Recursive Nonlinear Filter for a Continuous-Discrete Time Model: Separation of Parameters and Observations

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Abstract

A new nonlinear filtering algorithm is proposed for the model where the state is a randomly perturbed nonlinear dynamical system and the measurements are made at discrete time moments in Gaussian noise. It is shown that the approximate scheme based on the algorithm converges to the optimal filter and the error of the approximation is computed. The algorithm makes it possible to shift off line the most time consuming operations related to solving the Fokker-Planck equations and computing the integrals with respect to the filtering density.

Key words: Diffusion processes, Fokker-Planck equation, Nonlinear filtering, Real time.

1 Introduction

In the continuous-discrete time filtering model, an unobserved continuous time state process is estimated from the noisy measurements made at discrete time moments. This model seems of special interest from the point of view of applications, because many real life processes evolve in continuous time while the digital devices used to process the measurements require discrete time data. The case of continuous time observations was studied in [1, 2], see also [3].

The desired solution of the filtering problem is an algorithm that provides the best mean square estimate of the given functional of the state process in the form suitable

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for on-line implementation. In the linear case, such a solution is given by the Kalman filter [4, 5].

It is worth mentioning that the exact solution of the continuous-discrete time filtering problem is known for a wide class of nonlinear models [6, 4]. Specifically, let $X = (X(t))_{t\geq 0}$ be the state process and assume that the measurements $z(k)$ are made at moments t_k. If $f = f(x)$ is a function such that $\mathbf{E}|f(X(t))|^2 < \infty$ for all $t \geq 0$, then the best mean square estimate $\hat{f}(k)$ of $f(X(t_k))$ given the observations $z_i,\ i \leq k$ can be written as

$$\hat{f}(k) = \frac{\int p_k(x) f(x) dx}{\int p_k(x) dx}. \tag{1.1}$$

The function $p_k(x)$, called *the unnormalized filtering density* (UFD), satisfies a recursive equation of the predictor-corrector type: in between the measurements, $p_k(x)$ evolves according to the Fokker-Planck equation corresponding to the state process (prediction), and after each observation the value of p_k is updated using the conditional distribution of the measurements given the state process (correction). This procedure provides an algorithm for computing the filtering density (both unnormalized and normalized) and the optimal filter $\hat{f}(k)$.[1] Unfortunately, the implementation of this optimal algorithm is practically impossible when the dimension of the state process is greater than three. One reason is that on each step of the algorithm it is necessary to solve the Fokker-Planck equation. Even though the parameters (coefficients) of the equation might be known a priori, the solution cannot be obtained in advance because the initial condition depends on the previous measurements. On-line solution of such an equation can require too much time. Another reason is that on-line computations of the integrals in (1.1) even for simple functions f may be very time consuming if the dimension of the state process is large. These computations alone can rule out the on-line application of the algorithm.

On-line solution of the Fokker-Planck equation can be avoided if the UFD admits a finite dimensional sufficient statistics; in particular, this is the case when the model is linear Gaussian. The first nonlinear example of this sort was discovered by Beneš [7], and a more general class of UFD admitting finite dimensional sufficient statistics was studied by Daum [8]. Unfortunately, for a given nonlinear model, it is usually not clear whether the sufficient statistics exists. The practical algorithms based on this approach use approximations similar to the extended Kalman filter [9], so as a rule the error of such approximations is unknown.

The objective of the current work is to develop a recursive numerical algorithm for computing $\hat{f}(k)$ in which the on-line part is as simple as possible; in particular, no differential equations are to be solved on-line. The starting point in the derivation is the equation satisfied by $p_k(x)$ in the general nonlinear model, and the approach is based on the technique known as the parameterization of the unnormalized filtering density [4].

[1] If the opbjective is an estimate of $X(t_k)$, then the maximum likelihood estimates (MLEs) sometimes provide better results than the conditional mean, e.g. when the filtering density is multimodal. Note that, for computing the MLE, the UFD is sufficient since it attains its maxima at the same points as the normalized density.

In the proposed algorithm (Section 3) both time consuming operations of solving the Fokker-Planck equation and computing the integrals are performed off line, which makes the algorithm suitable for on-line implementation. Since the result is only an approximation of the optimal filter, the error of the approximation is computed.

2 The Filtering Problem

Consider the problem of estimating an $\mathbb{R}^d$-valued state process $X = X(t)$ that evolves in continuous time according to the following Ito stochastic differential equation

$$dX(t) = b(X(t))dt + \sigma(X(t))dW(t), \quad 0 \le t \le t_0, X(0) = x_0 \tag{2.1}$$

given $\mathbb{R}^r$-valued measurements $z = z(k)$ made at discrete time moments $t_k = k\Delta,\ k \ge 1$:

$$z(k) = h(X(t_k)) + v(k). \tag{2.2}$$

In the above, $W = (W(t))_{t\ge 0}$ is an $\mathbb{R}^{d_1}$-valued standard Brownian motion independent of the initial condition x_0; functions $b = b(x),\ \sigma = \sigma(x)$, and $h = h(x),\ x \in \mathbb{R}^d$, take values in $\mathbb{R}^d,\ \mathbb{R}^{d\times d_1}$, and $\mathbb{R}^r$ respectively; the sequence $\{v(k)\}_{k\ge 1}$ is independent of the state process and consists of i.i.d. Gaussian random vectors with zero mean and covariance $\mathbf{E}v(k)v(k)^T = (1/\Delta)I,\ I \in \mathbb{R}^{r\times r}$ is the identity matrix. The underlying probability space $(\Omega, \mathcal{F}, \mathrm{P})$ is assumed to be fixed.

The following regularity assumptions are made about the model (2.1), (2.2):

1. *the functions b, σ, and h are infinitely differentiable and bounded, and all their derivatives are also bounded;*

2. *the random vector x_0 has a density $p = p(x),\ x \in \mathbb{R}^d$, so that the function p is infinitely differentiable and, together with all its derivatives, decays at infinity faster than any power of $|x|$ (in other words, the density function p is an element of the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ of rapidly decreasing functions).*

Among the density functions satisfying the second assumption are the mixtures of the normal distributions and the function $p(x) = A\exp\left(-B\sqrt{1+|x-x_0|^2}\right),\ B > 0$, A- normalizing constant.

The nonlinear filtering model with unbounded coefficients is traditionally treated separately becuse it requires different and more complicated technical tools. For many practical purposes, though, the algorithm described in this note can be used even if the coefficients are not bounded because the actual computations are carried out in a bounded domain (see Section 5).

Let $f = f(x), x \in \mathbb{R}^d$, be a measurable scalar function such that $\mathbf{E}|f(X(t))|^2 < \infty$ for all $t \ge 0$. Then the filtering problem for (2.1), (2.2) can be stated as follows: find the best mean square estimate of $f(X(t_k))$ given the measurements $z(m),\ m = 1, \dots, k$. This estimate is called *the optimal filter* and will be denoted by $\hat{f}(k)$. For computational purposes, the optimal filter $\hat{f}(k)$ can be characterized as follows.

Denote by T_t the solution operator for the Fokker-Planck equation corresponding to the state process; in other words, $u(t,x) = T_t\varphi(x)$ is the solution of the equation

$$\begin{aligned} \frac{\partial u(t,x)}{\partial t} &= \frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial^2}{\partial x_i \partial x_j}\left(\sum_{l=1}^{d_1}\sigma_{il}(x)\sigma_{jl}(x)u(t,x)\right) - \sum_{i=1}^{d}\frac{\partial}{\partial x_i}(b_i(x)u(t,x)), \quad t>0, \\ u(0,x) &= \varphi(x). \end{aligned} \tag{2.3}$$

Next, define the sequence $p_k(x),\ x \in I\!R^d,\ k \geq 0$, by

$$\begin{aligned} p_0(x) &= p(x), \\ p_k(x) &= \exp\left(\Delta\sum_{l=1}^{r}h_l(x)z_l(k) - \frac{\Delta}{2}\sum_{l=1}^{r}h_l^2(x)\right)T_\Delta p_{k-1}(x). \end{aligned} \tag{2.4}$$

Then the optimal filter $\hat{f}(k)$ can be written as follows [4]:

$$\hat{f}(k) = \frac{\int_{I\!R^d} p_k(x)f(x)dx}{\int_{I\!R^d} p_k(x)dx}. \tag{2.5}$$

The numerator in (2.5) will be denoted by $\phi_k[f]$. With this notation, (2.5) becomes

$$\hat{f}(k) = \frac{\phi_k[f]}{\phi_k[1]}.$$

3 The Algorithm

Assume that the function f satisfies the following growth condition:

$$|f(x)| \leq C_f(1+|x|^\alpha),\ x \in I\!R^d, \tag{3.1}$$

for some $\alpha,\ C_f > 0$. Then $\mathbf{E}|f(X(t))|^2 < \infty$ for all $t \geq 0$ [5, Theorem 4.6].

The objective is to construct a recursive algorithm for computing an approximation of $\hat{f}(k)$. The algorithm uses parameterization of $p_k(x)$ [4] as discussed in the Introduction. The derivation of the algorithm is given in Section 4.

To parameterize the function p_k, an orthonormal basis is introduced in $L_2(I\!R^d)$ as follows. Let

$$e_n(t) = \frac{1}{\sqrt{2^n\pi^{1/2}n!}}e^{-t^2/2}H_n(t),$$

where

$$H_n(t) = (-1)^n e^{t^2}\frac{d^n}{dt^n}e^{-t^2}, \quad n \geq 0, \tag{3.2}$$

is the n-th Hermite polynomial. Let J be the collection of d-dimensional multi-indices. For $l = (l_1, \ldots, l_d) \in J$ define

$$e_l(x_1, \ldots, x_d) = \prod_{i=1}^{d} e_{l_i}(x_i). \tag{3.3}$$

Then the set $\{e_l\}_{l\in J}$ is an orthonormal basis in $L_2(I\!R^d)$ [10].

For $i,j=1,\dots,r$ and $l,n\in J$ define the numbers

$$q_{ln}=\int_{I\!R^d} e_l(x)T_\Delta e_n(x)dx,\quad q^i_{ln}=\int_{I\!R^d} h_i(x)e_l(x)T_\Delta e_n(x)dx,$$
$$q^{ij}_{ln}=\int_{I\!R^d} h_j(x)h_i(x)e_l(x)T_\Delta e_n(x)dx,\ f_l=\int_{I\!R^d} f(x)e_l(x)dx,\ \psi_l(0)=\int_{I\!R^d} p(x)e_l(x)dx. \tag{3.4}$$

The growth condition (3.1) and the regularity assumptions from Section 2 imply that the above integrals are well defined.

Fix an integer $\kappa>0$. For $l\in J$ we say that $l\le\kappa$ if $\sum_{i=1}^d l_i\le\kappa$. The following is a recursive algorithm for computing an approximation to $p_k(x)$ and $\phi_k[f]$.

1. Off line (before the measurements are available):

— *for* $i,j=1,\dots,r$ *and* $l,n\le\kappa$, *compute* q_{ln}, q^i_{ln}, q^{ij}_{ln}, f_l, *and* $\psi_l(0)$;

— *set* $\bar p_0(x)=\sum_{l\le\kappa}\psi_l(0)e_l(x)$ *and* $\bar\phi_0[f]=\sum_{l\le\kappa}\psi_l(0)f_l$.

2. On line, k-th step (as the measurements become available): compute

$$\psi_l(k)=\sum_{n\le\kappa} Q_{ln}(z(k))\psi_n(k-1), \tag{3.5}$$

$l\le\kappa$, *where*

$$Q_{ln}(z(k))=q_{ln}+\Delta\sum_{i=1}^r z_i(k)q^i_{ln}+\Delta^2\sum_{i\ne j=1}^r z_i(k)z_j(k)q^{ij}_{ln}+\frac{\Delta}{2}\sum_{i=1}^r\left(z_i^2(k)\Delta-1\right)q^{ii}_{ln},$$

then compute

$$\bar p_k(x)=\sum_{l\le\kappa}\psi_l(k)e_l(x), \tag{3.6}$$

$$\bar\phi_k[f]=\sum_{l\le\kappa}\psi_l(k)f_l, \tag{3.7}$$

and

$$\bar f_k=\frac{\bar\phi_k[f]}{\bar\phi_k[1]}\ .$$

We would like to remark the following features of the algorithm:

(1) The time consuming operations of solving the partial differential equation (2.3) and computing integrals are performed off line;

(2) The overall amount of the off-line computations does not depend on the number of the on-line time steps;

(3) Formula (3.7) can be used to compute an approximation to $\hat f(k)$ (e.g. conditional moments) without the computations of $\bar p_k(x)$ and the related integrals;

(4) Only the Fourier coefficients ψ_l must be computed at every time step while the approximate filter $\bar f_k$ and UFD $\bar p_k(x)$ can be computed as needed, e.g. at the final time moment.

(5) The on-line part of the algorithm can be easily parallelized.

The convergence of the algorithm is established in the following theorem.

Theorem 3.1 *Suppose that $t_0 = M\Delta$ is fixed. If the regularity assumptions hold, then for every $M > 0$ there exist positive constants A_p, B_p, and C_p so that*

$$\max_{1\le k\le M} \mathbf{E}\|p_k - \bar{p}_k\|_{L_2(I\!R^d)} \le A_p\Delta + \frac{B_p}{\sqrt{\Delta}} \exp\left(-C_p(\ln\kappa)^2\right). \tag{3.8}$$

If in addition the function f satisfies (3.1) *, then for every $M > 0$ there are positive constants $A_\phi(C_f, \alpha)$, $B_\phi(C_f, \alpha)$ and $C_\phi(\alpha)$ so that*

$$\max_{1\le k\le M} \mathbf{E}|\phi_k[f] - \bar{\phi}_k[f]| \le A_\phi(C_f,\alpha)\Delta + \frac{B_\phi(C_f,\alpha)}{\sqrt{\Delta}} \exp\left(-C_\phi(\alpha)(\ln\kappa)^2\right). \tag{3.9}$$

All constants in the above inequalities also depend on t_0 and the parameters of the model.

This theorem is proved in Appendix. The result implies that in the limit $\lim_{\Delta\to 0}\lim_{\kappa\to\infty}$ both approximations converge in the mean.

Remark 1. *In the limit $\kappa \to \infty$, the order of the approximations (3.6) and (3.7) is Δ. On the other hand, if (2.2) is considered as an approximation of continuous time observations, then the order of the approximation is also Δ, which means that higher order approximations of $p_k(x)$ might be unnecessary.*

Remark 2. *Inequalities* (3.8) *or* (3.9) *imply that if the approximation error is to be made less than ϵ, then the asymptotic (as $\epsilon \to 0$) dependence of κ on ϵ is given by $\ln\kappa \asymp \sqrt{\ln 1/\epsilon}$. The results of numerical simulations indicate that for a sufficiently regular model the value $\kappa = 10$ provides a good approximation of the filtering density.*

The number of on-line operations per step in the proposed algorithm is $K_1 = N_\kappa^2(2r^2 + 2r + 1)$ flops, where $N_\kappa = (\kappa + d)!/(\kappa!\, d!)$ is the number of multi-indices $l = (l_1, \ldots, l_d)$ with $\sum_{i=1}^d l_i \le \kappa$. If $\kappa = 10$, $r = 1$, and $d = 6$, then $K_1 \approx 3 \cdot 10^8$. For comparison, the same number for the algorithm (2.4)–(2.5) is $K_2 = CN_s^{1+1/d}$, where N_s is the total number of points in the spatial domain and the constant $C > 1$.[2] If $N_s = 50^6$ and $d = 6$ (i.e. 50 points are taken in each direction), then $K_2 > 4 \cdot 10^{13}$. Due to this reduction in the number of the on-line operations and the possibility of an easy parallelization of the on-line computations, we believe that the proposed algorithm can be used in real time.

[2]We assume that a realistic algorithm to solve the general Fokker-Plank equation is the conjugate gradient method without pre-conditioning requiring $O(N_s^{1+1/d})$ operations [11]; exponential functions and integrals can be computed in $O(N_s)$ flops.

4 Derivation of the Algorithm

It follows from (2.4) and the Taylor formula that

$$\begin{aligned} p_k(x) &\approx \Big(\textstyle\prod_{i=1}^r \big(1 + \Delta \cdot h_i(x) z_i(k) + \frac{\Delta}{2} h_i^2(x)(z_i^2(k)\Delta - 1)\big)\Big) T_\Delta p_{k-1}(x) \\ &\approx \Big(1 + \Delta \textstyle\sum_{i=1}^r h_i(x) z_i(k) + \frac{\Delta}{2} \sum_{i=1}^r h_i^2(x)\big(z_i^2(k)\Delta - 1\big) \\ &+ \Delta^2 \textstyle\sum_{i \neq j=1}^r h_i(x) h_j(x) z_i(k) z_j(k)\Big) T_\Delta p_{k-1}(x) := \Phi_k(x) T_\Delta p_{k-1}(x). \end{aligned}$$

Define $\tilde{p}_k(x)$ by

$$\tilde{p}_0(x) = p(x), \quad \tilde{p}_k(x) = \Phi_k(x) T_\Delta \tilde{p}_{k-1}(x), \ k \geq 1. \tag{4.1}$$

Then $\max\limits_{1 \leq k \leq M} \mathbf{E} \|p_k - \tilde{p}_k\|_{L_2(\mathbb{R}^d)} \leq C\Delta$ (proved in the Appendix).

Next, write $\tilde{p}_k(x) = \sum_{l \in J} \tilde{\psi}_l(k) e_l(x)$. Substitution in (4.1) yields

$$\sum_{l \in J} \tilde{\psi}_l(k+1) e_l(x) = \sum_{l \in J} \Phi_{k+1}(x) T_\Delta e_l(x) \tilde{\psi}_l(k)$$

or

$$\tilde{\psi}_l(k+1) = \sum_{n \in J} \Big(\int_{\mathbb{R}^d} e_l(x) \Phi_{k+1}(x) T_\Delta e_n(x) dx \Big) \tilde{\psi}_n(k).$$

After that, define $\psi_l(k), \ l \leq \kappa$, by

$$\begin{aligned} \psi_l(0) &= \int_{\mathbb{R}^d} p(x) e_l(x) dx, \\ \psi_l(k) &= \sum_{n \leq \kappa} \Big(\int_{\mathbb{R}^d} e_l(x) \Phi_k(x) T_\Delta e_n(x) dx \Big) \psi_n(k-1). \end{aligned}$$

and set $\bar{p}_k(x) = \sum_{l \leq \kappa} \psi_l(k) e_l(x)$. Then $\{\psi_l(k)\}_{l \leq \kappa}$ satisfy (3.5) (direct computations) and $\max\limits_{1 \leq k \leq M} \mathbf{E} \|\tilde{p}_k - \bar{p}_k\|_{L_2(\mathbb{R}^d)} \leq C/(\kappa^\gamma \Delta)$ (proved in the Appendix).

5 Simulation Example

The proposed algorithm was tested on a two dimensional tracking problem with angle-only observations. The state process $X = (X_1(t), X_2(t))$ was defined by

$$\begin{aligned} dX_1(t) &= -189.33 X_2^3(t) + 9.16 X_2(t) + 0.001 dW_1(t), \\ dX_2(t) &= -1/3 + 0.03 dW_2(t), \end{aligned}$$

and the measurements $z = (z_k)$, by

$$z_k = \arcsin\left(\frac{X_2(t_k)}{\sqrt{X_1^2(t_k) + X_2^2(t_k)}} \right) + \frac{0.2}{\sqrt{\Delta}} V_k,$$

Table 1: On-line complexity

k	exact filter	approximate filter	ratio
100	$4.3 \cdot 10^9$	$5.9 \cdot 10^6$	730
150	$6.5 \cdot 10^9$	$8.6 \cdot 10^6$	755

Table 2: Quality of filtering by the approximate filter

k	N_{75}	N_{95}
100	100	100
150	96	100

where $\Delta = 0.01$, $(V_k)_{k\geq 1}$ are independent standard Gaussian random variables, and the initial state distribution $p_0(x_1, x_2) = \frac{1}{c_0}[\exp(-500(x_1-0.37)^2 - 500(x_2-0.31)^2) + \frac{3}{4}\exp(-1000(x_1+0.32)^2 - 1000(x_2-0.22)^2)]$, c_0 is the normalizing constant. It was shown in [12] that the extended Kalman filter fails for this model.

To compute the exact filter, the Fokker-Planck equation was solved according to the method of lines: the partial differential equation was reduced to a system of ordinary differential equations by approximating the spatial derivatives using the up-wind scheme [13], and then the resulting system of ODEs was solved using the Padé method [14]. The spatial domain was restricted to a rectangle $[-0.8; 0.8] \times [-0.5; 0.5]$ with $N_s = 80 \times 60$ points uniformly distributed in each direction. The same approximation was used to compute the coefficients q according to (3.4). The approximate filter was used with $N_\kappa = 66$ basis functions, which corresponds to $\kappa = 10$. The computations were performed on the SGI PowerChallenge machine.

The computational complexity of the algorithms was characterized by the number of on-line floating point operations (flops) to evaluate the filtering density at the given time moment $t_k = k\Delta$. The quality of filtering was characterized by the number of times N_β the state process was in the $\beta\%$ confidence interval as given by the computed density. To determine N_β, one realization of the state process was used with 100 independent measurement sequences.

The results indicate that the proposed algorithm reduces the complexity by a factor of more than 700 (Table 1), while providing sufficiently good quality of filtering (Table 2).

6 Conclusion

The optimal filter for a nonlinear diffusion process observed in discrete time requires on-line solution of the Fokker-Planck equation and subsequent evaluation of integrals in the spatial domain. Using the parametric representation of the unnormalized filtering density, these time consuming operations can be performed off line and the

results stored. In the resulting algorithm, the on-line computations can be organized recursively in time and are relatively simple even when the dimension of the state process is large. Moreover, certain functionals of the state process can be estimated without computing the unnormalized filtering density. As the sampling frequency of the observations increases, the computed estimate converges to the optimal in the mean.

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Appendix

Below, Theorem 3.1 is proved. It is assumed everywhere that the filtering problem is considered on a fixed time interval $[0, t_0]$, where $t_0 = M\Delta$.

1. *Change of measure.* Following [6], define

$$\Lambda_0 = 1, \quad \Lambda_k = \prod_{i=1}^{k} \exp\Big(\Delta \sum_{l=1}^{r} h_l(X_{t_i}) z_l(i) - \frac{\Delta}{2} \sum_{l=1}^{r} h_l^2(X_{t_i})\Big), \quad k = 1, \dots, M,$$

and introduce a new measure $\tilde{\mathrm{P}}$ on $(\Omega, \mathcal{F})$ by $d\tilde{\mathrm{P}} = (\Lambda_M)^{-1} d\mathrm{P}$.

Lemma A.1. *The measure $\tilde{\mathrm{P}}$ is a probability measure on $(\Omega, \mathcal{F})$. Moreover, under measure $\tilde{\mathrm{P}}$, the distribution of the process $(X(t))_{0 \le t \le M\Delta}$ does not change, $\{z(k)\}_{1 \le k \le M}$ is a sequence of i.i.d. Gaussian random variables independent of the state process, and $\tilde{\mathbf{E}} z(k) = 0$, $\tilde{\mathbf{E}} z(k) z(k)^T = (1/\Delta) I$, where $\tilde{\mathbf{E}}$ is the expectation with respect to measure $\tilde{\mathrm{P}}$.*

Proof. This lemma can be proved in the same way as the statements of Theorem 1 and Remark 1 in [15].

□

2. *Proof of Theorem 3.1.* In what follows, all constants can depend on the parameters of the model and on t_0. For brevity, $\|\cdot\|$ is used instead of $\|\cdot\|_{L_2(\mathbb{R}^d)}$.

Lemma A.2. *There is a constant C_h so that for every integrable random variable ξ,*

$$\mathbf{E}|\xi| \le C_h \sqrt{\tilde{\mathbf{E}}|\xi|^2}.$$

Proof. This follows from the definition of $\tilde{\mathbf{E}}$ and the Cauchy-Schwartz inequality.

□

Recall that $\tilde{p}_k(x)$ is defined by (4.1) as $\tilde{p}_0(x) = p(x),\ \tilde{p}_k(x) = \Phi_k(x)T_\Delta \tilde{p}_{k-1}(x),\ k \geq 1$, where

$$\Phi_k(x) = 1 + \Delta \sum_{i=1}^{r} \Big(h_i(x) z_i(k) + \frac{1}{2} h_i^2(x) \Big(z_i^2(k)\Delta - 1 \Big) \Big) + \Delta^2 \sum_{i \neq j = 1}^{r} h_i(x) h_j(x) z_i(k) z_j(k).$$

Also, define $\Psi_k(x) := \exp\Big(\Delta \sum_{l=1}^{r} h_l(x) z_l(k) - 0.5\Delta \sum_{l=1}^{r} h_l^2(x) \Big)$ so that $p_k(x) = \Psi_k(x) T_\Delta p_{k-1}(x)$. Then, by the previous lemma, inequality (3.8) will follow from

$$\sup_{1 \leq k \leq M} \tilde{\mathbf{E}} \| p_k - \tilde{p}_k \|^2 \leq \tilde{A}_p \Delta^2 \tag{A.1}$$

and

$$\sup_{1 \leq k \leq M} \tilde{\mathbf{E}} \| \tilde{p}_k - \bar{p}_k \|^2 \leq \frac{\tilde{B}_p}{\Delta} \exp\Big(-\tilde{C}_p (\ln \kappa)^2 \Big). \tag{A.2}$$

The proof of these inequalities requires some technical results summarized in the following lemma.

Lemma A.3.

$$\tilde{\mathbf{E}} \| p_k \|^2 \leq C_0 \| p \|^2,\ k \leq M, \tag{A.3}$$

$$\tilde{\mathbf{E}} \Phi_k(x) (\Psi_k(x) - \Phi_k(x)) = 0, \tag{A.4}$$

$$\tilde{\mathbf{E}} \| \Phi_k T_\Delta \varphi \|^2 \leq e^{C_1 \Delta} \tilde{\mathbf{E}} \| \varphi \|^2, \tag{A.5}$$

where φ is Z_{k-1}-measurable.

Proof. The regularity assumptions imply that the operator T_Δ is continuous in $L_2(\mathbb{R}^d)$:

$$\| T_\Delta \cdot \| \leq e^{C_2 \Delta} \| \cdot \|. \tag{A.6}$$

This inequality is well-known [16, 17]. Since $z(k)$ is independent of Z_{k-1} under measure $\tilde{\mathrm{P}}$, (A.3) follows by induction from (2.4) and (A.6).

To prove (A.4), define $\tilde{H}(x) := 2^{-n/2} H_n(x/\sqrt{2})$, where $H_n(x)$ is the n-th Hermit polynomial (3.2). Then

$$\Psi_k(x) = \prod_{l=1}^{r} \Big(\sum_{n \geq 0} \frac{h_l^n(x)}{n!} \Delta^{n/2} \tilde{H}_n(\sqrt{\Delta} z_l(k)) \Big). \tag{A.7}$$

Since $\tilde{\mathbf{E}} \tilde{H}_n(\sqrt{\Delta} z_l(k)) \tilde{H}_m(\sqrt{\Delta} z_s(k)) = n! \delta_{mn} \delta_{ls}$, the result follows.

Finally, (A.4) implies that $\tilde{\mathbf{E}} \Psi_k^2(x) = \tilde{\mathbf{E}} \Phi_k^2(x) + \tilde{\mathbf{E}} (\Psi_k(x) - \Phi_k(x))^2$, and consequently $\tilde{\mathbf{E}} \Phi_k^2(x) \leq \tilde{\mathbf{E}} \Psi_k^2(x)$. Then (A.5) can be derived from (A.6) in the same way as (A.3).

□

It follows from (A.4) that

$$\tilde{\mathbf{E}} \| p_k - \tilde{p}_k \|^2 \leq \tilde{\mathbf{E}} \| \Phi_k T_\Delta (p_{k-1} - \tilde{p}_{k-1}) \|^2 + \tilde{\mathbf{E}} \| (\Psi_k - \Phi_k) T_\Delta p_{k-1} \|^2. \tag{A.8}$$

Next, the definition of $\Phi_k(x)$ and (A.7) imply that

$$\tilde{\mathbf{E}} (\Psi_k(x) - \Phi_k(x))^2 \leq C_3 \Delta^3. \tag{A.9}$$

Combining (A.3), (A.5), (A.8), and (A.9) results in $\tilde{\mathbf{E}} \| p_k - \tilde{p}_k \|^2 \leq e^{C_1 \Delta} \tilde{\mathbf{E}} \| p_{k-1} - \tilde{p}_{k-1} \|^2 + C_4 \Delta^3$ with $C_4 = C_0 C_3 e^{C_2 t_0}$, and (A.1) follows by the Gronwall inequality.

To prove (A.2), denote by Π^κ the $L_2(\mathbb{R}^d)$ - orthogonal projection on $\{e_l\}_{l\le\kappa}$, where e_l is defined in (3.3). Then $\bar{p}_k = \Pi^\kappa \Phi_k(x) T_\Delta \bar{p}_{k-1}(x)$ and

$$\tilde{\mathbf{E}}\|\tilde{p}_k - \bar{p}_k\|^2 = \tilde{\mathbf{E}}\|\Pi^\kappa \Phi_k T_\Delta(\tilde{p}_{k-1} - \bar{p}_{k-1})\|^2 + \tilde{\mathbf{E}}\|(1-\Pi^\kappa)\Phi_k T_\Delta \tilde{p}_{k-1}\|^2. \tag{A.10}$$

By (A.5),

$$\tilde{\mathbf{E}}\|\Pi^\kappa \Phi_k T_\Delta(\tilde{p}_{k-1} - \bar{p}_{k-1})\|^2 \le e^{C_1\Delta}\tilde{\mathbf{E}}\|\tilde{p}_{k-1} - \bar{p}_{k-1}\|^2. \tag{A.11}$$

Next, since all the derivatives of the initial density $p(x)$ are square integrable with any polynomial weight, so are the derivatives of $\tilde{p}_k$ for all $k \le M$ – by the regularity assumptions and (2.4). This implies that the Fourier - Hermite coefficients of $\tilde{p}_k$ decay faster than algebraically (see [10, pp. 43–44] for $d = 1$; the same arguments can be applied in the general case), i.e. for every $\gamma > 0$ there is $C_5(\gamma) > 0$ so that

$$\tilde{\mathbf{E}}\|(1-\Pi^\kappa)\Phi_k T_\Delta \tilde{p}_{k-1}\|^2 \le \frac{C_5(\gamma)}{\kappa^{2\gamma}}. \tag{A.12}$$

Finally, it is shown in [18, Section 4.5.2] that the bound in (A.12) can be replaced by $C_6 \exp\left(-C_7(\ln\kappa)^2\right)$. Combining (A.10), (A.11), and (A.12) results in $\tilde{\mathbf{E}}\|\tilde{p}_k - \bar{p}_k\|^2 \le e^{C_1\Delta}\tilde{\mathbf{E}}\|\tilde{p}_{k-1} - \bar{p}_{k-1}\|^2 + C_6 \exp\left(-C_7(\ln\kappa)^2\right)$, and (A.2) follows by the Gronwall inequality.

To prove (3.7) note that, if $f \in L_2(\mathbb{R}^d)$, then (3.7) is implied by (3.6) and the Cauchy-Schwartz inequality. For general f satisfying (3.1), inequality (3.7) is proved in a similar way by first establishing (3.6) in a weighted space $L_2(\mathbb{R}^d, s)$ for sufficiently large s ([17], inequality of the type (A.6) still holds). Details of the argument can be found in [18, 1].