A note on characteristic functions.

By definition, the characteristic function of a (real-valued) random variable ξ is a (complex-valued) function φ of the real variable t:

$$\varphi(t) = \mathbb{E}e^{\sqrt{-1}\,\xi t}.$$

The definition immediately implies the following properties of ϕ:

(1) $\varphi(0) = 1$
(2) $|\varphi(t)| \leq 1$ for all t;
(3) φ is (uniformly) continuous [check out the uniform part].

The main necessary and sufficient result is known as the `Bochner-Khinchin theorem:` a complex-valued function ϕ of a real variable t is a characteristic function of some random variable if and only if all the following three properties hold

(1) $\varphi(0) = 1$
(2) φ is continuous for all t
(3) for every collection $t_1, \dots, t_n$ of real numbers the matrix $(\varphi(t_i - t_j),\ i, j = 1, \dots, n)$ is Hermitian and non-negative definite.

The third property is not so easy to verify. One famous sufficient condition is due to Polya: If $\varphi(t)$ is even, $\varphi(0) = 1$, φ is convex for $t > 0$, and $\lim_{t\to+\infty}\varphi(t) = 0$, then φ is a characteristic function of *an absolutely continuous* random variable. For more, see [1,3].

Here is a necessary condition [1, Theorem 4.1.1]: if φ is a characteristic function and $\varphi(t) = 1 + o(t^2), t \to 0$, then $\varphi(t) = 1$ for all t [indeed, the random variable with such a characteristic function must have zero mean and zero variance]. In particular, if $r > 2$, then $e^{-|t|^r}$ is not a characteristic function.

Another necessary condition is due to Marcinkiewitz (see [2], no proof...): if $\varphi(t) = e^{p(t)}$ is a characteristic function and $p = p(t)$ is a polynomial, then the degree of p is at most 2. For example, $e^{-t^2-t^4}$ is not a characteristic function.

One more necessary condition is a consequence of trig identities: $\Re\big(1 - \phi(t)\big) \geq \Re\big(1 - \phi(2t)\big)/4$. Indeed, $\Re(1 - \phi(t)\big) = \mathbb{E}\big(1 - \cos(t\xi)\big)$, and

$$4(1 - \cos(tx)) = 8\sin^2(tx/2) \geq 8\sin^2(tx/2)\cos^2(tx/2) = 2\sin^2(tx) = 1 - \cos(2tx).$$

For real φ, this becomes

$$3 + \varphi(2t) \geq 4\varphi(t).$$

An immediate consequence is that if $\varphi(t) = 1$ in some neighborhood of $t = 0$, then $\varphi(t) = 1$ in twice that neighborhood, and then, by induction, for all $t \in \mathbb{R}$, so that $\xi = 0$ with probability one.

Some other facts:

(1) If ξ is absolutely continuous, then $\lim_{|t|\to\infty}|\varphi(t)| = 0$ [Riemann-Lebesque];
(2) If $\int_{-\infty}^{\infty}|\varphi(t)|dt < \infty$, then ξ is absolutely continuous with pdf

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-\sqrt{-1}\,tx}\varphi(t)dt.$$

(3) If $\lim_{|t|\to\infty}|\varphi(t)| = 0$, then ξ is continuous: $\mathbb{P}(\xi = a) = 0$ for every $a \in \mathbb{R}$. Indeed, if $\mathbb{P}(\xi = a_0) = p_0$, then $\varphi(t)$ has a component $e^{\sqrt{-1}\,ta_0}$ that does not go to zero.

(4) It is possible to have a continuous random variable with $\limsup_{|t|\to\infty} |\varphi(t)| = 1$. For example, the random variable $\xi = \sum_{k=1}^{\infty} \varepsilon_k/k!$, where ε_k are iid taking values ± 1 with probability $1/2$, is continuous [because all converging sums of the form $\sum_k a_k \varepsilon_k$ are continuous], but, by the dominated convergence theorem,

$$\lim_{n\to\infty} \varphi(2\pi n!) = \lim_{n\to\infty} \prod_{k=1}^{\infty} \cos(2\pi n!/k!) = 1.$$

(5) With ε_k as above, it is known that a random variable $\xi = \sum_{k\geq 1} \varepsilon_k/a^k$ is singular if $a > 2$; if $a > 2$ is not an integer, then $|\varphi(t)| = O\Big(\big(\ln|t|\big)^{-r}\Big)$, $|t| \to \infty$ for some $r > 0$ [not obvious]; for $a = 2$, ξ is uniform on $(-1, 1)$ (and thus absolutely continuous) because

$$\frac{\sin t}{t} = \frac{\sin(t/2)}{t/2}\cos(t/2) = \frac{\sin(t/4)}{t/4}\cos(t/2)\cos(t/4) = \ldots = \frac{\sin(t/2^n)}{t/2^n}\prod_{k=1}^{n}\cos(t/2^k);$$

as $n \to \infty$, the right hand side converges to the characteristic function of ξ (with $a = 2$); the left hand side is the characteristic function of the uniform on $(-1, 1)$.

Question 1. Is $e^{-|t|-t^4}$ a characteristic function? [Technically, $|t| - t^4$ is not a polynomial...]

Question 2. Let $X_1, X_2, \ldots$ be iid random variables with support on the standard mid-third Cantor set; the cdf of X_1 is the Cantor ladder (or Devil's staircase...); the characteristic function of X_1 is $\varphi(t) = e^{\sqrt{-1}\,t/2} \prod_{k=1}^{\infty} \cos(t/3^k)$. Does there exist an $n > 1$ such that the sum $X_1 + \ldots + X_n$ is absolutely continuous?

References.

[1] E. Lukacs. Characteristic Functions. Griffin, 1970. [Chapter 4]

[2] A. N. Shiryaev. Probability. Springer, 1996. [Section II.12]

[3] N. G. Ushakov. Selected topics in characteristic functions. VSP—Utrecht, 1999. [Section 1.3]