A Summary of The Cameron-Martin-Girsanov-Meyer Theorem(s).[1]

Cameron-Martin

Let $W = W(t)$, $t \in [0,T]$, be a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let μ be the measure generated by W on the spaces $\mathcal{C}([0,T])$ of continuous functions on $[0,T]$:

$$\mu(A) = \mathbb{P}(W \in A),\ A \in \mathcal{B}\big(\mathcal{C}([0,T])\big).$$

Given a function $h \in \mathcal{C}([0,T])$, define the measure μ_h on $\mathcal{C}([0,T])$ by

$$\mu_h(A) = \mathbb{P}(W + h \in A),\ A \in \mathcal{B}\big(\mathcal{C}([0,T])\big).$$

Then the measure μ_h is absolutely continuous with respect to the measure μ if and only if the function h has the form $h(t) = \int_0^t h'(s)ds$ for some function $h' \in L_2((0,T))$; in that case,

$$\frac{d\mu_h}{d\mu}(x) = \exp\left(I_{h'}(x) - \frac{1}{2}\int_0^T |h'(t)|^2\,dt\right),\ x \in \mathcal{C}([0,T]). \tag{1}$$

The mapping $x \mapsto I_{h'}(x)$ is known as the Cameron-Martin functional or Paley-Wiener integral.

The original reference: Cameron, R. H.; Martin, W. T. (1944). "Transformations of Wiener Integrals under Translations". Annals of Mathematics. 45 (2): 386–396.

Girsanov

$W = W(t)$, $t \in [0,T]$, be a standard Brownian motion on a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, and let $h = h(t),\ t \geq 0$, be an adapted process satisfying

$$\mathbb{P}\left(\int_0^T h^2(t)\,dt < \infty\right) = 1.$$

Define the process

$$Z(t) = \exp\left(\int_0^t h(s)dW(s) - \frac{1}{2}\int_0^t h^2(s)\,ds\right),\ t \in [0,T].$$

IF

$$\mathbb{E}Z(T) = 1, \tag{2}$$

then the process $\widetilde{W}(t) = W(t) - \int_0^t h(s)\,ds,\ 0 \leq t \leq T$, is a standard Brownian motion on the stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \widetilde{\mathbb{P}})$, with $d\widetilde{\mathbb{P}} = Z(T)d\mathbb{P}$. In particular,

- If Φ_t is a bounded functional on $\mathcal{C}([0,t])$, $t \leq T$, then
$$\mathbb{E}\Phi_t(W) = \widetilde{\mathbb{E}}\Phi_t(\widetilde{W}) = \mathbb{E}\Big(\Phi_t(\widetilde{W})Z(T)\Big) = \mathbb{E}\Big(\Phi_t(\widetilde{W})Z(t)\Big), \tag{3}$$
where the last equality follows from the martingale property of Z.
- If τ is a stopping time, with $\mathbb{P}(\tau \leq T) = 1$, and ζ is an $\mathcal{F}_\tau$-measurable random variable, then
$$\widetilde{\mathbb{E}}\zeta = \mathbb{E}\Big(\zeta Z(\tau)\Big). \tag{4}$$

The original reference: Girsanov, I. V. (1960). "On transforming a certain class of stochastic processes by absolutely continuous substitution of measures". Theory of Probability and Its Applications. 5 (3): 285–301.

Note that $dZ(t) = h(t)Z(t)dW(t),\ Z(0) = 1$, so that Z is a non-negative local martingale (and thus a supermartingale). Condition (2) means that Z is, in fact, a martingale. `Novikov's condition`

$$\mathbb{E}\exp\left(\frac{1}{2}\int_0^T h^2(s)\,ds\right) < +\infty \tag{5}$$

is *sufficient* for (2) to hold; on the one hand, (5) is far from necessary, but, on the other hand, (5) cannot, in general, be relaxed even as far as the factor 1/2. A detailed discussion is in Section 6.2 of *Statistics of Random Processes* by Liptser and Shiryaev.

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An SODE version of Girsanov by Liptser and Shiryaev

Let $W = W(t)$, $t \in [0, T]$, be a standard Brownian motion on a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ and let $b = b(t, x)$, $\sigma = \sigma(t, x)$, $h = h(t, x)$ be *non-random* functions such that each of the following equations

$$dX = b(t, X)dt + \sigma(t, X)dW(t),\ dY = B(t, Y)dt + \sigma(t, Y)dW,\ \text{where } B(t, x) = b(t, x) + h(t, x)\sigma(t, x),$$

has a unique strong solution on $[0, T]$ for some non-random $T < \infty$. For $0 \leq t \leq T$ define the process

$$Z(t) = \exp\left(\int_0^t \frac{b\big(s, Y(s)\big) - B\big(s, Y(s)\big)}{\sigma^2\big(s, Y(s)\big)}\, dY(s) - \frac{1}{2}\int_0^t \frac{b^2\big(s, Y(s)\big) - B^2\big(s, Y(s)\big)}{\sigma^2\big(s, Y(s)\big)}\, ds\right). \tag{6}$$

If we solve for dY in terms of dW, then

$$Z(t) = \exp\left(-\int_0^t h(s, Y(s))dW(s) - \frac{1}{2}\int_0^t h^2(s, Y(s))\, ds\right),\ t \in [0, T].$$

IF we assume that $d\widetilde{\mathbb{P}} = Z(T)d\mathbb{P}$ defines a new probability measure, then, by Girsanov, $\widetilde{W}(t) = W(t) + \int_0^t h(s, Y(s))\, ds$ is a standard Brownian motion on $(\Omega, \mathcal{F}, \widetilde{\mathbb{P}})$. On the other hand,

$$dY = B(t, Y)dt + \sigma(t, Y)dW = b(t, Y)dt + \sigma(t, Y)(h(t, Y)dt + dW) = b(t, Y)dt + \sigma(t, Y)d\widetilde{W},$$

that is, the equation satisfied by X on $(\Omega, \mathcal{F}, \mathbb{P})$ is the same as the equation satisfied by Y on $(\Omega, \mathcal{F}, \widetilde{\mathbb{P}})$ and so $\mathbb{E}\Phi(X) = \widetilde{\mathbb{E}}\Phi(Y)$. The main result is that we do have $\mathbb{E}Z(T) = 1$; the details are in Chapter 7 of *Statistics of Random Processes* by Liptser and Shiryaev, in particular, Theorem 19. Accordingly, if $X(0) = Y(0)$, then

- Similar to (3), for every bounded measurable functional Φ on $\mathcal{C}([0, T])$,
$$\mathbb{E}\Phi(X) = \mathbb{E}\Big(\Phi(Y)Z(T)\Big); \tag{7}$$
- Similar to (4), for every bounded stopping time τ and an $\mathcal{F}_\tau$-measurable random variable ζ,
$$\widetilde{\mathbb{E}}\zeta = \mathbb{E}\Big(\zeta Z(\tau)\Big). \tag{8}$$

Meyer.

Let $\mathbb{F} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a stochastic basis with the usual assumptions, let $\widetilde{Z}$ be a non-negative random variable with $\mathbb{E}\widetilde{Z} = 1$, and let the martingale $Z = Z(t)$ be the right-continuous version of $\mathbb{E}(\widetilde{Z}|\mathcal{F}_t)$. Define the probability measure $\widetilde{\mathbb{P}}$ by $d\widetilde{\mathbb{P}} = Zd\mathbb{P}$. If M is a local martingale on $\mathbb{F} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ and the measures $\widetilde{\mathbb{P}}$ and $\mathbb{P}$ are equivalent, then the process $\widetilde{M} = \widetilde{M}(t)$ defined by

$$\widetilde{M}(t) = M(t) - \int_0^t \frac{d\,[Z, M](s)}{Z(s)},$$

is a local martingale on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \widetilde{\mathbb{P}})$.

For a cadlag semi-martingale X, the process $t \mapsto [X, X](t)$ is the quadratic variation of X: $[X, X](t) = \langle X^c\rangle_t + \sum_{s\leq t}\big(X(s) - X(s\text{-})\big)^2$; it is the "corrector" in the Itô formula applied to X^2:

$$X^2(t) = X^2(0) + 2\int_0^t X(s\text{-})\, dX(s) + [X, X](t);$$

similarly, $[X, Y] = (1/4)\big([X + Y, X + Y] - [X - Y, X - Y]\big)$. If X is a local square-integrable martingale, then $[X, X] = \langle X\rangle$, because in this case $X^2 - [X, X]$ is a local martingale [L-Sh-Mart, Thm. 1.8.1]; if $X(t) = \int_0^t f(s)dW(s)$ and $Y(t) = \int_0^t g(s)dW(s)$, then $[X, Y](t) = \langle X, Y\rangle_t = \int_0^t f(s)g(s)\, ds$. In particular, in the setting of the Girsanov theorem, $M = W$, $dZ = hZdW$, so that $d[Z, M](t) = h(t)Z(t)dt$.

Some examples.

Crossing a linear boundary. *If $W = W(t)$ is the standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ and $\tau = \inf\{t > 0 : W(t) = a + bt\}$, with $a, b > 0$ and $\inf\{\emptyset\} = +\infty$, then $\mathbb{P}(\tau < +\infty) = e^{-2ab}$.*

Indeed, for $c \in \mathbb{R}$, define $W_c(t) = W(t) + ct$, and, for a continuous adapted process $X = X(t)$, set $\tau_a(X) = \inf\{t > 0 : X(t) = a\}$. Then $\mathbb{P}(\tau < +\infty) = \mathbb{P}(\tau_a(W_{-b}) < +\infty)$. By (7) and (8), with $X = W_{-b}$ and $Y = W_b$ [so that $b(t,x) = -b, B(t,x) = b, \sigma(t,x) = 1$], for every $n > 0$,

$$\mathbb{P}\big(\tau_a(W_{-b}) < n\big) = \mathbb{E}\Big(I(\tau_a(W_b) < n)e^{-2bW_b\left(\tau_a(W_b)\right)}\Big) = \mathbb{E}\Big(I(\tau_a(W_b) < n)e^{-2ab}\Big),$$

where the last equality is a consequence of $X(\tau_a(X)) = a$. After passing to the limit as $n \to \infty$, we conclude that $\mathbb{P}(\tau < +\infty) = e^{-2ab}\mathbb{P}\big(\tau_a(W_b) < +\infty\big)$, and it remains to note that, with $b > 0$, the process W_b satisfies $\lim_{t\to\infty} W_b(t)/t = b$ and therefore reaches every fixed positive level a with probability one: $\mathbb{P}\big(\tau_a(W_b) < +\infty\big) = 1$.

A Cameron-Martin formula. *If $W = W(t)$ is the standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$, then*

$$\mathbb{E}\exp\left(-\frac{1}{2}\int_0^T W^2(t)\,dt\right) = \frac{1}{\sqrt{\cosh(T)}}. \tag{9}$$

Indeed, take a $b \in \mathbb{R}$ and consider $dX(t) = dW(t)$, $dY(t) = bY(t)dt + dW(t)$, $X(0) = Y(0) = 0$ [so that $b = 0$, $\sigma = 1$, $B(t,x) = bx$]. We will apply (7) with

$$Z(T) = \exp\left(-b\int_0^T Y(t)\,dY(t) + \frac{b^2}{2}\int_0^T Y^2(t)\,dt\right), \quad \Phi_T(X) = \exp\left(-\frac{1}{2}\int_0^T X^2(t)\,dt\right).$$

Then

$$\mathbb{E}\Phi_T(X) = \mathbb{E}\exp\left(-\frac{1}{2}\int_0^T Y^2(t)\,dt - b\int_0^T Y(t)\,dY(t) + \frac{b^2}{2}\int_0^T Y^2(t)\,dt\right).$$

Taking $b = -1$ we get

$$\mathbb{E}\Phi_T(X) = \mathbb{E}\exp\left(\int_0^T Y(t)\,dY(t)\right) = \mathbb{E}\exp\left(\frac{Y^2(T) - T}{2}\right), \tag{10}$$

where the second equality follows from the Itô formula.

Next, we note that

$$Y(t) = \int_0^t e^{-(t-s)}dW(s),$$

so that $Y(T)$ is a Gaussian random variable with mean zero and variance $\int_0^T e^{-2(T-s)}\,ds = (1 - e^{-2T})/2$. On the other hand, if ζ is a Gaussian random variable with mean zero and variance ρ^2, then, for every $0 < r < \rho^{-2}$,

$$\mathbb{E}e^{r\zeta^2/2} = \frac{1}{\sqrt{2\pi}\,\rho}\int_{-\infty}^{+\infty} e^{-(\rho^{-2}-r)x^2/2}\,dx = \frac{1}{\sqrt{\rho^2(\rho^{-2} - r)}} = \frac{1}{\sqrt{1 - r\rho^2}}.$$

Using this result with $r = 1$ and $\rho = (1 - e^{-2T})/2$, we conclude the computation in (10):

$$\mathbb{E}\Phi_T(X) = e^{-T/2}\left(1 - \frac{1 - e^{-2T}}{2}\right)^{-1/2} = \left(e^T - \frac{e^T - e^{-T}}{2}\right)^{-1/2} = \frac{1}{\sqrt{\cosh(T)}}.$$

Note that

- The choice $b = 1$ will not change the final result: we only need $b^2 = 1$ to cancel the integrals;
- Self-similarity of the Brownian motion ($t \mapsto \sqrt{\lambda}W(t/\lambda)$ is a standard Brownian motion for every $\lambda > 0$) implies a more general version of (9):

$$\mathbb{E}\exp\left(-\frac{\lambda^2}{2}\int_0^T W^2(t)\,dt\right) = \frac{1}{\sqrt{\cosh(\lambda T)}}.$$

The people

William Ted Martin (1911–2004): MIT math department chair 1947–1968.

Robert Horton Cameron (1908–1989): supervised 35 Ph.D. students in 30+ years at the University of Minnesota; one of the students was M. Donsker.

Igor Vladimirovich Girsanov (1934–1967): introduced the concept of a strong Feller process.

Paul-André Meyer (1934–2003): until 1952, his last name was Meyerowitz; *Probabilités et Potentiel*, joint with Claude Dellacherie, consists of five volumes; played the violin, viola, and the flute.

Some (informal) proofs.

Change of measure in conditional expectation. If $d\widetilde{\mathbb{P}} = Zd\mathbb{P}$ for some positive random variable Z with $\mathbb{E}Z = 1$, and $\mathcal{G}$ is a sigma-algebra of sub-sets of Ω, then, for every $\widetilde{\mathbb{P}}$-integrable random variable ζ,

$$\widetilde{\mathbb{E}}(\zeta|\mathcal{G}) = \frac{\mathbb{E}(\zeta Z|\mathcal{G})}{\mathbb{E}(Z|\mathcal{G})}. \tag{11}$$

Proof. Re-write (11) as $\widetilde{\mathbb{E}}(\zeta|\mathcal{G})\mathbb{E}(Z|\mathcal{G}) = \mathbb{E}(\zeta Z|\mathcal{G})$, multiply both sides by a $\mathcal{G}$-measurable random variable η, take the $\mathbb{E}$ expectation on both sides, keeping in mind that $\mathbb{E}(\eta\zeta Z) = \widetilde{\mathbb{E}}(\eta\zeta)$, and confirm that both sides are equal to $\widetilde{\mathbb{E}}(\eta\zeta)$.

The Itô formula. If $M = M(t)$ is a continuous martingale and $F = F(t,x)$ is a function that is continuously differentiable in t and twice continuously differentiable in x, then

$$F\big(t, M(t)\big) = F\big(0, M(0)\big) + \int_0^t F_t\big(s, M(s)\big)\,ds + \int_0^t F_x\big(M(s)\big)\,dM(s) + \frac{1}{2}\int_0^t F_{xx}\big(M(s)\big)\,d\langle M\rangle_s. \tag{12}$$

Proof. Let $0 = t_0 < t_1 < \cdots < t_n = t$, $\triangle t_k = t_{k+1} - t_k$, $\triangle M_k = M(t_{k+1}) - M(t_k)$. By Taylor formula

$$\begin{aligned} F\big(t_{k+1}, M(t_{k+1})\big) &= F\big(t_k + \triangle t_k, M(t_k) + \triangle M_k)\big) \\ &\approx F\big(t_k, M(t_k)\big) + F_t\big(t_k, M(t_k)\big)\triangle t_k + F_x\big(M(t_k)\big)\triangle M_k + \frac{1}{2}F_{xx}\big(M(t_k)\big)\big(\triangle M_k\big)^2, \end{aligned}$$

and then (12) follows from the equality $\langle M\rangle_t = \lim\limits_{\max \triangle t_k \to 0} \sum_k \big(\triangle M_k\big)^2$ (in probability)[2]

Similarly, for continuous semi-martingales X, Y, $d(XY) = XdY + YdX + d\langle X^c, Y^c\rangle$.

Lévy's characterization of the Brownian motion. If $W = W(t)$ is a continuous square-integrable martingale with $W(0) = 0$ and $\langle W\rangle_t = t$ (a.k.a. Wiener process), then W is a Gaussian process with mean zero and $\mathbb{E}|W(t) - W(s)|^2 = |t - s|$ (a.k.a. Brownian motion).

Proof. We only need to show that W is a Gaussian process, which will follow from

$$\mathbb{E}\Big(e^{\lambda\big(W(t)-W(s)\big)}|\mathcal{F}_s\Big) = e^{\lambda^2(t-s)/2}, \quad t > s, \tag{13}$$

for all $\lambda \in \mathbb{R}$, because (13) means that W has increments that are Gaussian and independent. Applying (12) to $F(t,x) = e^{\lambda x - (\lambda^2 t/2)}$, we conclude that $M(t) = F(t, W(t))$ is a martingale:

$$M(t) = 1 + \lambda\int_0^t M(s)dW(s),$$

and then (13) follows.

Proof of Girsanov's theorem. Using Itô formula,

$$d\widetilde{W} = dW - hdt,\ dZ = hZdW,\ d(\widetilde{W}Z) = Zd\widetilde{W} + \widetilde{W}dZ + hZdt = Z(1 + h\widetilde{W})dW,$$

so that the processes $t \mapsto Z(t)$ and $t \mapsto \widetilde{W}(t)Z(t)$ are martingales under $\mathbb{P}$:

$$\mathbb{E}\big(\widetilde{W}(t)Z(t)|\mathcal{F}_s\big) = \widetilde{W}(t)Z(t),\ \mathbb{E}\big(Z(t)|\mathcal{F}_s\big) = Z(s),$$

and then, using (11) with $\zeta = \widetilde{W}(t)$ and $\mathcal{G} = \mathcal{F}_s$, we conclude that $t \mapsto \widetilde{W}$ is a (continuous) martingale under $\widetilde{\mathbb{P}}$. Next, because *quadratic variation* of $\widetilde{W}$ is t, Itô formula applied to $Y(t) = \widetilde{W}^2(t) - t$ gives $dY = 2\widetilde{W}d\widetilde{W}$, that is, the process $t \mapsto \widetilde{W}^2(t) - t$ is a martingale under $\widetilde{\mathbb{P}}$. Then Lévy's characterization implies that $\widetilde{W}$ is a Brownian motion under $\widetilde{\mathbb{P}}$.

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[2]Karatzas-Shreve, *Brownian motion and stochastic calculus*, Theorem 1.5.8.