which immediately implies the first inequality in (2).

If $0 \leqq x < n$ then $\alpha_n(x) < \alpha_n(n)$, whence

$$\int_0^n \alpha_n(x) e^{-x} dx < \alpha_n(n) \int_0^n e^{-x} dx < \alpha_n(n). \tag{4}$$

Let λ_n denote the second term in (3). Integration by parts gives

$$\begin{aligned} \lambda_n &= e^{-n}\alpha_n(n) + \int_n^\infty \frac{n}{1+x} \alpha_{n-1}(x) e^{-x} dx \\ &< \alpha_n(n) + \int_{n-1}^\infty \alpha_{n-1}(x) e^{-x} dx = \alpha_n(n) + \lambda_{n-1}. \end{aligned}$$

Since $\lambda_0 = 1$, it follows that

$$\lambda_n < 1 + \sum_{k=1}^n \alpha_k(k).$$

Since the $\alpha_k(k)$ increase with k and since $2\alpha_n(n) > 1$, we must have

$$\lambda_n < (n+2)\alpha_n(n). \tag{5}$$

Combining (4) and (5), we see that

$$\nu_n < (n+3)\alpha_n(n),$$

which implies the second inequality in (2) as soon as it is noted that $(n+3)^{1/n} \to 1$.

References

1. M. G. Kendall and A. Stuart, The advanced theory of statistics, vol. I, Stechert-Hafner, New York, 1958.
2. J. A. Shohat and J. D. Tamarkin, The problem of moments, Amer. Math. Soc., 1943.

DISTRIBUTIONS OF PRODUCT AND QUOTIENT OF CAUCHY VARIABLES

PAUL R. RIDER, Aerospace Res. Labs., Wright-Patterson Air Force Base

1. Introduction. The Cauchy distribution,

$$f(x) = \frac{1}{\pi(1+x^2)}, \qquad -\infty < x < \infty, \tag{1}$$

has several peculiar properties. The distribution of arithmetic means of samples from it is identical with the distribution itself. The reciprocal of a Cauchy variable has the same distribution as the variable itself. The distribution of harmonic means is, consequently, the same as the original distribution.

This note derives the distributions of product and quotient of two independent Cauchy variables.

2. Distribution of product. The probability element of the joint distribution of two independent variables, x and y, from the Cauchy distribution is

$$\frac{dxdy}{\pi^2(1+x^2)(1+y^2)}. \tag{2}$$

In order to find the distribution of the product $u=xy$ we substitute $y=x^{-1}u$, $dy=x^{-1}du$ in (2), obtaining

$$\frac{|x|\,dxdu}{\pi^2(1+x^2)(u^2+x^2)},$$

which separates into partial fractions as follows:

$$\frac{1}{\pi^2(u^2-1)}\left[\frac{|x|}{1+x^2}-\frac{|x|}{u^2+x^2}\right]dxdu.$$

We now integrate with respect to x between the limits $-\infty$ and ∞. (It is convenient to integrate between 0 and ∞ and double the result.) For the distribution of the product u, we find

$$f(u)=\frac{\log u^2}{\pi^2(u^2-1)}. \tag{3}$$

As a check we note [1, p. 192, formula (6)] that $\int_{-\infty}^{\infty}f(u)du=1$.

3. Distribution of quotient. The distribution of the quotient of two independent Cauchy variables can be found by the same process. It turns out to be identical with the distribution of the product, which of course must be the case, since, as already stated, the distribution of the reciprocal of a Cauchy variable is identical with the distribution of the variable itself.

It may be noted that (3) is also the distribution of the product or quotient of the means (arithmetic or harmonic) of two independent samples from the distribution (1).

4. Nature of frequency curve. The function $f(u)$ becomes infinite as u approaches zero. Its value is indeterminate for $u=\pm 1$, but by l'Hôpital's rule it can be shown that $f(u)$ approaches $1/\pi^2$ as u approaches ± 1.

To investigate the appearance of the frequency curve further we find the derivative

$$f'(u)=\frac{2u}{\pi^2(u^2-1)^2}\left[\frac{u^2-1}{u^2}-\log u^2\right]. \tag{4}$$

To show that this derivative is nonnegative we consider first the case $u>0$. If we set $u^{-2}=V$ the expression in brackets in (4) can be written $1+\log V-V$. Considering the ratio $(1+\log V)/V$, we find by usual methods that its maximum value is 1, given by $V=1$. Thus,

$$(1 + \log V)/V \leqq 1, \quad \text{or} \quad 1 + \log V \leqq V.$$

In terms of u^2 we have $1 - \log u^2 \leqq 1/u^2$, which shows that the bracketed expression in (4) is (for $u > 0$) always negative or zero. It can be zero for $u = 1$. But l'Hôpital's rule shows that $f'(u)$ approaches $-1/\pi^2$ as u approaches 1. Consequently $f'(u)$ is always negative and, as u increases from 0 to ∞, $f(u)$ decreases from ∞ to 0.

A similar argument shows that for $u < 0$, $f'(u)$ is always positive and, as u increases from $-\infty$ to 0, $f(u)$ increases from 0 to ∞.

Combining results, we see that the frequency curve resembles the graph of $y = 1/x^2$.

All moments of the distribution are infinite.

Thanks of the author are hereby expressed to Dr. Robert S. DeZur of the Martin Company, Denver Division, for valuable suggestions during the preparation of this note.

Reference

1. D. Bierens de Haan, Nouvelles Tables d'Intégrales Définies, Stechert Hafner, New York, 1929.

ANOTHER PROOF OF THE INFINITE PRIMES THEOREM

M. Wunderlich, University of Colorado
(Now at the State University of New York at Buffalo)

Let F_n be the nth Fibonacci number. It is a well-known and easily proved result [1] that

$$F_{(m,n)} = (F_m, F_n), \tag{1}$$

where (m, n) as usual denotes the greatest common divisor. This property yields another proof of the infinite prime theorem.

Theorem. *There are infinitely many primes.*

Proof. Suppose $p_1, p_2, \cdots, p_k$ are all the prime numbers. Then consider

$$F_{p_1}, F_{p_2}, \cdots, F_{p_k}. \tag{2}$$

From (1), the numbers in (2) are pairwise relatively prime, and since there are only k primes, each of the numbers in (2) has only one prime factor. But this contradicts the fact that

$$F_{19} = 4181 = 113 \cdot 37.$$

Reference

1. N. N. Vorob'ev, Fibonacci Numbers, Blaisdell, New York, 1961, p. 30.