Some famous, but advanced, problems in probability[1]

The Buffon needle

The problem. A needle of length L is tossed in a random way on the floor with infinitely many parallel lines d units apart, $d > L$. What is the probability p that the needle crosses a line?

The origin. Georges-Louis Leclerc, compte de Buffon (1707–1788), French naturalist and intellectual; 1733: statement; 1777: solution.

The answer.

$$p = \frac{2L}{\pi d}.$$

Informal argument. By common sense, $p = cL/d$: the longer the needle and closer the lines, the more likely the needle to cross a line. All we need is the number c.

Note that, in the case of a needle, p is also the *average number* of intersections. An extra stretch of imagination suggests that the *average number* of intersections should not depend on the shape of the needle but only on its length, that is, should be the same for a needle or a *noodle* (a not-very-wild curve that can be simple closed), as long as both have the same length L. In other words, we accept that the average number of intersections should be cL/d for either a needle or a noodle.

Now take a noodle in the form of the circle of diameter d so that $L = \pi d$. Then the number of intersections is always 2, that is, $2 = c(\pi d)/d$, that is, $c = 2/\pi$.

Rigorous derivation. Introduce the distance X from the mid-point of the needle to the nearest line and the angle θ between the needle and a line. The needle intersects a line if

$$X \le \frac{L}{2}\sin\theta.$$

A random toss means that X is uniform on $(-d/2, d/2)$, θ is uniform on $(0, \pi)$, and X and θ are independent. The answer follows after a simple integration.

Statistical computation of π. From

$$p = \frac{2L}{\pi d},$$

we conclude that

$$\pi \approx \frac{2L}{d} \times \frac{\text{total number of tosses}}{\text{number of times the needle crossed a line}}$$

Recall

$$\pi = 3.141592654...$$

A physical experiment: Rudolf Wolf (1816–1893), Swiss; conducted the experiment between 1849 and 1853 with $d = 45\,mm$, $L = 36\,mm$. He made 5000 tosses, got 2532 intersections, resulting in

$$\pi \approx \frac{360,000}{2532 \cdot 45} \approx 3.159558$$

Very reasonable.

A "mathematical" experiment: Mario Lazzarini, Italian; conducted the experiment in 1901 with $d = 30\,mm$, $L = 25\,mm$. He made 3408 tosses, got 1808 intersections, resulting in

$$\pi \approx \frac{3408 \cdot 5}{1808 \cdot 3} \approx 3.14159292$$

Amazing...

Making sense of Lazzarini's experiment Fact 1: $3408 = 213 \cdot 16 = 71 \cdot 3 \cdot 16$, $1808 = 113 \cdot 16$, that is,

$$\frac{3408 \cdot 5}{1808 \cdot 3} = \frac{71 \cdot 3 \cdot 16 \cdot 5}{113 \cdot 16 \cdot 3} = \frac{355}{113}.$$

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Fact 2:

$$\pi \approx \frac{355}{113} \approx 3.14159292$$

is a famous rational approximation of π: the best there is if you limit the number of digits on top and bottom to three (even to four or five...).

So Lazzarini knew what he wanted ($113n/213$ crossings in n tosses) and ran a "sequential" experiment, checking every 213 tosses if he has a multiple of 113 crossings, until he got it on 16th try (this is the most probable explanation; we will never know for sure).

Note also that, to claim accuracy to 6th decimal in the outcome, one would need to claim similar accuracy in the measurements of d and L, which is not easily achievable, especially in 1901.

Further reading

- Lee Badger. Lazzarini's Lucky Approximation of π. *Mathematics Magazine*, Vol. 67 (1994), No. 2, pp. 83–91.
- Ivars Peterson. Buffon's Needling Ants, `http://www.sciencenews.org/view/generic/id/459/description/Buffons_Needling_Ants`

Related problems.

1. Consider a rectangular grid with vertical lines a units apart and horizontal lines b units apart. A needle of length $\ell < \min(a, b)$ is dropped at random on the grid. Let A represent the event that the needle crosses a vertical line and let B represent the event that the needle crosses a horizontal line. Confirm that $P(A) = 2\ell/(\pi a)$, $P(B) = 2\ell/(\pi b)$, and $P(A \cap B) = \ell^2/(\pi ab)$.

2. What happens for a longer needle, when tossed either on the strips or on a grid? Note that, in the case of a really long needle, one can talk about intersecting several lines, and, in the case of the grid, some probabilities can be equal to 1.

What does *uniformly at random* mean?

Bertrand's Paradox, after French mathematician Joseph Louis François Bertrand (1822–1900): what is the probability that a random chord in the unit circle is longer than $\sqrt{3}$, the side of an equilateral triangle inscribed into the circle? There are (at least) three possible answers:

(1) Take a point uniformly at random on a radius, then draw the chord through that point, perpendicular to the radius; then the probability is 1/2 because the point on the radius should be closer to the center of the circle than to the boundary;

(2) Take a point uniformly at random inside the unit disk, draw the radius through the point and the chord that goes through the point and is perpendicular to the radius; then the probability is 1/4 for the same reasons as in the first case, but now the favorable region is the disk of radius 1/2 and has the 1/4 of the total area;

(3) Given a point A on the circle, select another point B uniformly at random on the circle; then the probability is 1/3 by actually inscribing a regular triangle with one vertex at the A and looking at the part of the circle where the point B has to be.

Ambiguity in this case is caused by lack of a clearly defined uniform measure on the set of all chords. The choice of the interpretation can depend on the invariance (with respect to scaling, translation, rotation, etc.) properties of the result.

The four point problem of Sylvester, after English mathematician James Joseph Sylvester (1814–1897): what is the probability that four randomly selected points in the plane are vertices of a convex quadrilateral? This problem is not well posed because there is no uniform distribution on the plane: the plane has infinite area. One interpretation could be to solve the problem in a bounded region and then pass to the limit; in this case, the answer will depend on the region (2/3 for triangle, 25/36 for square, $1 - \frac{35}{12\pi^2}$ for disk, etc.) An alternative (and the original) solution is to look at the three points *that make the largest triangle* and then see where the fourth point has to be so that the resulting quadrilateral is convex; the answer in this case is 3/4. For details, see

Richard E. Pfiefer. The Historical Development of J. J. Sylvester's Four Point Problem, *Mathematics Magazine*, Vol. 62 (1989), No. 5, pp. 309–317.