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The Brunn-Minkowski Inequality in Gauss Space

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1. Summary

In [11] H. P. McKean explains why it is a fruitful idea to think of Wiener measure as the uniform distribution on an infinite-dimensional spherical surface $S^\infty(\infty^{\frac{1}{2}})$ of radius $\infty^{\frac{1}{2}}$. This interpretation of Wiener measure and, more generally, of an arbitrary Gauss measure will lead us to an inequality of the Brunn-Minkowski type. The inequality so obtained seems, for many reasons, to be a better one than that obtained in [4]. We shall give two applications of the Brunn-Minkowski inequality proved in Section 3. In Section 4 we give upper and lower bounds for hitting probabilities of Brownian motion. This has applications to the heat equation [17] but we will not go into this here. Finally, in Section 5, we will, under some appropriate conditions, compute

$$\lim_{t\to\infty} t^{-2} \log \mu(\varphi \geqq t),$$

where μ is a Gauss measure and φ is a measurable sublinear function, which is finite a.s. $[\mu]$. This extends work of Marcus and Shepp [9].

2. Notation

Let E be a *real, locally convex Hausdorff vector space* (l.c.s.) and μ a Borel probability measure on E. We define

$$\mu_*(A) = \sup\{\mu(K): K \text{ compact} \subset A\},$$

where A is an arbitrary subset of E. A Borel probability measure μ on E is said to be a Radon probability measure if $\mu_*(A) = \mu(A)$ for every Borel subset A of E. A Radon probability measure μ on E is, by definition, a Gauss measure if the image measure $\xi(\mu)$ is a Gauss measure on $\mathbb{R}$ for every ξ belonging to the topological dual E' of E. We shall say that a pair (E, μ) is a Gauss space if (i) E is a l.c.s. (ii) μ is a Gauss measure on E and (iii) the map

$$E' \ni \xi \to \int \xi^2 \, d\mu \in \mathbb{R}$$

is continuous when E' is equipped with the Mackey topology $\tau(E', E)$.[1] Assuming that (E, μ) is a Gauss space we deduce that μ has a barycentre m. We set $\mu_0(\cdot) = \mu(\cdot + m)$ and observe that (E, μ_0) is a Gauss space. Let $\mathscr{H}$ be the closure of E' in $L^2(\mu_0)$. Then for every $h \in \mathscr{H}$ there exists a unique $a \in E$ such that

$$\langle a, \xi \rangle = \int h \cdot \xi \, d\mu_0, \quad \xi \in E'. \tag{1.1}$$

We thus get a linear mapping $\Lambda: \mathscr{H} \to E$ by setting $\Lambda h = a$ when h and a are related as in (1.1). Note that Λ is injective. We define $\mathscr{K} = \text{range}(\Lambda)$ and

$$\langle a, b \rangle_{\mathscr{K}} = \int (\Lambda^{-1} a) \cdot (\Lambda^{-1} b) \, d\mu_0, \quad a, b \in \mathscr{K}.$$

The vector space $\mathscr{K}$, equipped with this Hilbert norm, is called the reproducing kernel Hilbert space of (E, μ). In the following we shall say that a triplet $(E, \mu; \mathscr{K})$ is a Gauss space thereby meaning that (E, μ) is a Gauss space having reproducing kernel Hilbert space $\mathscr{K}$. Let θ be the identity mapping of $\mathscr{K}$ into E. Note that $\xi \circ \theta$ is a continuous linear form on $\mathscr{K}$ for every $\xi \in E'$. Furthermore, let γ be the canonical cylinder Gauss measure on $\mathscr{K}$. A computation shows that $\gamma(\theta^{-1}(A)) = \mu_0(A)$ for every finite-dimensional cylinder set A in E.

The canonical Gauss measure in d-space is denoted by γ_d. We set $\Phi(\alpha) = \gamma_1(]-\infty, \alpha[)$, $-\infty \leqq \alpha \leqq \infty$.

If A and B denote subsets of a vector space, we write $A + B = \{z : z = x + y$ for some $x \in A$ and $y \in B\}$ and $A \ominus B = \{x : x \in A$ and $\{x\} + B \subset A\}$.

The definitions introduced in this section are very close to those given in [1] and [2].

3. The Main Result

We can now formulate the following Brunn-Minkowski inequality in Gauss space.

Theorem 3.1. *Let $(E, \mu; \mathscr{K})$ be a Gauss space and denote by $O_{\mathscr{K}}$ the closed unit ball in $\mathscr{K}$. Furthermore, let A be a μ-measurable subset of E and choose $\alpha \in [-\infty, +\infty]$ such that*

$$\mu(A) = \Phi(\alpha).$$

Then

$$\mu_*(A + \varepsilon O_{\mathscr{K}}) \geqq \Phi(\alpha + \varepsilon) \tag{3.1}$$

and

$$\mu_*(A \ominus \varepsilon O_{\mathscr{K}}) \leqq \Phi(\alpha - \varepsilon) \tag{3.2}$$

for every $\varepsilon > 0$.

Equality occurs in (3.1) *and* (3.2) *if A is a half-space.*

Note that the set $O_{\mathscr{K}}$ is of μ-measure zero if $\dim \mathscr{K} = +\infty$. (See e.g. [2, Cor. (IX, 1; 2)] or [6, Th. 2.4].)

The proof of Theorem 3.1 leans heavily on an observation of Poincaré [14], which has proved to be very useful [11]. In fact, denote by σ_{n-1} the uniform

[1] It can be proved that (i) and (ii) imply (iii).

distribution on the $(n-1)$-dimensional spherical surface $S^{n-1}(n^{\frac{1}{2}})$ of radius $n^{\frac{1}{2}}$. If M is a Lebesgue measurable subset of $\mathbb{R}^d$, and $n \geqq d$, we set

$$C_{d,n}(M) = \{x \in S^{n-1}(n^{\frac{1}{2}}): \operatorname{proj}_{\mathbb{R}^d}(x) \in M\}.$$

Poincaré [14] then noticed that

$$\lim_{n\to\infty} \sigma_{n-1}(C_{d,n}(M)) = \gamma_d(M). \tag{3.3}$$

Proof of Theorem 3.1. It, clearly, suffices to prove (3.1) since $(A \ominus \varepsilon O_{\mathscr{K}}) + \varepsilon O_{\mathscr{K}} \subset A$. The proof of the inequality (3.1) will now be divided into two steps.

Step 1. *$E = \mathbb{R}^d$ and $\mu = \gamma_d$.*

Proof of Step 1. It can be assumed that $-\infty < \alpha \leqq +\infty$, and that A is a Borel set in $\mathbb{R}^d$. We choose $\beta \in]-\infty, \alpha[$ arbitrarily but fixed, and set $B =]-\infty, \beta]$. From (3.3) we thus have

$$\sigma_{n-1}(C_{d,n}(A)) > \sigma_{n-1}(C_{1,n}(B))$$

for all n large enough.

If M is a subset of $\mathbb{R}^d$ we denote by $C^{\varepsilon}_{d,n}(M)$ the set of all points of $S^{n-1}(n^{\frac{1}{2}})$ having a spherical distance at most equal to ε to $C_{d,n}(M)$. Let O_d be the closed unit ball in $\mathbb{R}^d$. It is obvious that

$$C_{d,n}(A + \varepsilon O_d) \supset C^{\varepsilon}_{d,n}(A).$$

Using the Brunn-Minkowski inequality on $S^{n-1}(n^{\frac{1}{2}})$ [15], we thus have

$$\sigma_{n-1}(C_{n,d}(A + \varepsilon O_d)) > \sigma_{n-1}(C^{\varepsilon}_{1,n}(B))$$

for all n large enough. By letting n tend to infinity and observing that $S^{n-1}(n^{\frac{1}{2}})$ becomes very flat locally for large n, we get from (3.3) that

$$\gamma_d(A + \varepsilon O_d) \geqq \Phi(\beta + \varepsilon).$$

Since $\beta < \alpha$ is arbitrary, we have the inequality (3.1) from the fact that $O_d = O_{\mathscr{K}}$.

Step 2. *The general case.*

Proof of Step 2. There is no loss of generality to assume that the barycentre of μ equals 0. It can also be assumed that $-\infty < \alpha \leqq +\infty$. Let $\beta \in]-\infty, \alpha[$ be arbitrary but fixed. Since μ is a Radon probability measure there exists a compact subset K of A such that $\mu(K) > \Phi(\beta)$. To prove (3.1) it is enough to establish the inequality

$$\mu(K + \varepsilon O_{\mathscr{K}}) \geqq \Phi(\beta + \varepsilon). \tag{3.4}$$

To this end we denote by $\mathscr{F}(K)$ the family of all weakly closed finite-dimensional cylinder subsets of E containing K. More explicitly this means that $F \in \mathscr{F}(K)$ if and only if there exist a positive integer n, $\xi_1, \ldots, \xi_n \in E'$, and a closed subset M of $\mathbb{R}^n$ such that

$$F = \{x \in E: (\xi_1(x), \ldots, \xi_n(x)) \in M\} \tag{3.5}$$

and $F \supset K$. The Hahn-Banach separation theorem easily gives that

$$\bigcap [F: F \in \mathscr{F}(K)] = K.$$

(Compare [4].)

We will now make use of the representation of μ explained in Section 2. It is obvious that

$$\gamma(\theta^{-1}(F)) > \Phi(\beta)$$

for every $F \in \mathscr{F}(K)$. Suppose $F \in \mathscr{F}(K)$ is defined by (3.5). We have already remarked that $\xi_j \circ \theta$, $j = 1, \ldots, n$ are bounded linear forms on $\mathscr{H}$. We denote by L the linear submanifold of $\mathscr{H}$ spanned by these elements. Step 1 then yields

$$\gamma(\theta^{-1}(F) + \varepsilon(O_{\mathscr{H}} \cap L)) > \Phi(\beta + \varepsilon).$$

Hence

$$\mu(F + \varepsilon O_{\mathscr{H}}) > \Phi(\beta + \varepsilon)$$

for every $F \in \mathscr{F}(K)$.

We now claim that $O_{\mathscr{H}}$ is a weakly compact subset of E. To this end it is enough to show that the mapping $\theta: \mathscr{H}_\sigma \to E_\sigma$ is continuous. This follows at once from the identity

$$\langle a, \xi \rangle = \langle a, \Lambda \xi \rangle_{\mathscr{H}}, \quad a \in \mathscr{H}, \quad \xi \in E'.$$

Since $O_{\mathscr{H}}$ is a weakly compact subset of E it is readily seen that

$$\bigcap [F + \varepsilon O_{\mathscr{H}}: F \in \mathscr{F}(K)] \subset K + \varepsilon O_{\mathscr{H}}.$$

Furthermore, since μ is a Radon probability measure, we have

$$\mu(\bigcap [F + \varepsilon O_{\mathscr{H}}: F \in \mathscr{F}(K)]) = \inf_{F \in \mathscr{F}(K)} \mu(F + \varepsilon O_{\mathscr{H}}).$$

Summing up, we obtain (3.4). The case when equality occurs in (3.1) and (3.2) follows by direct computation. This proves Theorem 3.1.

The rest of this paper is devoted to various applications of Theorem 3.1.

4. Hitting Probabilities for Brownian Motion

Let $(B_t)_{t \geq 0}$ be the standard d-dimensional Brownian motion process starting at a fixed point $z \in \mathbb{R}^d$ [7], and let M be a Borel subset of $\mathbb{R}^d$. We denote by T_M the first hitting time of M, that is

$$T_M = \inf\{t > 0: B_t \in M\}.$$

Note that T_M is an $\bar{\mathbb{R}}_+$-valued random variable [3, p. 54]. We define

$$M_\varepsilon = \{z \in \mathbb{R}^d: \text{dist.}(z, M) < \varepsilon\}$$

and

$$M_{-\varepsilon} = \{z \in \mathbb{R}^d: \text{dist.}(z, \sim M) \geqq \varepsilon\}$$

for $\varepsilon > 0$.

With these conventions we shall prove

Theorem 4.1. *Suppose τ is a given positive real number and choose $\alpha \in [-\infty, +\infty]$ such that*

$$P_z(T_M < \tau) = \Phi(\alpha).$$

Then

$$P_z(T_{M_\varepsilon} < \tau) \geqq \Phi(\alpha + \varepsilon \cdot \tau^{-\frac{1}{2}})$$

and

$$P_z(T_{M_{-\varepsilon}} < \tau) \leqq \Phi(\alpha - \varepsilon \cdot \tau^{-\frac{1}{2}})$$

for every $\varepsilon > 0$.

Proof. The second inequality follows immediately from the first one since $(M_{-\varepsilon})_\varepsilon \subset M$.

A convenient model for the d-dimensional Brownian motion process starting at the origin is the stochastic process $(W, (C(\mathbb{R}_+))^d, (x_t)_{t \geqq 0})$, where W is Wiener measure on $(C(\mathbb{R}_+))^d$ and x_t is the coordinate map $x \to x(t)$ of $(C(\mathbb{R}_+))^d$ into $\mathbb{R}^d$. It is well known and easy to prove that an element $a \in (C(\mathbb{R}_+))^d$ belongs to the reproducing kernel Hilbert space of the Gauss space $((C(\mathbb{R}_+))^d, W)$ if and only if there exists an element $b \in (L^2(\mathbb{R}_+))^d$ such that

$$a(t) = \int_0^t b(s)\, ds, \qquad t \geqq 0. \tag{4.1}$$

We also have

$$\|a\|_{\mathscr{H}}^2 = \int_0^\infty |b(s)|^2\, ds.$$

Theorem 3.1 thus gives

$$P_z(\{T_M < \tau\} + \varepsilon \cdot \tau^{-\frac{1}{2}} O_{\mathscr{H}}) \geqq \Phi(\alpha + \varepsilon \cdot \tau^{-\frac{1}{2}})$$

for every $\varepsilon > 0$. We now claim that

$$\{T_M < \tau\} + \varepsilon \cdot \tau^{-\frac{1}{2}} O_{\mathscr{H}} \subset \{T_{M_\varepsilon} < \tau\}. \tag{4.2}$$

In fact, assume that y belongs to the left-hand side and let $y = x + \varepsilon \cdot \tau^{-\frac{1}{2}} a$, where $T_M(x) < \tau$ and $a \in O_{\mathscr{H}}$. Then $x(t_0) \in M$ for some $t_0 \in]0, \tau[$ and

$$|a(t_0)| \leqq t_0^{\frac{1}{2}} \cdot \left(\int_0^{t_0} |b(s)|^2\, ds\right)^{\frac{1}{2}},$$

if a and b are related as in (4.1). Hence $y(t_0) \in M_\varepsilon$, which proves (4.2) and the theorem.

5. Tail Probabilities for Sublinear Functions

Let $(X_k)_{k \in \mathbb{N}}$ be a real-valued Gaussian stochastic process and assume that $P(\sup |X_k| < +\infty) = 1$. Under these assumptions Marcus and Shepp [9, Th. 2.5] prove that

$$\lim_{t\to\infty} t^{-2} \log P(\sup |X_k| \geqq t) = -(2 \sup \operatorname{Var}(X_k))^{-1}.$$

In this section our main result is Theorem 5.2, which extends the theorem of Marcus and Shepp. Before formulating our result we must give some definitions and preliminary results.

Let E be an l.c.s. A function $\varphi: E \to]-\infty, +\infty]$ is said to be an ${}^0\bar{\mathbb{R}}$-valued sublinear function if

$$\varphi(x+y) \leqq \varphi(x) + \varphi(y), \qquad x, y \in E,$$

and

$$\varphi(\lambda x) = \lambda \varphi(x), \qquad \lambda \geqq 0, \qquad x \in E.$$

Here $0 \cdot +\infty = 0$. If, in addition, the function φ is symmetric, we shall say that φ is an $\bar{\mathbb{R}}_+$-valued seminorm.

Theorem 5.1. *Let $(E, \mu; \mathscr{H})$ be a Gauss space and denote by m the barycentre of μ. Furthermore, assume that E is a Souslin space and let φ be an ${}^0\bar{\mathbb{R}}$-valued universally Borel measurable sublinear function on E such that $\mu(\varphi < +\infty) > \frac{1}{2}$.*

Then

a) *if, in addition,*

$$\varphi(-m) < +\infty, \tag{5.1}$$

the restriction of φ to $\mathscr{H}$ is finite-valued and

$$\|\varphi\|_{\mathscr{H}} = \sup_{a \in O_{\mathscr{H}}} \varphi(a) < +\infty.$$

b) *if, in addition, φ is an $\bar{\mathbb{R}}_+$-valued seminorm the condition* (5.1) *is automatically fulfilled.*

We recall that a topological space is a Souslin space if it is a continuous image of a separable, metric, and complete space [1].

Easy examples show that the conclusion of part a), in general, is wrong without the assumption (5.1).

The proof of Theorem 5.1 is based on two lemmas.

Lemma 5.1. *Let E be a Banach space and φ a real-valued universally Borel measurable sublinear function on E.*

Then φ is continuous.

Under the stronger assumption that φ is Borel measurable the conclusion of Lemma 5.1 is an immediate consequence of [10, Th. 3] and Baire's theorem. It can be of some interest to point out that our proof is independent of Baire's theorem. For example Lemma 5.1, obviously, contains (a weak form of) the Banach-Steinhaus theorem. Our method of proof is to some extent similar to the proof of Douady's lemma given by L. Schwartz [16]. (See also Note Added in Proof at the very end of this paper.)

Proof of Lemma 5.1. We first assume that φ is a seminorm. For short, we shall write $\mathbb{R}^{\mathbb{N}} = \mathbb{R}^\infty$, $[-1, 1]^{\mathbb{N}} = I^\infty$, and $\ell^\infty(\mathbb{N}) = \ell^\infty$, respectively. Let $\sigma_k > 0$,

$k \in N$, and define a linear mapping $u\colon \mathbb{R}^\infty \to \mathbb{R}^\infty$ by setting $u(t)=(t_k\sigma_k)$, $t=(t_k)\in\mathbb{R}^\infty$. We denote by ν the infinite product measure $\Pi_{\mathbb{N}}\gamma_1$, and set $\mu=u(\nu)$. It is, clearly, possible to choose the σ_k such that $\mu(\ell^\infty)=1$. Now let $e_k\in E$, $k\in\mathbb{N}$, be an arbitrary sequence in E such that $\Sigma\|e_k\|<+\infty$, where $\|\cdot\|$ denotes the norm in E. It is enough to show that

$$\sup \sigma_k \varphi(e_k) < +\infty. \tag{5.2}$$

We now define $\psi(t)=\varphi(\Sigma t_k e_k)$, $t=(t_k)\in l^\infty$, and then extend ψ to ℓ^∞ so that ψ becomes homogeneous of degree one. Finally, set $\psi=+\infty$ on $\mathbb{R}^\infty\setminus\ell^\infty$. Then ψ is a universally Borel measurable $\overline{\mathbb{R}}_+$-valued seminorm on $\mathbb{R}^\infty$ and $\psi<+\infty$ a.s. $[\mu]$. Since μ is a Gauss measure, we have that $\psi\in L^1(\mu)$. (See e.g. [8], [12], or [4].) Now set $t^{(k)}=(0,\dots,0,t_k,0,\dots)$ when $(t_k)\in\mathbb{R}^\infty$ and t_k is placed in the $(k+1)$-th coordinate. We have

$$\psi(t^{(k)})\leqq\tfrac{1}{2}\psi(t)+\tfrac{1}{2}\psi(2t^{(k)}-t), \qquad t\in\mathbb{R}^\infty. \tag{5.3}$$

Note also that

$$\int\psi(2t^{(k)}-t)\,d\mu(t)=\int\psi(t)\,d\mu(t)=C<+\infty$$

and

$$\int\psi(t^{(k)})\,d\mu(t)=(2/\pi)^{\frac{1}{2}}\cdot\sigma_k\varphi(e_k).$$

An integration of the inequality (5.3) with respect to μ thus yields (5.2).

To prove the general case set $\varphi_0(x)=\max(0,\varphi(x))+\max(0,\varphi(-x))$, $x\in E$. Then φ_0 is a real-valued universally Borel measurable seminorm and the case already proved shows that φ_0 is continuous. Therefore φ must be bounded from above in a neighbourhood of the origin, which is equivalent to continuity. This proves Lemma 5.1.

To prove Theorem 5.1 we also need

Lemma 5.2. *Let $(E,\mu;\mathscr{H})$ be a Gauss space and assume that E is a Souslin space. Then*

$$\mathscr{H}\subset\mathbb{R}_+\cdot(A-A)$$

for every μ-measurable subset A of E such that $\mu(A)>0$.

Lemma 5.2 is proved by LePage [13] under slightly different assumptions. See also [2, Lemma (1 X, 1; 5)]. The proof will not be repeated here.

Proof of Theorem 5.1. We shall first prove part a). Let $C=\{\varphi<+\infty\}$. The set C is a convex cone of μ-measure $p>\frac{1}{2}$. In particular, $\mu_0(-m+C)=p$, which yields $\mu_0(m-C)=p$. The set $(-m+C)\cap(m-C)$ is thus of positive μ_0-measure and Lemma 5.2 implies that

$$\mathscr{H}\subset\mathbb{R}_+\cdot(-m+C)\cap(m-C).$$

Using (5.1) we now easily get that φ is finite-valued on $\mathscr{H}$.

We shall now prove that the map $\varphi|_{\mathscr{H}}=\varphi\circ\theta$ of $(\mathscr{H},\|\ \|_{\mathscr{H}})$ into $\mathbb{R}$ is continuous. To this end we first observe that the σ-algebra generated by the weakly open subsets of E is identical with the Borel σ-algebra $\mathfrak{B}(E)$ in E since E is a Souslin space

[1]. Furthermore, we know that the map $\theta: \mathscr{H}_\sigma \to E_\sigma$ is continuous. The function $\varphi \circ \theta$ is thus universally weakly Borel measurable and therefore also universally strongly Borel measurable on $\mathscr{H}$. Lemma 5.1 now proves part a).

We shall now prove part b). Suppose to the contrary that $\varphi(-m) = +\infty$, and set $A = \{x \in E: \varphi(-m+x) = +\infty\}$. Then $A \cup (-A) = E$. Hence $\mu_0(-A) \geqq \frac{1}{2}$ or, equivalently, $\mu(m-A) \geqq \frac{1}{2}$. But since φ is symmetric, we then have that $\varphi = +\infty$ on $m-A$, a set of μ-measure $\geqq \frac{1}{2}$. This contradiction proves part b) and concludes the proof of Theorem 5.1.

We can now formulate the main result of this section.

Theorem 5.2. *Let $(E, \mu; \mathscr{H})$ be a Gauss space and denote by m the barycentre of μ. Furthermore, assume that E is a Souslin space and let φ be an ${}^0\overline{\mathbb{R}}$-valued universally Borel measurable sublinear function on E such that $\mu(\varphi < +\infty) > \frac{1}{2}$ and $\varphi(-m) < +\infty$. Then*

$$\lim_{t \to \infty} t^{-2} \log \mu(\varphi \geqq t) = -(2\|\varphi\|_{\mathscr{H}}^2)^{-1}.$$

In particular, $\mu(\varphi < +\infty) = 1$.

Proof. Choose $\delta > 0$ arbitrarily. Then there exists a $t_\delta > 0$ such that

$$-(1+\delta) \cdot t^2/2 \leqq \log(1 - \Phi(t)) \leqq -(1-\delta) \cdot t^2/2 \tag{5.4}$$

for all $t \geqq t_\delta$.

Let us first assume that $\|\varphi\|_{\mathscr{H}} > 0$. By assumption, there exists a $t_0 < +\infty$ such that $\mu(\varphi < t_0) > 0$. Now choose $\alpha \in \mathbb{R}$ so that

$$\mu(\varphi < t_0) \geqq \Phi(\alpha).$$

Let $\varepsilon > 0$ be given and note that

$$\{\varphi < t_0\} + \varepsilon O_{\mathscr{H}} \subset \{\varphi < t_0 + \varepsilon \|\varphi\|_{\mathscr{H}}\}$$

Theorem 3.1 thus implies that

$$\mu(\varphi < t_0 + \varepsilon \|\varphi\|_{\mathscr{H}}) \geqq \Phi(\alpha + \varepsilon).$$

Hence

$$\mu(\varphi \geqq t) \leqq 1 - \Phi(\alpha + \|\varphi\|_{\mathscr{H}}^{-1} \cdot (t - t_0))$$

for every $t \geqq t_0$. The inequality (5.4) now tells us that

$$\overline{\lim_{t \to \infty}}\, t^{-2} \log \mu(\varphi \geqq t) \leqq -(2\|\varphi\|_{\mathscr{H}}^2)^{-1}.$$

In particular, this shows that $\mu(\varphi < +\infty) = 1$.

Now let $1 > \delta > 0$ be as in (5.4) and choose $a \in O_{\mathscr{H}}$, $\|a\|_{\mathscr{H}} = 1$, such that

$$\varphi(a) > (1-\delta)\|\varphi\|_{\mathscr{H}}.$$

To prove Theorem 5.2, under the additional assumption that $\|\varphi\|_{\mathscr{H}}>0$, it is enough to establish the inequality

$$\lim_{t\to+\infty} t^{-2}\log\mu(\varphi\geqq t)\geqq -(1+\delta)\cdot(2(1-\delta)^2\|\varphi\|_{\mathscr{H}}^2)^{-1}. \tag{5.5}$$

To this end it can be assumed that $\mu=\mu_0$. In fact, since

$$\{\varphi\geqq t\}\supseteq m+\{\varphi\geqq t+\varphi(-m)\},$$

we have that $\mu(\varphi\geqq t)\geqq\mu_0(\varphi\geqq t+\varphi(-m))$ and the assertion is obvious.

From now on it will thus be assumed that $\mu=\mu_0$ and we shall prove (5.5). Therefore define E-valued Gaussian random variables U and V, respectively, by setting $U(x)=a(\Lambda^{-1}a)(x)$, $V(x)=x-U(x)$, $x\in E$. It follows at once that $\xi\circ U$ and $\xi\circ V$ are orthogonal in $L^2(\mu)$ for every $\xi\in E'$. Since E is a Souslin space, we deduce that U and V are independent based over the probability space $(E,\mathscr{B}(E),\mu)$. By first using Fubini's theorem and then Jensen's inequality, we have

$$\log\mu(\varphi\geqq t)\geqq\int\log\mu(U\in -v+\{\varphi\geqq t\})\,V(\mu)(dv). \tag{5.6}$$

Furthermore, since $\varphi<+\infty$ a.s. $[\mu]$, it is easy to show that the set

$$M=\{v\in E\,|\,\varphi(-v)<+\infty\}$$

is of $V(\mu)$-measure one. If $v\in M$, we have that $\{U\in -v+\{\varphi\geqq t\}\}\supseteq\{\varphi(U)\geqq t+\varphi(-v)\}\cap\{\Lambda^{-1}a\geqq t_\delta\}$, where t_δ is as in (5.4). Hence

$$\{U\in -v+\{\varphi\geqq t\}\}\supseteq\{\Lambda^{-1}a\in[\max((t+\varphi(-v))/\varphi(a),t_\delta),+\infty[\,\},$$

which yields

$$t^{-2}\log\mu\{U\in -v+\{\varphi\geqq t\}\}\geqq -\tfrac{1}{2}(1+\delta)\cdot\left[\max\left(\frac{1}{\varphi(a)}+\frac{\varphi(-v)}{t\,\varphi(a)},\frac{t_\delta}{t}\right)\right]^2,\quad t\geqq t_\delta,$$

for all $v\in M$. Since M is of $V(\mu)$-measure one, we know that

$$\int\varphi^2(-v)\,V(\mu)(dv)<+\infty.$$

By dominated convergence, we thus get (5.5) from (5.6) and the definition of a.

One the other hand, if $\|\varphi\|_{\mathscr{H}}=0$, the first part of the proof shows that $\mu(\varphi\leqq t_0)=1$ for a suitable t. The result is thus obvious in this case. This proves Theorem 5.2.

The following theorem is mainly included as an example of an application of Theorem 5.2. (Compare [9, Th. 2.6].)

Theorem 5.3. *Let $(B_t)_{t\geqq 0}$ be the d-dimensional Brownian motion process starting at the origin, and let λ be a σ-finite positive Borel measure on $\mathbb{R}_+$.*

Set

$$X=\int|B_t|\,d\lambda(t).$$

Then $P(X<+\infty)=1$ if and only if

$$\int t^{\frac{1}{2}}\,d\lambda(t)<+\infty.$$

Furthermore, if this condition is fulfilled it holds

$$\lim_{t\to\infty} t^{-2}\log P(X\geqq t)=-\tfrac{1}{2}\left(\int\lambda^2([s,+\infty[)\,ds\right)^{-2}.$$

Proof. We choose the same representation of $(B_t)_{t\geqq 0}$ as in Section 4. The first part of the theorem follows easily from [4, Th. 4.2] or [5, Th. 6.1]. Using Theorem

5.2 and suitable parts of the proof of Theorem 4.1, we only have to show that

$$\sup \int \left| \int_0^t b(s)\, ds \right| d\lambda(t) = \int_0^\infty \lambda^2([s, +\infty[)\, ds,$$

where the sumpremum is taken over all $b \in (L^2([0, +\infty[))^d$ such that

$$\int_0^\infty |b(s)|^2\, ds \leqq 1.$$

This follows at once from the Cauchy-Schwarz inequality.

Note Added in Proof. The author is grateful to the referee for pointing out a completely different proof of Lemma 5.1. It can be assumed that E is a separable Banach space. In fact, assume that φ is an $\mathbb{R}$-valued universally Borel measurable seminorm on E and set $E_n = \{\varphi \leqq n\}$. Since $U E_n = E$, all the sets E_n cannot be a zero Haar set. Therefore, the set $E_n - E_n$ contains a neighbourhood of the origin for a suitable n. (See Theorem 7.3 in J. P. R. Christensen, Topology and Borel Structure, Mathematics Studies 10, North-Holland 1974). The seminorm φ is thus bounded in a neighbourhood of the origin.

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