Abstract Wiener Space[1]

Starting point:[2]

- A sufficiently rich probability space $\mathfrak{F} = (\Omega, \mathcal{F}, \mathbb{P})$;
- A locally convex linear (over $\mathbb{R}$) topological space $\mathbb{X}$ with (topological) dual $\mathbb{X}^*$;
- A random element $\boldsymbol{X}$ on $\mathfrak{F}$ with values in $\mathbb{X}$ such that, for every $f \in \mathbb{X}^*$, the random variable $f(\boldsymbol{X})$ is Gaussian with mean zero;
- The measure μ on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ defined by $\mu(A) = \mathbb{P}(\boldsymbol{X} \in A)$ and the corresponding Hilbert space $L_2(\mathbb{X}, \mu)$.

Basic constructions.

(1) `Covariance operator` $\mathbf{K}$ of $\boldsymbol{X}$ is a continuous linear mapping from $\mathbb{X}^*$ to $\mathbb{X}$: for $f, g \in \mathbb{X}^*$,
$$\mathbb{E}\big(f(\boldsymbol{X})g(\boldsymbol{X})\big) = f\big(\mathbf{K}(g)\big) = g\big(\mathbf{K}(f)\big).$$

(2) The `canonical embedding operator` $\boldsymbol{\imath}^*$ of $\mathbb{X}^*$ into $L_2(\mathbb{X}, \mu)$: if $f \in \mathbb{X}^*$, then $f \in L_2(\mathbb{X}, \mu)$, because
$$\int_{\mathbb{X}} |f(x)|^2 \, \mu(dx) = \mathbb{E}|f(\boldsymbol{X})|^2 < \infty.$$

(3) The Hilbert space $\mathbb{X}^*_\mu$ of `measurable linear functionals` is the closure of the image $\boldsymbol{\imath}^*(\mathbb{X}^*)$ in $L_2(\mathbb{X}, \mu)$; this space is identified with its (topological) dual by the Riesz representation theorem. If $g \in \mathbb{X}^*_\mu$, then $g(\boldsymbol{X})$ is defined and is a zero-mean Gaussian random variable (being a mean-square limit of zero-mean Gaussian random variables).

(4) The `dual operator` $\boldsymbol{\imath} : \mathbb{X}^*_\mu \to \mathbb{X}$ for the operator $\boldsymbol{\imath}^* : \mathbb{X}^* \mapsto \mathbb{X}^*_\mu$:
$$f\big(\boldsymbol{\imath}(g)\big) = \mathbb{E}\big(f(\boldsymbol{X})g(\boldsymbol{X})\big) \equiv \big(\boldsymbol{\imath}^* f, g\big)_{\mathbb{X}^*_\mu}.$$

(5) The `Cameron-Martin space` $H_\mu = \boldsymbol{\imath}(\mathbb{X}^*_\mu)$, which is a separable Hilbert space with inner product
$$\big(x, y\big)_{H_\mu} = \big(\boldsymbol{\imath}^{-1}(x), \boldsymbol{\imath}^{-1}(y)\big)_{L_2(\mathbb{X},\mu)} \equiv \mathbb{E}\big((\boldsymbol{\imath}^{-1}(x))(\boldsymbol{X})(\boldsymbol{\imath}^{-1}(y))(\boldsymbol{X})\big), \tag{1.1}$$
and is compactly embedded into $\mathbb{X}$.

As a result,
$$\mathbf{K} = \boldsymbol{\imath}\boldsymbol{\imath}^*, \quad \mathbf{K}(\mathbb{X}^*) \subset H_\mu \subset \mathbb{X},$$
and H_μ is a *reproducing kernel Hilbert space* with kernel
$$K(f, g) = \mathbb{E}\big(f(\boldsymbol{X})g(\boldsymbol{X})\big), \ f, g \in \mathbb{X}^*.$$

In the above setting, the `abstract Wiener space` is the triple $(\mathbb{X}, H_\mu, \mu)$. In the original construction of Leonard Gross[3] (around 1965), the *starting point* is the triple, in which $\mathbb{X}$ is separable Banach space and H_μ is a separable Hilbert space that is densely and continuously embedded into $\mathbb{X}$; the key point is *existence* of the corresponding measure μ.

The main example is $\mathbb{X} = \mathcal{C}((0, T))$, the space of continuous functions on $[0, T]$, and $\boldsymbol{X} = W$, a standard Brownian motion. Then $H_\mu = \{f \in H_1((0, T)) : f(0) = 0\}$, where
$$H_1((0, T)) = \{f : f(t) = f(0) + \int_0^t f'(s)\, ds, \ \int_0^T |f'(s)|^2 \, ds < \infty.\}$$

The *abstract* **Cameron-Martin theorem** becomes as follows: given a non-random $x \in \mathbb{X}$, the distribution μ of $\boldsymbol{X}$ and the distribution μ_x of $\boldsymbol{X} + x$ [defined by $\mu_x(A) = \mathbb{P}(\boldsymbol{X} + x \in A)$] are mutually absolutely continuous if and only if $x \in H_\mu$, and in that case,
$$\frac{d\mu_x}{d\mu}(y) = \exp\left(\big(\boldsymbol{\imath}^{-1}(x)\big)(y) - \frac{\|x\|^2_{H_\mu}}{2}\right), \ y \in \mathbb{X}. \tag{1.2}$$

[1]Sergey Lototsky, USC

[2]Some stretch of imagination might be necessary to resolve potential questions about existence.

[3]b. 1931, Professor Emeritus at Cornell

Otherwise, the distributions are mutually singular [supported on disjoint sets].

Two related results.

Theorem. [J. Hájek (1958), J. Feldman (1958)] Two Gaussian measures on a locally convex linear topological are either equivalent or singular.

Theorem. [S. Kakutani (1948)] Two infinite product measures are equivalent if and only if the series

$$\sum_{k\geq 1} \ln \int \sqrt{\frac{d\nu_k}{d\mu_k}}\, d\mu_k \tag{1.3}$$

converges.

The Gaussian product measure on a separable Hilbert space.[4]

Let $\mathbb{X}$ be a separable Hilbert space with inner product $(\cdot,\cdot)$, norm $\|\cdot\|$, and an orthonormal basis $\{\varphi_k,\ k\geq 1\}$, and let $\boldsymbol{X}$ be a Gaussian random element with values in $\mathbb{X}$: $\boldsymbol{X} = \sum_k q_k\xi_k\varphi_k$ where $\xi_k,\ k\geq 1$, are iid $\mathcal{N}(0,1)$, $q_k>0$, and $\sum_k q_k^2<\infty$:

$$\mathbb{E}\|\boldsymbol{X}\|^2 = \sum_k q_k^2.$$

Then

(1) $\mathbb{X}^* = \mathbb{X}$;

(2) The operator $\mathbf{K}$ is positive and symmetric, with $\boldsymbol{\imath} = \boldsymbol{\imath}^* = \mathbf{K}^{1/2}$:

$$\mathbf{K}(f) = \sum_k q_k^2 f_k\varphi_k, \qquad f_k = (f,\varphi_k);$$

(3) The distribution μ of $\boldsymbol{X}$ in H is a product measure of $\mathcal{N}(0,q_k^2)$;

(4) For $x\in\mathbb{X}$, the distribution $\nu=\mu_x$ of $\boldsymbol{X}+x$ is the product measure of $\mathcal{N}(x_k,q_k^2)$ so that

$$\frac{d\nu_k}{d\mu_k}(t) = \exp\left(\frac{2tx_k - x_k^2}{2q_k^2}\right), \quad t\in\mathbb{R}. \tag{1.4}$$

With $\zeta_k\sim\mathcal{N}(0,q_k^2)$,

$$\int\sqrt{\frac{d\nu_k}{d\mu_k}}\, d\mu_k = e^{-x_k^2/(4q_k^2)}\mathbb{E}e^{\zeta_k x_k/(2q_k^2)} = e^{-x_k^2/(8q_k^2)}, \tag{1.5}$$

and (1.3) becomes

$$\sum_k q_k^{-2}x_k^2<\infty. \tag{1.6}$$

(5) The abstract construction leads to the Cameron-Martin space

$$H_\mu = \mathbf{K}^{1/2}(\mathbb{X}) = \{f\in\mathbb{X}: \sum_k q_k^{-2}f_k^2<\infty\},$$

which is consistent with (1.6).

(6) Equality (1.4) is consistent with (1.2): just as in (1.5), we write $y_k = t/q_k$ to get

$$\prod_k \frac{d\nu_k}{d\mu_k}(y_k) = \exp\left((\mathbf{K}^{-1/2}x, y) - \frac{1}{2}\|\mathbf{K}^{-1/2}x\|^2\right).$$

[4] Sometimes referred to as the "trivial example".