# MATH 445

## MOTION IN AN ATTRACTING INVERSE-SQUARE CENTRAL FIELD

**The setting:** the three-dimensional space with an attracting point object (mass or charge) placed at the origin of a coordinate system.

**Equation of motion**:

$$\boldsymbol{r}''(t) = -\frac{c}{r^3}\boldsymbol{r},$$

$c > 0$ is a constant, $\boldsymbol{r}$ is the position vector, $r = \|\boldsymbol{r}\|$.

**The goal:** To show that the trajectories are conic sections (ellipse, hyperbola, or parabola)

**Step 1: trajectories are in the same plane**.
We note that

$$\frac{d}{dt}(\boldsymbol{r}(t)\times\boldsymbol{r}'(t)) = \boldsymbol{r}'(t)\times\boldsymbol{r}'(t) + \boldsymbol{r}(t)\times\boldsymbol{r}''(t) = \boldsymbol{0} - \frac{c}{r^3}\boldsymbol{r}(t)\times\boldsymbol{r}(t) = \boldsymbol{0}.$$

Therefore

$$\boldsymbol{r}(t)\times\boldsymbol{r}'(t) = \boldsymbol{r}(0)\times\boldsymbol{r}'(0) = \boldsymbol{h},$$

a constant vector, determined by the initial conditions.

Thus $\boldsymbol{r}(t)$ is perpendicular to $\boldsymbol{h}$ for all $t \geq 0$, meaning that the motion is in the plane that passes through the origin and has $\boldsymbol{h}$ as the normal vector.

**Step 2: an alternative representation of $\boldsymbol{h}$.** Denote by $\hat{\boldsymbol{r}}(t)$ the vector $\boldsymbol{r}(t)/r(t)$, the unit vector in the direction of $r(t)$. Note that, in the central field, you are never hitting the origin, so $\hat{\boldsymbol{r}}(t)$ is always defined.

Then the claim is that

$$\boldsymbol{h} = r^2(t)\hat{\boldsymbol{r}}(t)\times\hat{\boldsymbol{r}}'(t)$$

for all $t \geq 0$.

Indeed,

$$\hat{\boldsymbol{r}}'(t) = \frac{d}{dt}\left(\frac{\boldsymbol{r}(t)}{r(t)}\right) = \frac{\boldsymbol{r}'(t)}{r(t)} - \frac{r'(t)\,\boldsymbol{r}(t)}{r^2(t)},$$

and therefore

$$\hat{\boldsymbol{r}}(t)\times\hat{\boldsymbol{r}}'(t) = \frac{\hat{\boldsymbol{r}}(t)\times\boldsymbol{r}'(t)}{r(t)} - \frac{r'(t)\,\hat{\boldsymbol{r}}(t)\times\boldsymbol{r}(t)}{r^2(t)} = \frac{\hat{\boldsymbol{r}}(t)\times\boldsymbol{r}'(t)}{r(t)} = \frac{\boldsymbol{r}(t)\times\boldsymbol{r}'(t)}{r^2(t)} = \frac{\boldsymbol{h}}{r^2(t)}.$$

**Step 3: an expression for $(\boldsymbol{r}(t)\times\boldsymbol{h})'$.**
First note that since $\boldsymbol{h}$ is a constant vector,

$$(\boldsymbol{r}'(t)\times\boldsymbol{h})' = \boldsymbol{r}''(t)\times\boldsymbol{h}$$

Next, we use the equality

$$\boldsymbol{r}''(t) = -\frac{c}{r^2(t)}\hat{\boldsymbol{r}}(t),$$

from the equation of motion, as well as the equality

$$\boldsymbol{h} = r^2(t)\hat{\boldsymbol{r}}(t)\times\hat{\boldsymbol{r}}'(t)$$

from the previous step, and the equality

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}$$

which is a general property of the cross product. Then

$$(\boldsymbol{r}'(t) \times \boldsymbol{h})' = \boldsymbol{r}''(t) \times \boldsymbol{h} = -\frac{c}{r^2(t)}\hat{\boldsymbol{r}}(t) \times \big(r^2(t)\hat{\boldsymbol{r}}(t) \times \hat{\boldsymbol{r}}'(t)\big)$$

$$= -c\hat{\boldsymbol{r}}(t) \times \big(\hat{\boldsymbol{r}}(t) \times \hat{\boldsymbol{r}}'(t)\big) = -c(\hat{\boldsymbol{r}}(t) \cdot \hat{\boldsymbol{r}}'(t))\hat{\boldsymbol{r}}(t) + c(\hat{\boldsymbol{r}}(t) \cdot \hat{\boldsymbol{r}}(t))\hat{\boldsymbol{r}}'(t).$$

Finally, we use the remarkable fact that

$$0 = \frac{d}{dt}1 = \frac{d}{dt}\|\hat{\boldsymbol{r}}(t)\|^2 = \frac{d}{dt}(\hat{\boldsymbol{r}}(t) \cdot \hat{\boldsymbol{r}}(t)) = 2\hat{\boldsymbol{r}}(t) \cdot \hat{\boldsymbol{r}}'(t)$$

that is

$$\hat{\boldsymbol{r}}(t) \cdot \hat{\boldsymbol{r}}'(t) = 0.$$

As a result,

$$(\boldsymbol{r}'(t) \times \boldsymbol{h})' = c(\hat{\boldsymbol{r}}(t) \cdot \hat{\boldsymbol{r}}(t))\hat{\boldsymbol{r}}'(t) = c\,\hat{\boldsymbol{r}}'(t),$$

or

$$(\boldsymbol{r}'(t) \times \boldsymbol{h}) = \boldsymbol{b} + c\,\hat{\boldsymbol{r}}(t)$$

where $\boldsymbol{b} = (\boldsymbol{r}'(0) \times \boldsymbol{h}) - \hat{\boldsymbol{r}}(0)$ is a constant vector determined by the initial conditions.

**Step 4: selecting the vectors $\boldsymbol{h}$ and $\boldsymbol{b}$:** Note that $\boldsymbol{b}$ is in the plane spanned by $\boldsymbol{r}(0)$ and $\boldsymbol{r}'(0)$, and is therefore perpendicular to $\boldsymbol{h}$. Therefore we can choose our coordinate system so that $\boldsymbol{b} = b\,\hat{\boldsymbol{\imath}}$ and $\boldsymbol{h} = h\,\hat{\boldsymbol{\kappa}}$. With this coordinate system, the motion will take place in the $(x, y)$ plane.

**Step 5: two ways to write $\boldsymbol{r}(t) \cdot (\boldsymbol{r}'(t) \times \boldsymbol{h})$**

On the one hand, using the properties of the triple scalar product and the results of the first step,

$$\boldsymbol{r}(t) \cdot (\boldsymbol{r}'(t) \times h) = (\boldsymbol{r}(t) \times \boldsymbol{r}'(t)) \cdot \boldsymbol{h}) = \boldsymbol{h} \times \boldsymbol{h} = \|\boldsymbol{h}\|^2 = h^2.$$

On the other hand, from the previous step,

$$\boldsymbol{r}(t) \cdot (\boldsymbol{r}'(t) \times \boldsymbol{h}) = \boldsymbol{r}(t) \cdot \boldsymbol{b} + c\boldsymbol{r}(t) \cdot \hat{\boldsymbol{r}}(t) = br(t)\cos\theta(t) + cr(t) = r(t)(b\cos\theta + c),$$

where $b = \|\boldsymbol{b}\|$ and $\theta$ is the angle between $\boldsymbol{r}$ and $\boldsymbol{b} = \hat{\boldsymbol{\imath}}$ (the polar angle)

**Step 6: we are done:**

$$r(t) = \frac{h^2}{c + b\cos\theta(t)}$$

or, with $A = h^2/c$, $\varepsilon = b/c$, and omitting the time,

$$r = \frac{A}{1 + \varepsilon\,\cos\theta}$$

which is the equation of a conic section in the polar coordinates:

- **circle** if $\varepsilon = 0$
- **ellipse** if $\varepsilon \in (0, 1)$
- **parabola** if $\varepsilon = 1$
- **hyperbola** if $\varepsilon > 1$