PDEs in a bounded domain.[1]

Heat equation on an interval.

(1) $$u_t = au_{xx},\ u = u(t,x), \quad t > 0,\ x \in (0, L)$$

$u(0,x) = \varphi(x),\ u(t,0) = u(t,L) = 0$. Try

(2) $$u(t,x) = \sum_{k=1}^{\infty} F_k(t)G_k(x):$$

`separation of variables`. Then

(3) $$u(0,x) = \sum_k F_k(0)G_k(x),\ u(t,0) = \sum_k F_k(t)G_k(0),\ u(t,L) = \sum_k F_k(t)G_k(L),$$

so that, in particular, we expect

$$G_k(0) = G_k(L) = 0$$

for all k. Also, by linearity, each function $u_k(t,x) = G_k(t)F_k(x)$ must satisfy the heat equation, that is,

(4) $$F_k'(t)G_k(x) = aF_k(t)G_k''(x),\ \frac{F_k'(t)}{aF_k(t)} = \frac{G_k''(x)}{G_k(x)} = b_k$$

for some numbers b_k. In particular, we need (not identically zero) functions $G_k = G_k(x)$ satisfying

$$G_k''(x) = b_k G_k(x),\ G_k(0) = G_k(L) = 0.$$

If $c_k > 0$, then

$$G_k(x) = A_k \cosh(\sqrt{c_k}\, x) + B_k \sinh(\sqrt{c_k}\, x),$$

so that $G_k(0) = A_k = 0$ and, because $\sinh(t) \neq 0$ for (real) $t \neq 0$, the condition $G_k(L) = 0$ implies $B_k = 0$ as well. In other words, if $b_k > 0$, then G_k must be identically zero: not what we want.

If $b_k = 0$, then

$$G_k(x) = A_k + B_k x.$$

and again $G_k(0) = 0$ implies $A_k = 0$, and then $G_k(L) = 0$ implies $B_k = 0$. In other words, $b_k = 0$ does not work either.

Finally, assume that $b_k = -\lambda_k^2 < 0$. Then

$$G_k''(x) + \lambda_k^2 G_k(x) = 0$$

so that

$$G_k(x) = A_k \cos(\lambda_k x) + B_k \sin(\lambda_k x).$$

From $G_k(0) = 0$ we get $A_k = 0$; from $G_k(L) = 0$ and $B_k \neq 0$, we get

$$\sin(\lambda_k L) = 0,$$

that is

(5) $$\lambda_k = \frac{\pi}{L}\, k,\ k \in \mathbb{N},\ c_k = -\lambda_k^2.$$

We now put together everything we got so far and re-write (2) as

(6) $$u(t,x) = \sum_{k=1}^{\infty} F_k(t) \sin(\pi k x/L)$$

Also, by (4) and (5),

$$F_k'(t) = -\left(\frac{\pi k}{L}\right)^2 F_k(t).$$

Then

(7) $$F_k(t) = F_k(0)e^{-(\pi k/L)^2 t},$$

that is, (6) becomes

(8) $$u(t,x) = \sum_{k=1}^{\infty} F_k(0)e^{-(\pi k/L)^2 t} \sin(\pi k x/L).$$

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Finally, putting $t = 0$ in (8),

$$u(0, x) = \varphi(x) = \sum_{k=1}^{\infty} F_k(0) \sin(\pi k x/L)$$

so that [after multiplying both sides by a particular $\sin(\pi k x/L)$ and integrating from 0 to L]

(9) $$F_k(0) = \frac{2}{L}\int_0^L \varphi(x)\sin(\pi k x/L)\,dx.$$

In other words, the "solution" of (1) is given by (8) and (9), as long as the we can expand the initial condition φ in a Fourier sine series. If φ is continuous on $[0, L]$ and $\varphi(0) = \varphi(L) = 0$, then the "solution" is a classical solution: you can substitute it into the equation to get an identity, and the initial condition is satisfied in the sense that $u(0+, x) = \varphi(x)$.

Wave equation on an interval.

(10) $$u_{tt} = c^2 u_{xx},\ u = u(t, x),\quad t > 0,\ x \in (0, L)$$

$u(0, x) = \varphi(x)$, $\quad u_t(0, x) = \psi(x)$, $\quad u(t, 0) = u(t, L) = 0$. The solution procedure is identical to the one used to solve the heat equation. Write

(11) $$u(t, x) = \sum_{k=1}^{\infty} F_k(t) G_k(x),$$

conclude that

$$\frac{F_k''(t)}{c^2 F_k(t)} = \frac{G_k''(x)}{G_k(x)} = b_k$$

and then

$$b_k = -\left(\frac{\pi k}{L}\right)^2,\ G_k(x) = \sin(\pi k x/L),$$

so that, from

$$F_k''(t) + c^2(\pi k/L)^2 F_k(t) = 0,$$

we get

$$F_k(t) = A_k \cos(c\pi k t/L) + B_k \sin(c\pi k t/L),$$

that is,

(12) $$u(t, x) = \sum_{k=1}^{\infty} \big(A_k \cos(c\pi k t/L) + B_k \sin(c\pi k t/L)\big) \sin(\pi k x/L).$$

Put $t = 0$ to get

$$\varphi(x) = \sum_{k=1}^{+\infty} A_k \sin(\pi k x/L),$$

that is,

(13) $$A_k = \frac{2}{L}\int_0^L \varphi(x)\sin(\pi k x/L)\,dx.$$

Similarly,

$$\psi(x) = \frac{c\pi}{L}\sum_{k=1}^{+\infty} k B_k \sin(\pi k x/L),$$

that is,

(14) $$B_k = \frac{2}{c\pi k}\int_0^L \psi(x)\sin(\pi k x/L)\,dx.$$

The final answer is given by the combination of (12), (13), and (14). It is clearly a "solution". When is it a classical solution?

A note about [musical] string. All string instruments are (approximately) described by (10). The difference is in

- length of the string L;
- linear mass density of the string ρ [measured in mass per unit length];

- tension of the string τ [measured in the units of force].

The quantities ρ and τ combine nicely to provide the propagation speed:

$$c = \sqrt{\frac{\tau}{\rho}}.$$

The sound we hear comes (mostly) from the base frequency, corresponding to $k = 1$:

$$\omega_1 = \frac{c\pi}{L};$$

The frequencies $\omega_k = k\omega_1$ represent overtones; very roughly speaking, controlling those overtones is a major part of both the quality of the instrument and the quality of the musician playing the instrument.

Looking at the formula

$$\omega_1 = \frac{\pi}{L}\sqrt{\frac{\tau}{\rho}},$$

we can now understand the basic math behind the string section of an orchestra: the frequency gets lower as the string gets longer and heavier [violin-viola-cello-bass]; fine-tuning the string is achieved by changing the tension τ.

A note about battle ropes. The starting point could be the equation

$$u_{tt} = c^2 u_{xx},\ u = u(t,x),\ t > 0,\ x \in (0, L), \tag{15}$$

with boundary conditions $u(t, 0) = f(t),\ u(t, L) = 0$. The most basic question to address is existence of `standing wave` solutions, that is, solutions of the form

$$u_{sv}(t,x) = F(t)H(x),$$

where the functions F and H are periodic. A natural way to satisfy boundary conditions is to consider a function

$$u_n(t,x) = f(t)\cos\left(\frac{\pi}{L}\left(\frac{1}{2}+n\right)x\right),\quad n = 0, 1, 2, \ldots, \tag{16}$$

which could be a standing wave solution. The main point here is that

$$\cos(0) = 1,\quad \cos\left(\frac{\pi}{L}\left(\frac{1}{2}+n\right)L\right) = 0$$

so the function u_n satisfies the boundary conditions. Plugging u_n into (15) equation, we conclude that u_n is indeed a standing wave solution if

$$f''(t) = -c^2\left(\frac{\pi}{L}\left(\frac{1}{2}+n\right)\right)^2 f(t)$$

that is, if

$$f(t) = A_n\cos\omega_n t + B_n \sin\omega_n t,\quad \omega_n = \frac{c\pi}{L}\left(\frac{1}{2}+n\right).$$

How realistic is this result? I would argue: not much, for (at least) two reasons:

(1) Typically, the system starts from rest, that is, $f(t) = f'(t) = 0$, which is not possible with the above setting;
(2) Real-life system has damping; for battle ropes you probably call it "resistance".

As a result, a more realistic model of battle ropes could be a damped wave equation

$$u_{tt} + \gamma u_t = c^2 u_{xx} \tag{17}$$

with zero initial conditions $u(0,x) = u_t(0,x) = 0$ and boundary conditions $u(t,0) = f(t),\ u(t,L) = 0$ so that a `compatibility condition` holds: $f(0) = f'(0) = 0$. The extra term γu_t, with $\gamma > 0$, represents damping [or resistance]. How can we solve this equation?

Again, we start with the function u_n from (16) and define

$$v(t,x) = u(t,x) - u_n(t,x). \tag{18}$$

If the function u is a solution of (17), then the function v must be a solution of the `inhomogeneous` wave equation

$$v_{tt} + \gamma v_t = v_{xx} + B(t,x) \tag{19}$$

with zero initial and boundary conditions. The function B is

$$B(t,x) = c^2(u_n)_{xx}(t,x) - \gamma(u_n)_t(t,x) - (u_n)_{tt}(t,x).$$

The solution of (19) can then be written using the `variation of parameters` formula. [We will discuss the general version of the formula later; writing out the corresponding solution for (19) and analyzing it is up to you].

Laplace equation in a rectangle: an example

$$u_{xx} + u_{yy} = 0,\ 0 < x < 1,\quad 0 < y < \pi,\quad u = u(x,y),$$

$u(1,y) = 1$, $u(0,y) = u(x,0) = u(x,\pi) = 0$. Write

$$u(x,y) = \sum_{k=1}^{\infty} F_k(x) G_k(y)$$

with $G_k(0) = G_k(\pi) = F_k(0) = 0$. Then

$$-\frac{F_k''(x)}{F_k(x)} = \frac{G_k''(y)}{G_k(y)} = b_k = -\lambda_k^2 :$$

we need non-zero solutions of

$$G_k''(y) - b_k G_k(y) = 0,\quad G_k(0) = G_k(\pi) = 0,$$

which, as we know, is only possible for $b_k = -\lambda_k^2$. In fact, we also know that

$$\lambda_k = k,\quad G_k(y) = \sin(ky).$$

After that,

$$F_k''(x) - k^2 F_k(x) = 0,\quad F_k(x) = A_k \sinh(kx) + B_k \cosh(kx).$$

Because $F_k(0) = 0$, we have $B_k = 0$, $F_k(x) = A_k \sinh(kx)$, and

$$u(x,y) = \sum_k A_k \sinh(kx) \sin(ky).$$

From

$$u(1,y) = 1 = \sum_k A_k \sinh(k) \sin(ky)$$

we conclude [after multiplying by a particular $\sin(ky)$ and integrating from 0 to π]

$$A_{2k} = 0,\ A_{2k+1} = \frac{4}{\pi(2k+1)\sinh(2k+1)}.$$

As a result,

$$u(x,y) = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\sinh((2k+1)x)\sin((2k+1)y)}{(2k+1)\sinh(2k+1)}.$$

Can you see that, for $(x,y) \in (0,1) \times (0,\pi)$, the function u is infinitely differentiable?

Poisson equation in a square: an example

$$u_{xx} + u_{yy} = -1,\quad u = u(x,y),\ (x,y) \in G,\quad u|_{\partial G} = 0,$$

where G is a square:

$$G = (0,\pi) \times (0,\pi).$$

Now we have to consider the eigenvalue problem for the Laplacian in G [also known as the `Helmholtz equation`],

$$U_{xx} + U_{yy} = -\lambda_{k,\ell}^2 U,\ U|_{\partial G} = 0,$$

look for the solution in the form $U(x,u) = F_k(x) G_\ell(y)$ so that

$$\frac{F_k''(x)}{F_k(x)} + \frac{G_\ell''(y)}{G_\ell(y)} = -\lambda_{k,\ell}^2,$$

then conclude from the boundary conditions that

$$F_k(x) = \sin(kx),\ G_\ell(y) = \sin(\ell y),\ \lambda_{k,\ell} = k^2 + \ell^2,\quad k,\ell \in \mathbb{N}.$$

After that, the solution will be of the form

$$u(x,y) = \sum_{k,\ell} A_{k,\ell} \sin(kx) \sin(\ell y),$$

which, after substitution into the original equation gives

$$\sum_{k,\ell} A_{k,\ell}(k^2 + \ell^2) \sin(kx) \sin(\ell y) = 1. \tag{20}$$

As we already know, for $x \in (0,\pi)$ and $y \in (0,\pi)$,

$$\frac{4}{\pi}\sum_k \frac{\sin((2k+1)x)}{2k+1} = 1 = \frac{4}{\pi}\sum_\ell \frac{\sin((2\ell+1)y)}{2\ell+1}$$

and therefore

$$1 = 1\times 1 = \frac{16}{\pi^2}\sum_{k,\ell} \frac{\sin((2k+1)x)}{2k+1}\,\frac{\sin((2\ell+1)y)}{2\ell+1}. \tag{21}$$

Comparing (20) and (21), we conclude that

$$u(x,y) = \frac{16}{\pi^2}\sum_{k,\ell} \frac{\sin((2k+1)x)\,\sin((2\ell+1)y)}{(2k+1)(2\ell+1)((2k+1)^2+(2\ell+1)^2)}.$$

Can you see from this formula that $u(x,y) > 0$ for all $(x,y) \in G$? Can you see from this formula that u is infinitely differentiable in G?

The big (and general) picture: orthogonal expansion in eigenfunctions of self-adjoint operators

As a motivation, let us take another look at the computations for the heat equation on the interval, specifically, those leading to the functions G_k and numbers λ_k.

Denote by $\mathbb{X}$ the collection of twice-continuously differentiable functions $f = f(x)$ on $[0,L]$ such that $f(0) = f(L) = 0$. For $f \in \mathbb{X}$, define the `operator` $\mathcal{A}$ by

$$\mathcal{A}[f](x) = -f''(x).$$

Also, for $f,g \in \mathbb{X}$, define the `inner product`

$$(f,g) = \int_0^L f(x)\overline{g(x)}\,dx; \quad \|f\|^2 = (f,f);$$

we allow the possibility that the functions f,g can take complex values. Note that

$$(g,f) = \overline{(f,g)} \tag{22}$$

Then the operator $\mathcal{A}$ is

(1) `non-negative definite`: for $f \in \mathbb{X}$, we integrate by parts

$$(\mathcal{A}[f],f) = -\int_0^L f''(x)\overline{f(x)}\,dx = -\int_0^L \bar{f(x)}\,df'(x) - f'(x)\overline{f(x)}\Big|_{x=0}^{x=L} + \int_0^L f'(x)\overline{f'(x)}\,dx = \|f'\|^2 \geq 0;$$

remember that $f(0) = f(L) = 0$.

(2) `symmetric`: for $f,g \in \mathbb{X}$, we integrate by parts twice

$$(\mathcal{A}[f],g) = -\int_0^L f''(x)\bar{g(x)}\,dx = -\int_0^L \overline{g(x)}\,df'(x) - f'(x)\overline{g(x)}\Big|_{x=0}^{x=L} + \int_0^L f'(x)\overline{g'(x)}\,dx$$
$$= f(x)\overline{g'(x)}\Big|_{x=0}^{x=L} - \int_0^L f(x)\overline{g''(x)}\,dx = (f,\mathcal{A}[g]);$$

once again, we use that $f(0) = f(L) = g(0) = g(L) = 0$.

Next, we say that $G \in \mathbb{X}$ is an `eigenfunction` of $\mathcal{A}$ if $\|G\| > 0$ and there exists a number $\mu \in \mathbb{C}$ such that

$$\mathcal{A}[G](x) = \mu G(x),\ x \in (0,L).$$

The following result could be familiar from linear algebra: a symmetric matrix has real eigenvalues, and the eigenvectors corresponding to different eigenvalues are orthogonal; the eigenvalues of a symmetric non-negative definite matrix are non-negative. Turns out, this is not just about matrices.

Proposition If $\mathcal{A}$ is symmetric, that is,

$$(\mathcal{A}[f],g) = (f,\mathcal{A}[g]), \tag{23}$$

then $\mu \in \mathbb{R}$ and, if $\mathcal{A}[G_1] = \mu_1 G_1,\ \mathcal{A}[G_2] = \mu_2 G_2$ with $\mu_1 \neq \mu_2$, then

$$(G_1,G_2) = 0.$$

If, in addition $\mathcal{A}$ is non-negative definite, that is

$$(\mathcal{A}[f],f) \geq 0,\ f \in \mathbb{X}, \tag{24}$$

then $\mu \geq 0$.

Proof. Taking $f = g = G$ in (23) and using (22),

$$\mu\|G\|^2 = (\mathcal{A}[G], G) = (G, \mathcal{A}[G]) = (G, \mu G) = \overline{\mu}\|G\|^2.$$

Because $\|G\| > 0$ by the definition of the eigenfunction, we get $\mu = \overline{\mu}$, that is, $\mu \in \mathbb{R}$.

Next, take $f = G_1, g = G_2$ in (23). Then

$$\mu_1(G_1, G_2) = \mu_2(G_1, G_2).$$

Because $\mu_1 \neq \mu_2$, we conclude that $(G_1, G_2) = 0$.

Finally, take $f = G$ in (24). Then $\mu\|G\|^2 \geq 0$, that is, $\mu \geq 0$. □

Under some additional technical assumptions, which hold in all examples we will ever encounter, the collection of the eigenfunctions G_k is complete, in the sense that all "reasonable" functions $f = f(x)$ can be written as

$$f(x) = \sum_k f_k G_k(x) \tag{25}$$

for some numbers $f_k \in \mathbb{C}$. The following table summarizes how all the expansions we know [Fourier sine, Fourier cosine, and usual Fourier] are particular cases of (25). In fact, the only thing that changes is the boundary conditions.

Operator	Eigenfunctions G_k	Expansion
$\mathcal{A}[f](x) = -f''(x),\ f(0) = f(L) = 0$	$\sin(\pi kx/L),\ k \in \mathbb{N}$	Fourier sine
$\mathcal{A}[f](x) = -f''(x),\ f'(0) = f'(L) = 0$	$\cos(\pi kx/L),\ k \in \mathbb{Z}_+$	Fourier cosine
$\mathcal{A}[f](x) = -f''(x),\ f(0) = f(L),\ f'(0) = f'(L)$	$\exp(\mathrm{i}\pi kx/L),\ k \in \mathbb{Z}$	Fourier

One other example that is doable in closed form is

$$\mathcal{A}[f](x) = -f''(x),\ f(0) = 0,\ f'(L) = 0.$$

Indeed, direct computations show that $\mathcal{A}$ is symmetric and positive definite. Then a quick repetition of the computations from the analysis of the heat equation leads to the relations

$$G_k''(x) + \lambda^2 G_k(x) = 0,\ G_k(0) = G_k'(L) = 0,$$

which imply

$$G_k(x) = \sin(\lambda_k x),$$

and $\cos(\lambda_k L) = 0$, that is

$$\lambda_k = \frac{\pi}{2L} + \frac{\pi}{L}k,\ k \in \mathbb{Z}_+. \tag{26}$$

As a quick concept check, verify that the eigenfunctions of

$$\mathcal{A}[f](x) = -f''(x),\ f'(0) = f(L) = 0$$

[now the derivative is zero at the left point] are

$$G_k(x) = \cos(\lambda_k x)$$

with the same λ_k as in (26). Note that neither of the resulting expansions (25) are truly Fourier, but might still be useful, for example, to understand the Brownian motion [a random process] or the clarinet [a musical instrument].