# Two computations

1. $\boxed{\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}}$

Step 1 $\quad \sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$

Step 2 $\quad \frac{\sin x}{x} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k}$

$$= 1 - \frac{x^2}{6} + \frac{x^4}{120} - \dots \qquad (A)$$

Step 3 – the key; originally noticed by Euler

$$\frac{\sin x}{x} = 0 \text{ when } x = \pi n,\ n = \pm 1, \pm 2, \dots$$

So it is reasonable to write

$$\frac{\sin x}{x} = \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{\pi^2 n^2}\right)$$

$$= \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right) \cdots \qquad (B)$$

[Similar to factoring polynomials].

Step 4 Now compare the coefficients of $x^2$ in (A) and (B):

$$-\frac{x^2}{6} = -\frac{x^2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

The result follows.

Step 5 To practice: in the same way, confirm that $\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}$

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2. $\boxed{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}} dx = 1.}$

Step 1 $\quad I = \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}} dx$

Step 2 $\quad I^2 = \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}} dx \int_{-\infty}^{+\infty} e^{-\frac{y^2}{2}} dy$

$$= \iint_{\mathbb{R}^2} e^{-\frac{x^2+y^2}{2}} dx\,dy.$$

Step 3 polar coordinates in $\mathbb{R}^2$

$(r, \theta)$: $\quad x^2 + y^2 = r^2; \quad dx\,dy = r\,dr\,d\theta$

So that $\quad I^2 = \int_0^{2\pi} \left(\int_0^{\infty} e^{-r^2/2} \, r\,dr\right) d\theta \qquad u = r^2/2$

$$= \underbrace{\left(\int_0^{2\pi} d\theta\right)}_{= 2\pi} \underbrace{\left(\int_0^{\infty} e^{-u} du\right)}_{= 1} = 2\pi$$

or $I^2 = 2\pi$, so that $\boxed{I = \sqrt{2\pi}}$

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