RECURSIVE MULTIPLE WIENER INTEGRAL EXPANSION FOR NONLINEAR FILTERING OF DIFFUSION PROCESSES

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Abstract

A recursive in time Wiener chaos representation of the optimal nonlinear filter is derived for a time homogeneous diffusion model with uncorrelated noises. The existing representations are either not recursive or require a prior computation of the unnormalized filtering density, which is time consuming. An algorithm is developed for computing a recursive approximation of the filter, and the error of the approximation is obtained. When the parameters of the model are known in advance, the on-line speed of the algorithm can be increased by performing part of the computations off line.

Key Words: nonlinear filtering, Wiener chaos, recursive filters.

1 INTRODUCTION

In a typical filtering model, a non-anticipative functional $f_t(x)$ of the unobserved signal process $(x(t))_{t\geq 0}$ is estimated from the observations $y(s),\ s\leq t$. The best mean square estimate is known to be the conditional expectation $\mathbf{E}[f_t(x)|y(s),\ s\leq t]$, called the optimal filter. When the observation noise is additive, the Kallianpur-Striebel formula (Kallianpur (1980), Liptser and Shiryayev (1992)) provides the representation of the optimal filter as follows:

$$\mathbf{E}[f_t(x)|y(s),\ s\leq t]=\frac{\phi_t[f]}{\phi_t[1]},$$

where $\phi_t[\cdot]$ is a functional called *the unnormalized optimal filter*. In the particular case $f_t(x)=f(x(t))$, there are two approaches to computing $\phi_t[f]$.

In the first approach (Lo and Ng (1983), Mikulevicius and Rozovskii (1995), Ocone (1983)), the functional $\phi_t[f]$ is expanded in a series of multiple integrals with respect to the observation process. This approach can be used to obtain representations of general functionals, but these representations are not recursive in time. In fact, there is no closed form differential equation satisfied by $\phi_t[f]$.

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In the second approach (Kallianpur (1980), Liptser and Shiryayev (1992), Rozovskii (1990)), it is proved that, under certain regularity assumptions, the functional $\phi_t[f]$ can be written as

$$\phi_t[f] = \int f(x)u(t,x)dx \tag{1.1}$$

for some function $u(t,x)$, called *the unnormalized filtering density.* Even though the computation of $u(t,x)$ can be organized recursively in time, and there are many numerical algorithms to do this (Budhiraja and Kallianpur (1995), Elliott and Glowinski (1989), Florchinger and LeGland (1991), Ito (1996), Lototsky et al. (1996), etc.), these algorithms are time consuming because they involve evaluation of $u(t,x)$ at many spatial points. Moreover, computation of $\phi_t[f]$ using this approach requires subsequent evaluation of the integral (1.1).

The objective of the current work is to develop a recursive in time algorithm for computing $\phi_t[f]$ without computing $u(t,x)$. The analysis is based on the multiple integral representation of the unnormalized filtering density (Lototsky et al. (1996), Mikulevicius and Rozovskii (1995), Ocone, (1983)) with subsequent Fourier series expansion in the spatial domain. For simplicity, in this paper we consider a one-dimensional diffusion model with uncorrelated noises. In the proposed algorithm, the computations involving the parameters of the model can be done separately from those involving the observation process. If the parameters of the model are known in advance, this separation can substantially increase the on-line speed of the algorithm.

2 REPRESENTATION OF THE UNNORMALIZED OPTIMAL FILTER

Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a complete probability space, on which standard one-dimensional Wiener processes $(V(t))_{t\geq 0}$ and $(W(t))_{t\geq 0}$ are given. Random processes $(x(t))_{t\geq 0}$ and $(y(t))_{t\geq 0}$ are defined by the equations

$$\begin{aligned} x(t) &= x_0 + \int_0^t b(x(s))ds + \int_0^t \sigma(x(s))dV(s), \\ y(t) &= \int_0^t h(x(s))ds + W(t). \end{aligned} \tag{2.1}$$

In applications, $x(t)$ represents the unobserved state process subject to estimation from the observations $y(s),\ s \leq t$. The σ - algebra generated by $y(s),\ s \leq t$, will be denoted by $\mathcal{F}_t^y$.

The following is assumed about the model (2.1):

(**A1**) The Wiener processes $(V(t))_{t\geq 0}$ and $(W(t))_{t\geq 0}$ are independent of x_0 and of each other;

(**A2**) The functions $b(x)$, $\sigma(x)$, and $h(x)$ are infinitely differentiable and bounded with all the derivatives;

(**A3**) The random variable x_0 has a density $p(x)$, $x \in \mathbf{R}$, so that the function $p = p(x)$ is infinitely differentiable and, together with all the derivatives, decays at infinity faster than any power of x.

Let $f = f(x)$ be a measurable function such that

$$|f(x)| \leq L(1 + |x|^{k_0}) \tag{2.2}$$

for some $k_0 \geq 0$ and $L > 0$. Assumptions (**A2**) and (**A3**) imply that $\mathbf{E}|f(x(t))|^2 < \infty$ for all $t \geq 0$ (Liptser and Shiryayev, 1992). Suppose that $T > 0$ is fixed. It is known (Kallianpur (1980), Liptser and Shiryayev (1992)) that the best mean square estimate of $f(x(t))$ given $y(s),\ s \leq t \leq T$, is $\hat{f}(x(t)) = \mathbf{E}[f(x(t))|\mathcal{F}_t^y]$, and this estimate can be written by the Kallianpur-Striebel formula as follows:

$$\hat{f}(x(t)) = \frac{\tilde{\mathbf{E}}[f(x(t))\rho(t)|\mathcal{F}_t^y]}{\tilde{\mathbf{E}}[\rho(t)|\mathcal{F}_t^y]}, \tag{2.3}$$

where

$$\rho(t) = \exp\Big\{ \int_0^t h(x(s))dy(s) - \frac{1}{2}\int_0^t |h(x(s))|^2 ds \Big\},$$

and $\tilde{\mathbf{E}}$ is the expectation with respect to measure $\tilde{\mathrm{P}}(\bullet) := \int_{\bullet}(\rho(T))^{-1}d\mathrm{P}$. Moreover, under measure $\tilde{\mathrm{P}}$, the observation process $(y(t))_{0\leq t\leq T}$ is a Wiener process independent of $(x(t))_{0\leq t\leq T}$.

The conditional expectation $\tilde{\mathbf{E}}[f(x(t))\rho(t)|\mathcal{F}_t^y]$ will be referred to as the unnormalized optimal filter and will be denoted by $\phi_t[f]$. In this section, a recursive in time representation of $\phi_t[f]$ is derived for an arbitrary function f satisfying (2.2).

It is known (Rozovskii, 1990) that, under assumptions (**A1**) – (**A3**), there exists a random field $u(t,x),\ t \geq 0,\ x \in \mathbf{R}$, called the unnormalized filtering density, such that

$$\phi_t[f] = \int_{\mathbf{R}} u(t,x)f(x)dx. \tag{2.4}$$

Denote by $P_t\,\varphi(x)$ the solution of the equation

$$\begin{aligned}&\frac{\partial v(t,x)}{\partial t} = \frac{1}{2}\frac{\partial^2(\sigma^2(x)v(t,x))}{\partial x^2} - \frac{\partial(b(x)v(t,x))}{\partial x},\ t > 0;\\ &v(0,x) = \varphi(x),\end{aligned}$$

and consider $0 = t_0 < t_1 < \ldots < t_M = T$, a partition of $[0,T]$ with steps $\Delta_i = t_i - t_{i-1}$ (this partition will be fixed hereafter). The following theorem gives a recursive representation of the unnormalized filtering density at the points of the partition.

THEOREM 2.1. (cf. Mikulevicius and Rozovskii (1995), Ocone (1983)) Under assumptions (**A1**) – (**A3**),

$$\begin{aligned}&u(t_0,x) = p(x),\\ &u(t_i,x) = P_t u(t_{i-1},\cdot)(x) +\\ &\sum_{k\geq 1}\int_0^{\Delta_i}\int_0^{s_k}\ldots\int_0^{s_2} P_{t-s_k}h\ldots hP_{s_1}u(t_{i-1},\cdot)(x)dy^{(i)}(s_1)\ldots dy^{(i)}(s_k),\ \mathrm{P-a.s.}\end{aligned} \tag{2.5}$$

for $i = 1,\ldots,M$, where $y^{(i)}(t) = y(t + t_{i-1}) - y(t_{i-1}),\ 0 \leq t \leq \Delta_i$.

Proof. This follows from Theorem 2.3 in Lototsky et al. (1996) and Theorem 3.1 in Ito (1951).

To simplify the further presentation, the following notations are introduced. For an $\mathcal{F}^y_{t_{i-1}}$ - measurable function $g = g(x, \omega)$ and $0 \le t \le \Delta_i$,

$$
\begin{aligned}
&F^{(i)}_0(t, g)(x) := P_t g(x),\\
&F^{(i)}_k(t, g)(x) := \int_0^t \int_0^{s_k} \dots \int_0^{s_2} P_{t-s_k} h \dots h P_{s_1} g(x) dy^{(i)}(s_1) \dots dy^{(i)}(s_k),\ k \ge 1.
\end{aligned}
\tag{2.6}
$$

With these notations, (2.5) becomes

$$
u(t_i, x) = \sum_{k \ge 0} F^{(i)}_k(\Delta_i, u(t_{i-1}, \cdot))(x),\ i = 1, \dots, M. \tag{2.7}
$$

It is known (Ladyzhenskaia et al. (1968), Rozovskii, (1990)) that [1]

$$
||P_t \cdot ||_0 \le e^{Ct} || \cdot ||_0. \tag{2.8}
$$

Then it follows by induction that for every $t \in [0, \Delta_i]$, $i = 1, \dots, M$, and $k \ge 0$, the operator $g \mapsto F^{(i)}_k(t, g)$ is linear and bounded from $L_2(\Omega, \tilde{\mathrm{P}}; L_2(\mathbf{R}))$ to itself and $\tilde{\mathbf{E}}||F^{(i)}_k(t, g)||_0^2 \le e^{Ct}[(Ct)^k / k!]\tilde{\mathbf{E}}||g||_0^2$. This, in particular, implies that $u(t_i, \cdot) \in L_2(\mathbf{R})$ P- and $\tilde{\mathrm{P}}$- a.s.

In the following theorem, the unnormalized filtering density is expanded with respect to an orthonormal basis in $L_2(\mathbf{R})$. With a special choice of the basis, this expansion will be used later to construct the recursive representation of the unnormalized optimal filter.

THEOREM 2.2. If $\{e_n\}_{n \ge 0}$ is an orthonormal basis in $L_2(\mathbf{R})$ and random variables $\psi_n(i),\ n \ge 0,\ i = 0, \dots, M$, are defined recursively by

$$
\begin{aligned}
&\psi_n(0) = (p, e_n)_0,\\
&\psi_n(i) = \sum_{k \ge 0} \Big(\sum_{l \ge 0} (F^{(i)}_k(\Delta_i, e_l), e_n)_0 \psi_l(i-1) \Big),\ i = 1, \dots, M,
\end{aligned}
\tag{2.9}
$$

then

$$
u(t_i, \cdot) = \sum_{n \ge 0} \psi_n(i) e_n,\ \mathrm{P-a.s.} \tag{2.10}
$$

Proof. The proof is carried out by induction. Representation (2.10) is obvious for $i = 0$. If it is assumed for some $i - 1 \ge 0$, then (2.7) and the continuity of $F^{(i)}_k(\Delta_i, \cdot)$ imply that

$$
\begin{aligned}
&\psi_n(i) := (u(t_i, \cdot), e_n)_0 = \sum_{k \ge 0} (F^{(i)}_k(\Delta_i, u(t_{i-1}, \cdot)), e_n)_0 =\\
&\sum_{k \ge 0} \Big(\sum_{l \ge 0} (F^{(i)}_k(\Delta_i, e_l), e_n)_0 \psi_l(i-1) \Big),
\end{aligned}
$$

and (2.10) follows with $\psi_n(i)$ given by (2.9).

REMARK. Direct computations show that the infinite sums in (2.9) can be interchanged even though the double sum need not converge absolutely. The absolute

[1] $|| \cdot ||_0$ and $(\cdot, \cdot)_0$ are the norm and the inner product in $L_2(\mathbf{R})$. The value of the constant C depends only on the parameters of the model and is usually different in different places.

convergence holds if $\sum_n \sqrt{\tilde{\mathbf{E}}|\psi_n(i)|^2} < \infty$, which is the case when $\{e_n\}$ is the Hermite polynomial basis (2.11). For practical computations, both infinite sums in (2.9) must be truncated. These truncations are studied in Section 3.

To get a representation of $\phi_t[f]$, it now seems natural, according to (2.4), to multiply both sides of (2.10) by $f(x)$ and integrate, but this cannot be done in general because (2.10) is an equality in $L_2(\mathbf{R})$ and f need not be square integrable. The difficulty is resolved by choosing a special basis $\{e_n\}$ so that integral $\int_{\mathbf{R}} f(x)e_n(x)dx$ can be defined for every function f satisfying (2.2).

Specifically, let $\{e_n\}$ be the Hermite basis in $L_2(\mathbf{R})$ (Gottlieb and Orszag (1977), Hille and Phillips (1957)):

$$e_n(x) = \frac{1}{\sqrt{2^n \pi^{1/2} n!}} e^{-x^2/2} H_n(x), \tag{2.11}$$

where $H_n(x)$ is the n th Hermite polynomial defined by

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}, \quad n \geq 0.$$

Then the following result is valid.

THEOREM 2.3. If assumptions (**A1**) – (**A3**) and (2.2) hold and e_n is defined by (2.11), then

$$\phi_{t_i}[f] = \sum_{n\geq 0} f_n \psi_n(i), \ \ \mathrm{P} - \text{a.s.}, \tag{2.12}$$

where $f_n = \int_{\mathbf{R}} f(x)e_n(x)dx$ and $\psi_n(i)$ is given by (2.9).
Proof. Condition (2.2) and fast decay of $e_n(x)$ at infinity imply that f_n is well defined for all n. Then (2.12) will follow from (2.4) and (2.10) if the series $\sum_{n\geq 0} f_n\psi_n(i)$ is P - a.s. absolutely convergent for all $i = 0, \ldots, M$. Since measures P and $\tilde{\mathrm{P}}$ are equivalent, it suffices to show that

$$\sum_{n\geq 0} |f_n| \, \tilde{\mathbf{E}}|\psi_n(i)| < \infty. \tag{2.13}$$

Arguments similar to those in Hille and Phillips (1957), paragraph (21.3.3), show that

$$\int_{\mathbf{R}} |x|^{k_0} |e_n(x)| dx \leq C n^{(2k_0+1)/4},$$

which implies that

$$|f_k| \leq C n^{(2k_0+1)/4}. \tag{2.14}$$

On the other hand, it follows from the proof of Theorem 2.6 in Lototsky et al. (1996) that for every integer γ there exists a constant $C(\gamma)$ such that

$$\tilde{\mathbf{E}}|\psi_n(i)| \leq \sqrt{\tilde{\mathbf{E}}|\psi_n(i)|^2} \leq \frac{C(\gamma)}{n^\gamma}. \tag{2.15}$$

Taking γ sufficiently large and combining (2.14) and (2.15) results in (2.13).

REMARK. It is known (Hille and Phillips (1957), paragraph (21.3.2)) that $\sup_x |e_n(x)| \leq C n^{-1/12}$. Together with (2.15), this inequality implies that, for the Hermite basis, the series in (2.10) converges uniformly in $x \in \mathbf{R}$, P - a.s.

3 RECURSIVE APPROXIMATION OF THE UNNORMALIZED OPTIMAL FILTER

It was already mentioned that the infinite sums in (2.9) must be approximated by truncating the number of terms, if the formula is to be used for practical computations. Multiple integrals in (2.6) must also be approximated. The effects of these approximations are studied below.

For simplicity, it is assumed that the partition of $[0,T]$ is uniform ($\Delta_i = \Delta$ for all $i = 1, \ldots, M$). With obvious modifications, the results remain valid for an arbitrary partition.

Given a positive integer κ, define random variables $\psi_{n,\kappa}(i)$, $n = 0, \ldots, \kappa$, $i = 0, \ldots, M$, by

$$\begin{aligned}&\psi_{n,\kappa}(0) = (p, e_n)_0,\\ &\psi_{n,\kappa}(i) = \sum_{l=0}^{\kappa} \Big((P_\Delta e_l, e_n)_0 + (P_\Delta h e_l, e_n)_0 [y(t_i) - y(t_{i-1})] + \\ &(1/2)(P_\Delta h^2 e_l, e_n)_0 [(y(t_i) - y(t_{i-1}))^2 - \Delta] \Big) \psi_{n,\kappa}(i-1),\ i = 1, \ldots, M.\end{aligned} \tag{3.1}$$

Then the corresponding approximations to $u(t_i, x)$ and $\phi_{t_i}[f]$ are

$$\begin{aligned}u_\kappa(t_i, x) &= \sum_{n=0}^{\kappa} \psi_{n,\kappa}(i) e_n(x),\\ \phi_{t_i,\kappa}[f] &= \sum_{n=0}^{\kappa} \psi_{n,\kappa}(i) f_n.\end{aligned} \tag{3.2}$$

The errors of these approximations are given in the following theorem.

THEOREM 3.1. If assumptions (**A1**) – (**A3**) and (2.2) hold and the basis $\{e_n\}$ is chosen according to (2.11) then

$$\max_{1 \le i \le M} \sqrt{\tilde{\mathbf{E}} ||u_\kappa(t_i, \cdot) - u(t_i, \cdot)||_0^2} \le C\Delta + \frac{C(\gamma)}{\kappa^{\gamma - 1/2}\Delta}, \tag{3.3}$$

$$\max_{1 \le i \le M} \sqrt{\tilde{\mathbf{E}} |\phi_{t_i,\kappa}[f] - \phi_{t_i}[f]|^2} \le C\Delta + \frac{C(\gamma)}{\kappa^{\gamma - 1/2}\Delta}. \tag{3.4}$$

Proof. To simplify the presentation, the notation $|||\cdot|||_0 := \sqrt{\tilde{\mathbf{E}} ||\cdot||_0^2}$ will be used. All constants are denoted by C.

1. Proof of (3.3) is carried out in three steps.

Step 1. Define

$$\begin{aligned}u^1(t_0, x) &:= p(x),\\ u^1(t_i, x) &:= \sum_{k=0}^{2} F_k^{(i)}(\Delta, u(t_{i-1}, \cdot))(x).\end{aligned}$$

It is proved in Lototsky et al. (1996), Theorem 2.4, that

$$\max_{0 \le i \le M} |||u(t_i, \cdot) - u^1(t_i, \cdot)|||_0 \le C\Delta. \tag{3.5}$$

Step 2. Define

$$\begin{aligned}
&\bar{F}_0^{(i)}(\Delta, g)(x) := P_\Delta g(x),\\
&\bar{F}_1^{(i)}(\Delta, g)(x) := [y(t_i) - y(t_{i-1})]P_\Delta h g(x),\\
&\bar{F}_2^{(i)}(\Delta, g)(x) := (1/2)[(y(t_i) - y(t_{i-1}))^2 - \Delta]P_\Delta h^2 g(x),
\end{aligned}$$

and then by induction

$$\begin{aligned}
&\bar{u}^1(t_0, x) := p(x),\\
&\bar{u}^1(t_i, x) = \sum_{k=0}^{2} \bar{F}_k^{(i)}(\Delta, \bar{u}^1(t_{i-1}, \cdot))(x).
\end{aligned}$$

Since $y(t) - y(t_{i-1}),\ t > t_{i-1}$, is independent of $\mathcal{F}^y_{t_{i-1}}$ under measure $\tilde{\mathrm{P}}$, it follows that

$$\begin{aligned}
&|||\bar{u}^1(t_i, \cdot) - u^1(t_i, \cdot)|||_0^2 = |||P_\Delta(\bar{u}^1(t_{i-1}, \cdot) - u^1(t_{i-1}, \cdot))|||_0^2 +\\
&|||\sum_{k=0}^{2} \bar{F}_k^{(i)}(\Delta, \bar{u}^1(t_{i-1}, \cdot)) - F_k^{(i)} u^1(t_{i-1}, \cdot))|||_0^2.
\end{aligned} \tag{3.6}$$

Next,

$$\begin{aligned}
&|||\sum_{k=0}^{2} \bar{F}_k^{(i)}(\Delta, \bar{u}^1(t_{i-1}, \cdot)) - F_k^{(i)}(\Delta, u^1(t_{i-1}, \cdot))|||_0^2 \le\\
&4\sum_{k=1}^{2} \Big(|||\bar{F}_k^{(i)}(\Delta, \bar{u}^1(t_{i-1}, \cdot) - u^1(t_{i-1}, \cdot))|||_0^2 +\\
&|||\bar{F}_k^{(i)}(\Delta, u^1(t_{i-1}, \cdot)) - F_k^{(i)}(\Delta, u^1(t_{i-1}, \cdot))|||_0^2\Big).
\end{aligned} \tag{3.7}$$

It follows from (2.8) and the definition of $\bar{F}_k^{(i)}$ that

$$\begin{aligned}
&4|||\bar{F}_k^{(i)}(\Delta, \bar{u}^1(t_{i-1}, \cdot) - u^1(t_{i-1}, \cdot))|||_0^2 \le\\
&\frac{(C\Delta)^k}{k!} e^{C\Delta} |||\bar{u}^1(t_{i-1}, \cdot) - u^1(t_{i-1}, \cdot)|||_0^2,\ k = 1, 2.
\end{aligned} \tag{3.8}$$

The Taylor formula and the definition of P_t imply

$$||P_{\Delta - s} h P_s g - P_\Delta h g||_0^2 \le C s^2 ||g||_{\mathbf{H}^2}^2,$$

where $||\cdot||_{\mathbf{H}^2}$ is the norm in the corresponding Sobolev space. Then

$$\begin{aligned}
&|||\bar{F}_1^{(i)}(\Delta, u^1(t_{i-1}, \cdot)) - F_1^{(i)}(\Delta, u^1(t_{i-1}, \cdot))|||_0^2 \le C\tilde{\mathbf{E}}||u^1(t_{i-1}, \cdot)||_{\mathbf{H}^2}^2 \Delta^3;\\
&|||\bar{F}_2^{(i)}(\Delta, u^1(t_{i-1}, \cdot)) - F_2^{(i)}(\Delta, u^1(t_{i-1}, \cdot))|||_0^2 \le C\tilde{\mathbf{E}}||u^1(t_{i-1}, \cdot)||_{\mathbf{H}^2}^2 \Delta^4.
\end{aligned} \tag{3.9}$$

Finally, the continuity of operator P_t in $\mathbf{H}^2(\mathbf{R})$ (Ladyzhenskaia et al. (1968), Rozovskii, (1990)) implies

$$\tilde{\mathbf{E}}||u^1(t_{i-1}, \cdot)||_{\mathbf{H}^2}^2 \le e^{CT} ||p||_{\mathbf{H}^2}^2 \le C. \tag{3.10}$$

Combining inequalities (3.6) – (3.10) results in

$$|||\bar{u}^1(t_i, \cdot) - u^1(t_i, \cdot)|||_0^2 \le e^{C\Delta} |||\bar{u}^1(t_{i-1}, \cdot) - u^1(t_{i-1}, \cdot)|||_0^2 + C\Delta^3$$

(at least for sufficiently small Δ), which, by the Gronwall Lemma, implies

$$\max_{0\le i\le M} |||\bar{u}^1(t_i,\cdot) - u^1(t_i,\cdot)|||_0^2 \le C\Delta^2. \tag{3.11}$$

Step 3. The same arguments as in the proof of Theorem 2.6 in Lototsky et al. (1996) show that

$$|||\bar{u}^1(t_i,\cdot) - u_\kappa(t_i,\cdot)|||_0 \le \frac{C(\gamma)}{\kappa^{\gamma-1/2}\Delta}. \tag{3.12}$$

Combining (3.5), (3.11), (3.12), and the triangle inequality results in (3.3).

2. The natural way of proving (3.4) is to use (3.3) and the Cauchy inequality. To deal with the technical difficulty that $f \notin L_2(\mathbf{R})$, the following spaces are introduced (Rozovskii (1990), Sec. 4.3): for $r \in \mathbf{R}$, $L_2(\mathbf{R}, r) = \{\varphi : \int_{\mathbf{R}} \varphi^2(x)(1+x^2)^r dx < \infty\}$. The weighted Sobolev spaces $\mathbf{H}^n(\mathbf{R}, r)$ are defined in a similar way. Then $L_2(\mathbf{R}, r)$ is a Hilbert space with inner product

$$(\varphi_1, \varphi_2)_r := \int_{\mathbf{R}} \varphi_1(x)\varphi_2(x)(1+x^2)^r dx$$

and the corresponding norm $||\cdot||_r$. If $\varphi_1 \in L_2(\mathbf{R}, r)$ and $\varphi_2 \in L_2(\mathbf{R}, -r)$, then $\int_{\mathbf{R}}\varphi_1(x)\varphi_2(x)dx$ is well defined and will be denoted by $(\varphi_1, \varphi_2)_0$. Condition (2.2) implies that $f \in L_2(\mathbf{R}, -r)$ for all $r > k_0 + 1/2$. On the other hand, assumptions **(A2)** and **(A3)** imply that $u(t,\cdot) \in L_2(\mathbf{R}, r)$ for all $r \in \mathbf{R}$ (Rozovskii, 1990, Theorem 4.3.2), and the same is true for $u^1(t_i,\cdot)$, $\bar{u}^1(t_i,\cdot)$, and $u_\kappa(t_i,\cdot)$.

Fix an even integer $r > k_0 + 1/2$ and define $\beta(x) := (1+x^2)^{r/2}$. Notation $|||\cdot|||_r := \sqrt{\tilde{\mathbf{E}}||\cdot||_r^2}$ will also be used.

By the Cauchy inequality,

$$\begin{aligned} &\sqrt{\tilde{\mathbf{E}}|\phi_{t_i,\kappa}[f] - \phi_{t_i}[f]|^2} \equiv \sqrt{\tilde{\mathbf{E}}(u_\kappa(t_i,\cdot) - u(t_i,\cdot), f)_0^2} \le \\ &\sqrt{||f||_{-r}^2|||u_\kappa(t_i,\cdot) - u(t_i,\cdot)|||_r^2} \le ||f||_{-r}(|||u(t_i,\cdot) - u^1(t_i,\cdot)|||_r + \\ &|||u^1(t_i,\cdot) - \bar{u}^1(t_i,\cdot)|||_r + |||\bar{u}^1(t_i,\cdot) - u_\kappa(t_i,\cdot)|||_r). \end{aligned} \tag{3.13}$$

.

Since the operator P_t is linear bounded from $\mathbf{H}^n(\mathbf{R}, r)$ to itself (Ladyzhenskaia et al. (1968), Rozovskii (1990)), the arguments of steps 1 and 2 can be repeated to conclude that

$$|||u(t_i,\cdot) - u^1(t_i,\cdot)|||_r + |||u^1(t_i,\cdot) - \bar{u}^1(t_i,\cdot)|||_r \le C\Delta. \tag{3.14}$$

Next, it follows from the proof of Theorem 2.6 in Lototsky et al. (1996) that for every positive integer γ there exists $C(\gamma)$ such that for all $i = 0, \ldots, M$

$$\tilde{\mathbf{E}}(\bar{u}^1(t_i,\cdot), e_n)_0^2 \le \frac{C(\gamma)}{n^{2\gamma+r}}. \tag{3.15}$$

Similarly, by (3.12), there is $C(\gamma)$ so that

$$|||\bar{u}^1(t_i,\cdot) - u_\kappa(t_i,\cdot)|||_0^2 \le \frac{C(\gamma)}{\kappa^{2\gamma+r-1}\Delta^2}. \tag{3.16}$$

On the other hand, repeated application of the relations $e_n' = (\sqrt{n}e_{n-1} - \sqrt{n+1}e_{n+1})/\sqrt{2}$ and $-e_n'' + (1+x^2)e_n = 2(n+1)e_n$ shows that

$$(g, e_n)_{r/2}^2 \le C \sum_{m=n-r/2}^{n+r/2} m^r (g, e_m)_0^2$$

(if $m < 0$, the corresponding term in the sum is set to be zero), and consequently

$$\sum_{n\ge 0} (g, e_n)_{r/2}^2 \le C \sum_{n \ge 0} n^r (g, e_n)_0^2.$$

Combining the last inequality with the identities

$$||g||_r^2 = ||g\beta||_0^2 = \sum_n (g\beta, e_n)_0^2 = \sum_n (g, e_n)_{r/2}^2$$

results in

$$|||\bar{u}^1(t_i,\cdot) - u_\kappa(t_i,\cdot)|||_r^2 = \sum_{n\ge 0} \tilde{\mathbf{E}}(\bar{u}^1(t_i,\cdot) - u_\kappa(t_i,\cdot), e_n)_{r/2}^2 \le$$
$$C \sum_{n\ge 0} n^r \tilde{\mathbf{E}}(\bar{u}^1(t_i,\cdot) - u_\kappa(t_i,\cdot), e_n)_0^2 = C \sum_{n=0}^{\kappa} n^r \tilde{\mathbf{E}}(\bar{u}^1(t_i,\cdot) - u_\kappa(t_i,\cdot), e_n)_0^2 +$$
$$C \sum_{n>\kappa} n^r \tilde{\mathbf{E}}(\bar{u}^1(t_i,\cdot), e_n)_0^2.$$

Now, (3.15) and (3.16) imply

$$|||\bar{u}^1(t_i,\cdot) - u_\kappa(t_i,\cdot)|||_r^2 \le C\kappa^r \sum_{n=1}^{\kappa} \tilde{\mathbf{E}}(\bar{u}^1(t_i,\cdot) - u_\kappa(t_i,\cdot), e_n)_0^2 +$$
$$C(\gamma) \sum_{n>\kappa} n^{-2\gamma} \le \kappa^r |||\bar{u}^1(t_i,\cdot) - u_\kappa(t_i,\cdot)|||_0^2 + \frac{C(\gamma)}{\kappa^{2\gamma-1}} \le \frac{C(\gamma)}{\kappa^{2\gamma-1}\Delta^2}.$$

Together with (3.13) and (3.14), the last inequality implies (3.4).

REMARK. The constants in (3.3) and (3.4) are determined by the bounds on the functions b, σ, h, and p and their derivatives and by the length T of the time interval. The constants in (3.4) also depend on L and k_0 from (2.2).

The error bounds in (3.3) and (3.4) involve two asymptotic parameters: Δ (the size of the partition of the time interval) and κ (the number of the spatial basis functions). With the appropriate choice of these parameters, the errors can be made arbitrarily small.

In Lototsky et al. (1996), the multiple integrals (2.6) were approximated using the Cameron-Martin version of the Wiener chaos decomposition. The analysis was carried out only for the unnormalized filtering density, but the results can be extended to the unnormalized optimal filter $\phi_t[f]$ in the same way as it is done in the present work. The overall error of approximation from Lototsky et al. (1996) has the same order in Δ and κ as (3.3), but the approximation formulas are more complicated.

Formulas (3.1) and (3.2) provide an effective numerical algorithm for computing both the unnormalized filtering density $u(t,x)$ and the unnormalized optimal filter $\phi_t[f]$ independently of each other. If the ultimate goal is an estimate of $f(x(t_i))$ (e.g.

estimation of moments of $x(t_i)$), it can be achieved with a given precision recursively in time without computing $u(t_i, x)$ as an intermediate step. This approach looks especially promising if the parameters of the model (i.e. functions b, σ, h and the initial density p) are known in advance. In this case, the values of $(P_\Delta e_l, e_n)_0$, $(P_\Delta h e_l, e_n)_0$, $(1/2)(P_\Delta h^2 e_l, e_n)_0$, and $f_n = (f, e_n)_0$, $n, l = 1, \ldots, \kappa$, can be pre-computed and stored. When the observations become available, the coefficients $\psi_n(i)$ are computed according to (3.1) and then $\phi_{t_i,\kappa}[f]$ is computed according to (3.2). As a result, the algorithm avoids performing on line the time consuming operations of solving partial differential equations and computing integrals. Moreover, only increments of the observation process are required at each step of the algorithm.

REFERENCES

Budhiraja, A. and Kallianpur, G. (1995). Approximations to the Solution of the Zakai Equations using Multiple Wiener and Stratonovich Integral Expansions, Technical Report 447, Center for Stochastic Processes, University of North Carolina, Chapel Hill, NC 27599-3260.

Elliott, R. J. and Glowinski, R. (1989). Approximations to solutions of the Zakai filtering equation, *Stoch. Anal. Appl., 7(2)*:145–168.

Florchinger, P. and LeGland, F. (1991). Time discretization of the Zakai equation for diffusion processes observed in correlated noise, *Stoch. and Stoch. Rep., 35(4)*:233–256.

Gottlieb, D. and Orszag, S. A. (1977). *Numerical Analysis of Spectral Methods: Theory and Applications*, CBMS-NSF Regional Conference, Series in Applied Mathematics, Vol.26.

Hille, E. and Phillips, R. S. (1957). *Functional Analysis and Semigroups*, Amer. Math. Soc. Colloq. Publ., Vol. XXXI.

Ito, K. (1951). Multiple Wiener integral, *J. Math. Soc. Japan, 3*:157–169.

Ito, K. (1996). Approximation of the Zakai equation for nonlinear filtering, *SIAM J. Cont. Opt.* (to appear).

Kallianpur, G. (1980). *Stochastic Filtering Theory*, Springer.

Ladyzhenskaia, O. A., Solonikov, V. A., and Ural'tseva, N. N. (1968). *Linear and quasi-linear equations of parabolic type*, American Mathematical Society, Providence, Rhode Island.

Liptser, R. S. and Shiryayev, A. N. (1992). *Statistics of Random Processes*, Springer.

Lo, J. T.-H. and Ng, S.-K. (1983). Optimal orthogonal expansion for estimation I: Signal in white Gaussian noise, *Nonlinear Stochastic Problems* (Bucy, R. and Moura, J., ed.), D. Reidel Publ. Company, pp. 291–309.

Lototsky, S., Mikulevicius, R., and Rozovskii, B. L. (1996). Nonlinear Filtering Revisited: A Spectral Approach, *SIAM Journal on Control and Optimization*, to appear.

Mikulevicius, R. and Rozovskii, B. L. (1995). Fourier-Hermite Expansion for Nonlinear Filtering, Festschrift in honor of A. N. Shiryayev.

Ocone, D. (1983). Multiple integral expansions for nonlinear filtering, *Stochastics, 10*:1–30.

Rozovskii, B. L. (1990). *Stochastic Evolution Systems*, Kluwer Academic Publishers.