Summary[1] of Asymptotic Integration.

Laplace's Method.

Let

$$F(n) = \int_a^b f(x) e^{nS(x)}\, dx,$$

where f and S are sufficiently smooth functions defined on $\mathbb{R}$, and $-\infty < a < b < +\infty$. Assume that there is a unique point $x^* \in [a,b]$ such that

$$\max_{x\in[a,b]} S(x) = S(x^*).$$

Then

$$F(n) = e^{nS(x^*)} \int_a^b f(x) e^{n\big(S(x)-S(x^*)\big)}\, dx,$$

and, because, for large n, $e^{n\big(S(x)-S(x^*)\big)}$ is very small when x is away from x^*, the asymptotic of $F(n)$, $n\to\infty$, depends on the behavior of $S(x)$ near x^*. More precisely, we approximate $S(x)-S(x^*)$ by the first non-trivial Taylor polynomial and then integrate the result to get the main term in the asymptotic expansion of F.

For example, assume that $x^* = a$ and $S'(a) < 0$. Then, using the notation $r = -S'(a)$,

(1.1)
$$S(x) - S(a) \approx -r(x-a), \quad \int_a^b f(x) e^{n\big(S(x)-S(x^*)\big)} dx \approx f(a)\int_a^b e^{-nr(x-a)}dx$$
$$\approx f(a)\int_0^{+\infty} e^{-nrx}dx = \frac{f(a)}{rn}.$$

Similarly, assume that $x^* = a$ and $S'(a) = 0,\ S''(a) < 0$. Then, using the notation $r = -S''(a)$,

(1.2)
$$S(x) - S(a) \approx -\frac{r(x-a)^2}{2}, \quad \int_a^b f(x) e^{n\big(S(x)-S(x^*)\big)} dx \approx f(a)\int_a^b e^{-nr(x-a)^2/2}dx$$
$$\approx f(a)\int_0^{+\infty} e^{nrx^2/2}dx = f(a)\sqrt{\frac{\pi}{2rn}};$$

recall that

(1.3)
$$\int_{-\infty}^{+\infty} e^{-cx^2}dx = \sqrt{\frac{\pi}{c}},\ c > 0.$$

The main error term in (1.1) is at most of order n^{-2}. Indeed, writing

$$f(x) \approx f(a) + f'(a)(x-a)$$

we see that, assuming $f'(a) \neq 0$, this error term is proportional to

(1.4)
$$\int_0^{+\infty} xe^{-nrx}dx = \frac{1}{n^2r^2}.$$

On the other hand, the main error term in (1.2) is at most of order n^{-1}. Indeed, writing

$$f(x) \approx f(a) + f'(a)(x-a)$$

we see that, assuming $f'(a) \neq 0$, this error term is proportional to

(1.5)
$$\int_0^{+\infty} xe^{-nrx^2/2}dx = \frac{1}{nr}.$$

The usual ϵ-δ arguments show that other approximation errors in (1.1) and (1.2) are much smaller; many are exponentially small as $n \to \infty$.

The cases $x^* = b$, $S'(b) > 0$ and $x^* = b$, $S'(b) = 0,\ S''(b) < 0$ are similar.

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Case	Arg Max x^* of S	Asymptotic of $\int_a^b f(x)e^{nS(x)}\,dx$, $n\to\infty$
1	$x^*=a,\ S'(a)<0$	$e^{nS(a)}\,\dfrac{f(a)}{n\lvert S'(a)\rvert}\,\big(1+O(n^{-1})\big)$
2	$x^*=a,\ S'(a)=0$	$e^{nS(a)}\,\dfrac{f(a)}{2}\sqrt{\dfrac{2\pi}{n\lvert S''(a)\rvert}}\,\big(1+O(n^{-1/2})\big)$
3	$x^*\in(a,b)$	$e^{nS(x^*)}\,f(x^*)\sqrt{\dfrac{2\pi}{n\lvert S''(x^*)\rvert}}\,\big(1+O(n^{-1})\big)$
4	$x^*=b,\ S'(b)>0$	$e^{nS(b)}\,\dfrac{f(b)}{nS'(b)}\,\big(1+O(n^{-1})\big)$
5	$x^*=b,\ S'(b)=0$	$e^{nS(b)}\,\dfrac{f(b)}{2}\sqrt{\dfrac{2\pi}{n\lvert S''(b)\rvert}}\,\big(1+O(n^{-1/2})\big)$

TABLE 1. **Basic Laplace's Method**

Finally, assume that $x^*\in(a,b)$ so that $S'(x^*)=0$, $S''(x^*)<0$. Then, using the notation $r=-S''(x^*)$,

$$S(x)-S(a)\approx-\frac{r(x-a)^2}{2},\qquad \int_a^b f(x)e^{n\left(S(x)-S(x^*)\right)}dx\approx f(x^*)\int_a^b e^{-nr(x-a)^2/2}dx$$
$$\approx f(x^*)\int_{-\infty}^{+\infty} e^{nrx^2/2}dx=f(x^*)\sqrt{\frac{2\pi}{rn}}.$$

The error term is at most of order $n^{-3/2}$. Indeed, writing

$$f(x^*)\approx f(x^*)+f'(x^*)(x-x^*)+\frac{1}{2}f''(x^*)(x-x^*)^2,$$

we see that, assuming $f''(x^*)\neq 0$, this error term is proportional to

$$\int_{-\infty}^{+\infty}(x+x^2)e^{-nrx^2/2}dx=\frac{1}{(nr)^{3/2}}.$$

Table 1 summarizes the results.

Note that the above computations suggest that if $f(x)$ is constant, then the approximation error will be smaller. Turns out this is not the case: in the end, it is all about the function S, but a slightly more sophisticated analysis is necessary. We first illustrate this for Case 1, with $f(x)\equiv 1$.

If $S(x)<S(a)$, $x\in(a,b]$, $S'(x)<0$, then

$$\int_a^b e^{nS(x)}\,dx=e^{nS(a)}\,\frac{1}{n|S'(a)|}\,\big(1+O(n^{-1})\big).$$

Indeed, integrate by parts:

$$\int_a^b e^{nS(x)}\,dx = \int_a^b \frac{1}{nS'(x)}\,d\left(e^{nS(x)}\right) = \left.\frac{e^{nS(x)}}{nS'(x)}\right|_{x=a}^{x=b} + \int_a^b \frac{S''(x)}{n(S'(x))^2}\,e^{nS(x)}\,dx.$$

Next, note that $e^{n(S(b)-S(a))}$ is exponentially small as $n\to\infty$, and, by assumption,

$$\sup_{x\in[a,b]} \frac{S''(x)}{(S'(x))^2} \le C < \infty$$

for some C. Finally,

$$S(x)-S(a) = S'(c(x))(x-a),\ \ c(x)\in[a,x],$$

so that

$$S(x)-S(a) \le -C_1(x-a),\ \ C_1>0,$$

and

$$\int_a^b \frac{S''(x)}{n(S'(x))^2} e^{n(S(x)-S(a))}\,dx \le C\int_a^b e^{-C_1 n(x-a)}\,dx \le \frac{C}{nC_1},$$

completing the proof. □

To investigate Case 3, we need an auxiliary result. Recall that if x^* is a (local) maximum of S, then Taylor approximation gives

$$S(x)-S(x^*) \approx -\frac{S''(x^*)}{2}(x-x^*)^2, \tag{1.6}$$

which was good enough to derive the main term in the approximation of

$$\int_a^b e^{nS(x)}\,dx,$$

but is not enough to study the approximation error. We need a suitable change of variables so that $\approx$ in (1.6) becomes an exact equality, at least in some neighborhood of x^*.

Assume that $S'(x^*)=0$ and $S''(x^*)<0$. Then one can construct

(1) a neighborhood U of the point x^*,
(2) a neighborhood V of the point 0,
(3) a function $g=g(y): V\to U$,

such that, for all $y\in V$,

$$S\big(g(y)\big) = -y^2 \tag{1.7}$$

and

$$g'(0) = \sqrt{\frac{2}{|S''(0)|}}. \tag{1.8}$$

To simplify the computations, we assume that $x^*=0$ and $S(x^*)=0$; otherwise, we can always consider the function

$$x\mapsto S(x+x^*)-S(x^*).$$

Then

$$S(x) = \int_0^x S'(r)\,dr = \int_0^x\int_0^t S''(t)\,dt\,dr = \int_0^x (x-t)S''(t)\,dt.$$

Define the function $h=h(x)$ by

$$h(x) = \int_0^1 (t-1)S''(xt)\,dt$$

so that

$$S(x) = -x^2 h(x). \tag{1.9}$$

By assumption, $S''(0) < 0$ and therefore $h(x) > 0$ in some neighborhood of $x^* = 0$. Moreover,

$$\frac{d}{dx}\left(x\sqrt{h(x)}\right)\Bigg|_{x=0} = \sqrt{h(0)} = \sqrt{-\frac{S"(0)}{2}} > 0. \tag{1.10}$$

By the inverse function theorem, the equation

$$x\sqrt{h(x)} = y, \tag{1.11}$$

has a unique solution $x = g(y)$ in some neighborhood U of $x^* = 0$, where the function $x \mapsto x\sqrt{h(x)}$ is defined and is monotonically increasing, and equality (1.9) shows that (1.11) is equivalent to $S(x) = -y^2$. With $g(0) = 0$, the function g is defined in some neighborhood of $y^* = 0$, whereas (1.10) implies (1.8), completing the proof.

□

As an illustration of (1.7), consider

$$S(x) = \ln(x+1) - x,$$

so that

$$x^* = 0,\ S''(0) = -1.$$

By direct computation, $S''(x) = -(1+x)^{-2}$,

$$\int \frac{1-t}{(tx+1)^2}\,dt = -\frac{1}{x^2}\ln(tx+1) - \frac{x+1}{x}\,\frac{1}{tx+1},$$

so that

$$h(x) = \frac{x - \ln(x+1)}{x^2},\ x \neq 0,\ h(0) = \frac{1}{2}.$$

In particular, $h(x) > 0$ for all $x > -1$. Equation (1.11) becomes

$$\operatorname{sgn}(x)\sqrt{x - \ln(x+1)} = y, \tag{1.12}$$

where

$$\operatorname{sgn}(x) = \frac{x}{|x|}.$$

Now we can establish the approximation error in Case 3, again with $f(x) \equiv 1$.

If there is a unique $x^* \in (a,b)$ such that $S(x^*) > S(x)$ for all $x \neq x^*$, then

$$\int_a^b e^{nS(x)}\,dx = e^{nS(x^*)}\sqrt{\frac{2\pi}{n|S''(x^*)|}}\,\big(1 + O(n^{-1})\big).$$

With no loss of generality, we assume that $x^* = 0$ and $S(x^*) = 0$. Using (1.7), we construct U, a neighborhood of $x^* = 0$, where, with a suitable function g, we have $S(g(y)) = -y^2$. Next, we select a neighborhood $U* = (a^*, b^*)$ of x^* in such as way that

(1) $U^* \subset U$;
(2) $S(a^*) = S(b^*) = -r^2$: this can be achieved by (1.7).

As before,

$$\int\limits_{(a,b)\setminus U^*} e^{n\left(S(x) - S(x^*)\right)}\,dx$$

is exponentially small as $n \to \infty$. Finally,

$$\int_{a^*}^{b^*} e^{nS(x)}\,dx = \int_{-r}^{r} e^{-y^2}\,g'(y)\,dy;$$

recall that we assumed $x^* = 0, S(x^*) = 0$. With

$$\bar{g}(y) = g'(y) + g'(-y),$$

we get

$$\int_a^b e^{nS(x)}\,dx \approx \int_0^\varepsilon \bar{g}(y)\,e^{-ny^2}\,dy = \sqrt{\frac{2\pi}{n|S''(0)|}}\Big(1 + O(n^{-1})\Big),$$

where the last equality follows from (1.8) after two integrations by parts, keeping in mind that $\bar{g}(0) = 2g'(0)$ and $\bar{g}'(0) = 0$. □

With little or no change, one or both end points of the interval (a, b) can be infinite. For example, if $S(a) > S(x)$ for all $x > a$, then, for $\varepsilon \in (0, b-a)$,

$$\int_{a+\varepsilon}^b e^{n(S(x)-S(a))}\,dx$$

is exponentially small as $n \to \infty$, and, as long as the integral converges, one can have $b = +\infty$ and still apply Case 1.

Stationary phase.

The main object:

(1.13) $$F(\lambda) = \int_a^b f(x)e^{\mathrm{i}\lambda S(x)}dx,\ \ f, S \in \mathcal{C}^\infty([a,b]),\ \ \lambda \to +\infty.$$

The main results:

(1) If $f \in \mathcal{C}_0^\infty([a,b])$ (compactly supported) and $S'(x) \neq 0$, $x \in [a,b]$, then

(1.14) $$F(\lambda) = O(\lambda^{-N}),\ \ \lambda \to +\infty,$$

for every $N > 0$. For the proof, keep integrating by parts using the operator $D_S = (1/S'(x))d/dx$ and observing that

$$D_S\Big(e^{\mathrm{i}\lambda S(x)}\Big) = \mathrm{i}\lambda e^{\mathrm{i}\lambda S(x)}.$$

(2) If there is a point $x_0 \in (a,b)$ such that $S'(x_0) = 0$ and $S''(x_0) \neq 0$ [STATIONARY PHASE point], then

(1.15) $$F(\lambda) = e^{\mathrm{i}\lambda S(x_0)}e^{\mathrm{i}(\pi/4)\,\mathrm{sign}(S''(x_0))}\sqrt{\frac{2\pi}{\lambda|S'(x_0)|}}\,f(x_0) + O(\lambda^{-1}),\ \ \lambda \to +\infty.$$

For the proof, write $f(x) = f(x_0) + (x-x_0)\frac{f(x)-f(x_0)}{x-x_0}$, $S(x) \approx S(x_0) + (1/2)S''(x_0)(x-x_0)^2$, and use (1.14) away from x_0 [only one integration by parts; the boundary terms are included in $O(\lambda^{-1})$]. The key technical computation is the FRESNEL INTEGRAL

$$\int_0^{+\infty} e^{\mathrm{i}x^2/2}dx = \sqrt{\frac{\pi}{2}}\,e^{\mathrm{i}\pi/4}.$$

If there are several points like this inside the interval, then the asymptotic of $F(\lambda)$ is the sum of the contributions (1.15) of each point.

Main examples.

1. `Bessel's function` of the first kind

$$J_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\mathrm{i}(nt - x\sin t)}\,dt = \sqrt{\frac{2}{\pi x}}\cos\Big(x - \frac{\pi n}{2} - \frac{\pi}{4}\Big) + O(1/x),\ \ x \to +\infty.$$

Here, an application of (1.15) is immediate, with $S(t) = \sin t$ and two points of stationary phase: $t = \pm\pi/2$, where $S = \mp 1$ and $S'' = \pm 1$; $f(t) = e^{int}$.

2. `Airy's function`

$$\mathrm{Ai}(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp\left(\mathrm{i}\left(\frac{t^3}{3}+tx\right)\right)\,dt \equiv \frac{\sqrt{|x|}}{2\pi}\int_{-\infty}^{+\infty} \exp\left(\mathrm{i}|x|^{3/2}\left(\frac{t^3}{3}-t\right)\right)\,dt$$
$$= \frac{1}{\sqrt{\pi}\,|x|^{1/4}}\ \sin\left(\frac{2}{3}|x|^{3/2}+\frac{\pi}{4}\right) + O(1/|x|),\ x\to-\infty.$$

Here, $S(t) = (t^3/3) - t$ and the stationary phase points are $t_1 = -1$, where $S = 2/3$, $S'' = -2$, and $t_2 = 1$, where $S = -2/3$, $S'' = 2$. Note also that, when comparing to the general setting (1.13), we have $f(t) = 1$, $\lambda = |x|^{3/2}$ (and there is an extra $|x|^{1/2}$ from the change of variables), but the interval is unbounded, so that, strictly speaking, in addition to (1.15), analysis of the "tails" is also necessary [it is possible to show that the "tails" are of order $|x|^{-3/2}$] .

Saddle Point/Steepest Descent.

The main point: generalizing Laplace and stationary phase by going into the complex plane.

The main background result: the Cauchy integral theorem (integral of an analytic function over a simple closed path is zero), so the integration path from point to point can be deformed in a continuous way.

The saddle point is the picture of the level curves for the real and imaginary parts of the function $f(z) = z^2$.

The main object is the path integral

$$F(\lambda) = \int_\gamma f(z)e^{\lambda S(z)}\,dz,$$

with real λ and analytic functions f and S. The path γ does not have to be closed.

The main result is

$$F(\lambda) = \int_{\tilde\gamma} f(z)e^{\lambda S(z)}\,dz = e^{\lambda S(z_0)}f(z_0)\sqrt{\frac{2\pi}{\lambda|S''(z_0)|}}\ \exp\left(\mathrm{i}\frac{\pi-\arg(S''(z_0))}{2}\right)\ \big(1+O(1/\lambda)\big),$$

where $\tilde\gamma$, the PATH OF STEEPEST DESCENT, is a continuous deformation of γ (with the same start and end points), and is such that

- z_0 is on $\tilde\gamma$, and is not start or end point;
- $S'(z_0) = 0$, $S''(z_0) \neq 0$;
- $f(z_0) \neq 0$.

Construction of a suitable $\tilde\gamma$ is a major part of the method.

While the idea of the method goes back to Riemann, the modern development was carried out in 1908 by P. Debye, who later got Nobel Prize in chemistry (1936).