}^{(m+1)\Delta t} A_{N-1}^{[-2,1]} U(s)ds.$$

We approximate the integral on the right-hand side of (6) by the *trapezoidal rule*:

(7) $$U((m+1)\Delta t) \approx U(m\Delta t) + \frac{a\Delta t}{2(\Delta x)^2}\Big(A_{N-1}^{[-2,1]} U(m\Delta t) + A_{N-1}^{[-2,1]} U((m+1)\Delta t)\Big).$$

To proceed, let us introduce the notation

$$r = \frac{a\Delta t}{2(\Delta x)^2}.$$

Now *define* the sequence of vectors $\bar{u}(m)$, $m = 0, \ldots, M$, by $\bar{u}_n(0) = f(n\Delta x)$ (f is the initial condition from (1)), and

$$\bar{u}(m+1) = \bar{u}(m) + r\left(A_{N-1}^{[-2,1]}\bar{u}(m) + A_{N-1}^{[-2,1]}\bar{u}(m+1)\right). \tag{8}$$

According to (7), $\bar{u}(m)$ can be considered an approximation of $U(m\Delta t)$, and therefore $\bar{u}_n(m)$ is an approximation of $u(n\Delta x, m\Delta t)$. Similar to the vector U, the vector $\bar{u}(m)$ in (8) has $N-1$ components $\bar{u}_1(m), \ldots, \bar{u}_{N-1}(m)$. The resulting approximation of the solution of equation (2) at time $m\Delta t$ and points $n\Delta x$, $n = 0, \ldots, N$, is $(0, \bar{u}_1(m), \ldots, \bar{u}_{N-1}(m), 0)$.

To compute $\bar{u}(m)$, note that (8) can be written as follows:

$$A_{N-1}^{[1+2r,-r]}\bar{u}(m+1) = A_{N-1}^{[1-2r,r]}\bar{u}(m), \tag{9}$$

where, similar to (5),

(10)

$$A_{N-1}^{[1+2r,-r]} = \begin{pmatrix} 1+2r & -r & 0 & \cdots & 0 \\ -r & 1+2r & -r & 0\cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & -r \\ 0 & \cdots & \cdots\ 0 & -r & 1+2r \end{pmatrix}, \quad A_{N-1}^{[1-2r,r]} = \begin{pmatrix} 1-2r & r & 0 & \cdots & 0 \\ r & 1-2r & r & 0\cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & r \\ 0 & \cdots & \cdots\ 0 & r & 1-2r \end{pmatrix}$$

(VERIFY THIS!!!)

and therefore

$$\bar{u}(m+1) = \left(A_{N-1}^{[1+2r,-r]}\right)^{-1} A_{N-1}^{[1-2r,r]}\bar{u}(m) \tag{11}$$

The method of computing an approximation of the solution of (1) according to (11) is called the Crank-Nicolson scheme. It was proposed in 1947 by the British physicists John Crank (b. 1916) and Phyllis Nicolson (1917–1968).