Some counting problems[1]

Ordered non-negative partitions of an integer: the number $S_{N,n}$ of *non-negative* solutions of

$$x_1 + x_2 + \ldots + x_n = N \tag{1}$$

is

$$S_{N,n} = \binom{N+n-1}{n-1}. \tag{2}$$

ONE WAY TO SEE IT: put $N+n-1$ boxes in a row, then put the plus sign in $n-1$ of them; this establishes a bijection between the box configurations and solutions of the equation (1). For example, with $N=9$, $n=4$, the following configuration

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corresponds to the ordered partition $9 = 2+0+6+1$.

The total number of these box configurations is given by (2).

VARIATION: ORDERED PARTITIONS WITH LOWER BOUND, that is, non-negative integer solutions of (1) satisfying $x_k \geq \ell_i$ for some positive integers ℓ_k. This is reduced to the original problem with the same n, but with $N - \sum_{i=1}^{n} \ell_i$ instead of N. The reason: corresponding box configuration will have only $N + n - \sum_{k=1}^{n} \ell_k$ places to put the pluses. Equivalently, replacing x_k with $x_k - \ell_k$ reduces the problem to an unrestricted one. For example,

$$S^{+}_{N,n} = \binom{N-1}{n-1}$$

is the number of *positive* solutions of (1) and corresponds to taking $\ell_i = 1$ for all i.

The general counting technique, establishing a bijection between the objects we want to count and the linear configurations of two types of objects, is known under names `stars and bars`, `sticks and stones`, `balls and bars`, and `dots and dividers` (and maybe more).

Multinomial formula:

$$(x_1 + \cdots + x_n)^N = \sum_{k_1+\ldots+k_n=N} \frac{N!}{k_1!k_2!\cdots k_n!} x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$$

Example:

$$\begin{aligned}
(a+b+c)^3 &= a^3 + b^3 + c^3 && [3 = 3+0+0] \\
&+ 3(ab^2 + ba^2 + ac^2 + ca^2 + bc^2 + cb^2) && [3 = 2+1+0] \\
&+ 6abc && [3 = 1+1+1]
\end{aligned}$$

FURTHER READING: Young tableau, Young diagram, etc.

Runs in Bernoulli trials: In a finite sequence of tosses of a coin, a `run of heads` is a consecutive appearance of heads. For example, each of the following three sequences

$$(a)\ HH\,TTT\,H\,T\,HH\,TTT \quad (b)\ T\,HH\,TTT\,H\,T\,HH\,TT \quad (c)\ TTT\,HH\,TTT\,H\,T\,HH \tag{3}$$

has three runs of heads and the total length 12. By convention, the number of runs of tails in each case is *four*: the first sequence started with H, and the convention is to say that the first run of tails has length 0; similarly, the third sequence ends with heads, and the convention is to say with the fourth run of tails has length 0 When the coin is fair and the tosses are independent, all 2^N sequences of length N are equally likely. Then, given that the sequence contains n heads and $m = N-n$ tails, the probability to have r runs of heads is

$$R_{n,m;r} = \frac{\binom{n-1}{r-1}\binom{m+1}{r}}{\binom{n+m}{n}}.$$

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Indeed, if x_k is the number of heads in the k-th run, we are counting the number of *positive* solutions of $x_1 + \cdots + x_r = n$; similarly, with tails, we have $y_1 + \cdots + y_{r+1} = m$, with $y_1 \geq 0$, $y_{r+1} \geq 0$ and $y_k \geq 1$ for $k = 2, \ldots, r$.

For each of the three sequences in (3), $N = 12, n = 5, m = 7, r = 3$, and

$$R_{5,7;3} = \frac{14}{33}.$$

By comparison,

$$R_{5,7;1} = \frac{1}{42} R_{5,7;3},$$

quantifying our intuition that each of the three sequences in (3) is a more "reasonable" outcome of 12 tosses of a fair coin than either $HHHHHTTTTTTT$ or $TTTTTTTHHHHH$. Given that each of the five sequences has the same probability to appear (namely, 2^{-12}), these considerations can lead to a much deeper investigation of what "random" actually means.

An antenna system: Have N antennas in a row, of which m are defective. The system works if no two defective antennas are next to each other. Then the probability $p_{N,m}$ that the system works is

$$p_{N,m} = \frac{\binom{N-m+1}{m}}{\binom{N}{m}}. \tag{4}$$

Indeed, the total number of ways to place m defective antennas among the total of N is $\binom{N}{m}$; the number of places for defective antennas in a working configuration is $N - m + 1$: $(d)w(d)w(d)w \cdots (d)w(d)$, where w represents a working antenna and (d) represents a possible location of a defective antenna; the number of working arrangements is therefore $\binom{N-m+1}{m}$.

Ice cream cone: Have N flavors, can put m scoops in the cone; order does not matter; repetitions are allowed. Then the total number $C_{N,m}$ of different cones is

$$C_{N,m} = \binom{N+m-1}{m}. \tag{5}$$

One way to see this: automated procedure for generating the cone using a sequence of black and white squares.

- order the flavor *somehow* by assigning a number to each flavor;
- each cone is represented by sequence like

$$1\blacksquare\square 2\blacksquare\blacksquare\square 3\square 4\square 5\blacksquare\square 6\square 7\square 8\blacksquare\blacksquare\blacksquare \tag{6}$$

- a black square $\blacksquare$ next to the number means a scoop of that flavor is included in the cone;
- a white square $\square$ mean move to the next flavor. In (6),
  - $N = 8, m = 7$;
  - flavors 3,4, 6, 7 are not included, 1 and 5 are scooped once, 2 is scooped twice, and the last flavor 8 is scooped three times.
- As a result, we have $N + m - 1$ white squares to begin, m of which will become black [giving the instruction to scoop].
- The total number of the resulting sequences is given by (5).