A Summary of Continuous Time Markov Chains

1. **Intuition**. For a *regular* continuous time Markov chain X with a countable state space $S_X = \{x_1, x_2, \dots\}$, the process sits in the state x_k for random time T_k, then jumps to another state x_l, sits there for random time T_l, then jumps to another state, and so on. The times T_i spent in each state are independent and exponentially distributed.

2. As a motivation, consider a two-state discrete time Markov chain with possible jumps between the states at times $t_k = k\triangle t$. To make the situation nontrivial, we need to assume the possibility of no jumps. More precisely, assume that the probability to jump from state 1 to state 2 is $q_1\triangle t$ and the probability to jump from state 2 to state 1 is $q_2\triangle t$, where q_1 and q_2 are positive numbers and $q_1\triangle t < 1$, $q_2\triangle t < 1$. Then, at each time $t_k = k\triangle t$, the probability to stay in state 1 is $1 - q_1\triangle t$ and the probability to stay in state 2 is $1 - q_2\triangle t$. If $X(0) = 1$ and $\triangle t = T/n$, then

$$\mathbb{P}\big(X(t) = 1, t \le T \big| X(0) = 1\big) = \mathbb{P}\big(X(t_k) = 1, k = 1, \dots, n \big| X(0) = 1\big) = \big(1 - (q_1 T/n)\big)^n \to e^{-q_1 T},\ n \to \infty.$$

Similarly, if $X(0) = 2$ and $\triangle t = T/n$, then

$$\mathbb{P}\big(X(t) = 2, t \le T \big| X(0) = 2\big) = \mathbb{P}\big(X(t_k) = 1, k = 1, \dots, n \big| X(0) = 2\big) = \big(1 - (q_2 T/n)\big)^n \to e^{-q_2 T},\ n \to \infty.$$

3. As a further motivation, consider the discrete time Markov chain with more than two states. Again, assume the possibility of not jumping. For the process at state i, the probability to jump to state j is $q_{ij}\triangle t$. Then, with $\triangle t$ sufficiently small, the probability of no jump is then $1 - q_i\triangle t$, where

$$q_i = \sum_{j:j\neq i} q_{ij}$$

By the same argument, if $\triangle t = T/n$ and we let $n \to \infty$, then the probability to stay in state i for time T is

$$\mathbb{P}\big(X(t_k) = i,\ k = 1, \dots, n | X(0) = i\big) = (1 - (q_i T/n))^n \to e^{-q_i T}.$$

4. Recall that $X = X(t),\ t \ge 0$, with values in a measurable space S_X is called a continuous time Markov process if, for every $t_n > t_{n-1} > t_{n-2} > \cdots > t_1 \ge 0$ and every measurable set $B \subseteq S_X$,

$$\mathbb{P}\big(X(t_n) \in B | X_{t_{n-1}}, X_{t_{n-1}}, \dots, X_{t_1}\big) = \mathbb{P}\big(X(t_n) \in B | X_{t_{n-1}}\big)$$

Alternatively, we can say that, for every $t > s \ge 0$,

$$\mathbb{P}\big(X(t) \in B | X_{[0,s]}\big) = \mathbb{P}\big(X(t) \in B | X_s\big),\quad X_{[0,s]} = \sigma\big(X(r),\ 0 \le r \le t\big).$$

In what follows, we assume that

1. $S_X = \{x_1, x_2, \dots\}$ (countable state space);
2. The trajectories of X are right-continuous;
3. The process is *time-homogeneous*, that is, for all $x_i, x_j \in S_X$ and all $t, s \ge 0$,

$$\mathbb{P}\big(X(t+s) = x_j | X(s) = x_i\big) = \mathbb{P}\big(X(t) = x_j | X(0) = x_i\big);$$

4. The functions $p_{ij}(t) = \mathbb{P}\big(X(t) = x_j | X(0) = x_i\big)$ are *continuous at zero*, that is,

$$\lim_{t \to 0^+} p_{ij}(t) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

By the Chapman-Kolmogorov equation,

$$p_{ij}(t+s) = \sum_k p_{ik}(t) p_{kj}(s) = \sum_k p_{ik}(s) p_{kj}(t).$$

Theorem 1.1.1. Under the above assumptions 1–4,

1. for every $i \neq j$, the limit $q_{ij} = \lim_{t \to 0^+} \frac{p_{ij}(t)}{t}$ exists and is finite;
2. for every i, the limit $q_i = \lim_{t \to 0^+} \frac{1-p_{ii}(t)}{t}$ exists, but can be $+\infty$;
3. $\sum_{j:j \neq i} q_{ij} \leq q_i$.

Definition 1.1.2.

(a) The process X is called `regular` if, for every i, q_i is finite and

$$\sum_{j:j \neq i} q_{ij} = q_i$$

(b) The state x_i of the process X is called `instantaneous` if $q_i = +\infty$.

Intuitively, x_i is instantaneous if

$$\mathbb{P}\big(X(t) = x_i | X(0) = x_i\big) = 0$$

for every $t > 0$.

Corollary 1.1.3. If X is regular, then

$$p_{ij}(t) = \delta_{ij} + q_{ij}t + o(t), \; t \to 0+,$$

where $q_{ii} = -q_i$.

For a regular Markov process X, define the matrices $\boldsymbol{P} = \boldsymbol{P}(t)$ with components $p_{ij}(t)$ (row i, column j) and $\boldsymbol{Q}$ with components q_{ij} (row i, column j).

Theorem 1.1.4. If X is regular, then $\dot{\boldsymbol{P}}(t) = \boldsymbol{Q}\boldsymbol{P}(t) = \boldsymbol{P}(t)\boldsymbol{Q}$ and $\boldsymbol{Q}$ is the `generator` of X. In particular, $\boldsymbol{P}(t) = e^{t\boldsymbol{Q}}$.

Embedded Markov chain

If X is regular and we define the sequence of random variables y_n and random times T_n by

$$T_0 = 0, y_0 = X(0), \, T_{n+1} = \inf\{t > T_n : X(t) \neq y_n\}, y_{n+1} = X(T_{n+1}),$$

then $y_n, n \geq 1$, is a Markov sequence with transition probabilities

$$\mathbb{P}(y_{n+1} = x_j | y_n = x_i) = \frac{q_{ij}}{q_i}$$

and

$$\mathbb{P}(T_{n+1} - T_n > t | y_n = x_i) = e^{-q_i t}.$$