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STOCHASTIC DIFFERENTIAL EQUATIONS: A WIENER CHAOS APPROACH

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Abstract. A new method is described for constructing a generalized solution for stochastic differential equations. The method is based on the Cameron-Martin version of the Wiener Chaos expansion and provides a unified framework for the study of ordinary and partial differential equations driven by finite- or infinite-dimensional noise with either adapted or anticipating input. Existence, uniqueness, regularity, and probabilistic representation of this Wiener Chaos solution is established for a large class of equations. A number of examples are presented to illustrate the general constructions. A detailed analysis is presented for the various forms of the passive scalar equation and for the first-order Itô stochastic partial differential equation. Applications to nonlinear filtering if diffusion processes and to the stochastic Navier-Stokes equation are also discussed.

Contents

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1. Introduction

Consider a stochastic evolution equation

$$du(t) = (\mathcal{A}u(t) + f(t))dt + (\mathcal{M}u(t) + g(t))dW(t), \tag{1.1}$$

where $\mathcal{A}$ and $\mathcal{M}$ are differential operators, and W is a noise process on a stochastic basis $\mathbb{F} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. Traditionally, this equation is studied under the following assumptions:

- (i) The operator $\mathcal{A}$ is elliptic, the order of the operator $\mathcal{M}$ is at most half the order of $\mathcal{A}$, and a special parabolicity condition holds.
- (ii) The functions f and g are predictable with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$, and the initial condition is $\mathcal{F}_0$-measurable.
- (iii) The noise process W is sufficiently regular.

Under these assumptions, there exists a unique predictable solution u of (1.1) so that $u \in L_2(\Omega \times (0,T); H)$ for $T > 0$ and a suitable function space H (see, for example, Chapter 3 of [42]). Moreover, there are examples showing that the parabolicity condition and the regularity of noise are necessary to have a square integrable solution of (1.1).

The objective of the current paper is to study stochastic differential equations of the type (1.1) without making the above assumptions (i)–(iii). We show that, with a suitable definition of the solution, solvability of the stochastic equation is essentially equivalent to solvability of a deterministic evolution equation $dv = (\mathcal{A}v + \varphi)dt$ for certain functions φ; the operator $\mathcal{A}$ does not even have to be elliptic.

Generalized solutions have been introduced and studied for stochastic differential equations, both ordinary and with partial derivatives, and definitions of such solutions relied on various forms of the Wiener Chaos decomposition. For stochastic ordinary differential equations, Krylov and Veretennikov [20] used multiple Wiener integral expansion to study Ito diffusions with non-smooth coefficients, and more recently, LeJan and Raimond [22] used a similar approach in the construction of stochastic flows. Various versions of the Wiener chaos appear in a number of papers on nonlinear filtering and related topics [2, 25, 33, 39, 46, etc.] The book by Holden et al. [12] presents a systematic approach to the stochastic differential equations based on the white noise theory. See also [10], [40] and the references therein.

For stochastic partial differential equations, most existing constructions of the generalized solution rely on various modifications of the Fourier transform in the infinite-dimensional Wiener Chaos space $L_2(\mathbb{W}) = L_2(\Omega, \mathcal{F}_T^W, \mathbb{P})$. The two main modifications are known as the S-transform [10] and the Hermite transform [12]. The key elements in the development of the theory are the spaces of the test functions and the corresponding distributions. Several constructions of these spaces were suggested by Hida [10], Kondratiev [17], and Nualart and Rozovskii [38]. Both S- and Hermite transforms establish a bijection between the space of generalized random elements and a suitable space of analytic functions. Using the S-transform, Mikulevicius and Rozovskii [33] studied stochastic parabolic equations with non-smooth coefficients, while Nualart and Rozovskii [38] and Potthoff et. al [40] constructed generalized solutions for the equations driven by space-time white noise in more than one spacial dimension. Many other types of equations have been studied, and the book [12] provides a good overview of literature the corresponding results.

In this paper, generalized solutions of (1.1) are defined in the spaces that are even larger than Hida or Kondratiev distribution. The Wiener Chaos space is a separable Hilbert space with a Cameron-Martin basis [3]. The elements of the space with a finite Fourier series expansion provide the natural collection of test functions $\mathcal{D}(L_2(\mathbb{W}))$, an analog of the space $\mathcal{D}(\mathbb{R}^d)$ of smooth compactly supported functions on $\mathbb{R}^d$. The corresponding space of distributions $\mathcal{D}'(L_2(\mathbb{W}))$ is the collection of generalized random elements represented by formal Fourier series. A generalized solution $u = u(t,x)$ of (1.1) is constructed as an element of $\mathcal{D}'(L_2(\mathbb{W}))$ so that the generalized Fourier coefficients satisfy a system of deterministic evolution equations, known as the propagator. If the equation is linear the propagator is a lower-triangular system. We call this solution a Wiener Chaos solution.

The propagator was first introduced by Mikulevicius and Rozovskii in [32], and further studied in [25], as a numerical tool for solving the nonlinear filtering problem. The propagator can also be derived for certain nonlinear equations; in particular, it was used in [31, 34, 35] to study the stochastic Navier-Stokes equation.

The propagator approach to defining the solution of (1.1) has two advantages over the S-transform approach. First, the resulting construction is more general: there are equations for which the Wiener Chaos solution is not in the domain of the S-transform. Indeed, it is shown in Section 14 that, for certain initial conditions, equation $du = u_x dW_t$ has a Wiener Chaos solution for which the S-transform is not defined. On the other hand, by Theorem 8.1 below, if the generalized solution of (1.1) can be defined using the S-transform, then this solution is also a Wiener Chaos solution. Second, there is no problem of inversion: the propagator provides a direct approach to studying the properties of Wiener Chaos solution and computing both the sample trajectories and statistical moments.

Let us emphasize also the following important features of the Wiener Chaos approach:

- The Wiener Chaos solution is a strong solution in the probabilistic sense, that is, it is uniquely determined by the coefficients, free terms, initial condition, and the Wiener process.
- The solution exists under minimal regularity conditions on the coefficients in the stochastic part of the equation and no special measurability restriction on the input.
- The Wiener Chaos solution often serves as a convenient first step in the investigation of the traditional solutions or solutions in weighted stochastic Sobolev spaces that are much smaller then the spaces of Hida or Kondratiev distributions.

To better understand the connection between the Wiener Chaos solution and other notions of the solution, recall that, traditionally, by a solution of a stochastic equation we understand a random process or field satisfying the equation for almost all elementary outcomes. This solution can be either strong or weak in the probabilistic sense.

Probabilistically strong solution is constructed on a prescribed probability space with a specific noise process. Existence of strong solutions requires certain regularity of the coefficients and the noise in the equation. The tools for constructing strong solutions often come from the theory of the corresponding deterministic equations.

Probabilistically weak solution includes not only the solution process but also the stochastic basis and the noise process. This freedom to choose the probability space and the noise process makes the conditions for existence of weak solutions less restrictive than the similar conditions for strong solutions. Weak solutions can be obtained either by considering the corresponding martingale problem or by constructing a suitable Hunt process using the theory of the Dirichlet forms.

There exist equations that have neither weak nor strong solutions in the traditional sense. An example is the bi-linear stochastic heat equation driven by a multiplicative space-time white noise in two or more spatial dimensions: the irregular nature of the noise prevents the existence of a random field that would satisfy the equation for individual elementary outcomes. For such equations, the solution must be defined as a generalized random element satisfying the equation after the randomness has been averaged out.

White noise theory provides one approach for constructing these generalized solutions. The approach is similar to the Fourier integral method for deterministic equations. The white noise solution is constructed on a special white noise probability space by inverting an integral transform; the special structure of the probability space is essential to carry out the inversion. We can therefore say that the white noise solution extends the notion of the probabilistically weak solution. Still, this extension is not a true generalization: when the equation satisfies the necessary regularity conditions, the connection between the white noise and the traditional weak solution is often not clear.

The Wiener chaos approach provides the means for constructing a generalized solution on a prescribed probability space. The Wiener Chaos solution is a formal Fourier series in the corresponding Cameron-Martin basis. The coefficients in the series are uniquely determined by the equation via the propagator system. This representation provides a convenient way for computing numerically the solution and its statistical moments. As a result, the Wiener Chaos solution extends the notion of the probabilistically strong solution. Unlike the white noise approach, this is a bona fide extension: when the equation satisfies the necessary regularity conditions, the Wiener Chaos solution coincides with the traditional strong solution.

After the general discussion of the Wiener Chaos space in Sections 4 and 5, the Wiener Chaos solution for equation (1.1) and the main properties of the solution are studied in Section 6. Several examples illustrate how the Wiener Chaos solution provides a uniform treatment of various types of equations: traditional parabolic, non-parabolic, and anticipating. In particular, for equations with non-predictable input, the Wiener Chaos solution corresponds to the Skorohod integral interpretation of the equation. The initial solution space $\mathcal{D}'(\mathbb{W})$ is too large to provide much *of* interesting information about the solution. Accordingly, Section 7 discusses various weighted Wiener Chaos spaces. These weighted spaces provide the necessary connection between the Wiener Chaos, white noise, and traditional solutions. This connection is studied in Section 8. In Section 9, the Wiener Chaos solution is constructed for degenerate linear parabolic equations and new regularity results are obtained for the solution. Probabilistic representation of the Wiener Chaos solution is studied in Section 10, where a Feynmann-Kac type formula is derived. Sections 11, 12, 13, and 14 discuss the applications of the general results to particular equations: the Zakai filtering equation, the stochastic transport equation, the stochastic Navier-Stokes equation, and a first-order Itô SPDE.

The following notation will be in force throughout the paper: Δ is the Laplace operator, $D_i = \partial/\partial x_i$, $i = 1, \ldots, d$, and summation over the repeated indices is assumed. The space of continuous functions is denoted by $\mathbf{C}$, and H_2^γ, $\gamma \in \mathbb{R}$, is the Sobolev space

$$\left\{ f : \int_{\mathbb{R}} |\hat{f}(y)|^2 (1+|y|^2)^\gamma dy < \infty \right\}, \text{ where } \hat{f} \text{ is the Fourier transform of } f.$$

2. Traditional Solutions of Linear Parabolic Equations

Below is a summary of the Hilbert space theory of linear stochastic parabolic equations. The details can be found in the books [41] and [42]; see also [19]. For a Hilbert space X, $(\cdot,\cdot)_X$ and $\|\cdot\|_X$ denote the inner product and the norm in X.

Definition 2.1. *The triple (V, H, V') of Hilbert spaces is called normal if and only if*

1. *$V \hookrightarrow H \hookrightarrow V'$ and both embeddings $V \hookrightarrow H$ and $H \hookrightarrow V'$ are dense and continuous;*
2. *The space V' is the dual of V relative to the inner product in H;*
3. *There exists a constant $C > 0$ so that $|(h, v)_H| \le C\|v\|_V \|h\|_{V'}$ for all $v \in V$ and $h \in H$.*

For example, the Sobolev spaces $(H_2^{\ell+\gamma}(\mathbb{R}^d), H_2^{\ell}(\mathbb{R}^d), H_2^{\ell-\gamma}(\mathbb{R}^d))$, $\gamma > 0$, $\ell \in \mathbb{R}$, form a normal triple.

Denote by $\langle v', v\rangle$, $v' \in V'$, $v \in V$, the duality between V and V' relative to the inner product in H. The properties of the normal triple imply that $|\langle v', v\rangle| \le C\|v\|_V\|v'\|_{V'}$, and, if $v' \in H$ and $v \in V$, then $\langle v', v\rangle = (v', v)_H$;

Let $\mathbb{F} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge0}, \mathbb{P})$ be a stochastic basis with the usual assumptions. In particular, the sigma-algebras $\mathcal{F}$ and $\mathcal{F}_0$ are $\mathbb{P}$-complete, and the filtration $\{\mathcal{F}_t\}_{t\ge0}$ is right-continuous; for details, see [23, Definition I.1.1]. We assume that $\mathbb{F}$ is rich enough to carry a collection $w_k = w_k(t),\ k \ge 1,\ t \ge 0$ of independent standard Wiener processes.

Given a normal triple (V, H, V') and a family of linear bounded operators $\mathcal{A}(t) : V \to V'$, $\mathcal{M}_k(t) : V \to H$, $t \in [0, T]$, consider the following equation:

$$u(t) = u_0 + \int_0^t (\mathcal{A}u(s) + f(s))ds + \int_0^t (\mathcal{M}_k u(s) + g_k(s))dw_k(s),\ 0 \le t \le T, \tag{2.1}$$

where $T < \infty$ is fixed and non-random and the summation convention is in force.

Assume that, for all $v \in V$,

$$\sum_{k\ge1} \|\mathcal{M}_k(t)v\|_H^2 < \infty,\ t \in [0, T]. \tag{2.2}$$

The input data u_0, f, and g_k are chosen so that

$$\mathbb{E}\left(\|u_0\|_H^2 + \int_0^T \|f(t)\|_{V'}^2 dt + \sum_{k\ge1}\int_0^T \|g_k(t)\|_H^2 dt\right) < \infty, \tag{2.3}$$

u_0 is $\mathcal{F}_0$-measurable, and the processes f, g_k are $\mathcal{F}_t$-adapted, that is, $f(t)$ and each $g_k(t)$ are $\mathcal{F}_t$-measurable for each $t \ge 0$.

Definition 2.2. *An $\mathcal{F}_t$-adapted process $u \in L_2(\mathbb{F}; L_2((0, T); V))$ is called a traditional, or square-integrable, solution of equation (2.1) if, for every $v \in V$, there exists a measurable sub-set Ω' of Ω with $\mathbb{P}(\Omega') = 1$, so that, the equality*

$$(u(t), v)_H = (u_0, v)_H + \int_0^t \langle \mathcal{A}u(s) + f(s), v\rangle ds + \sum_{k\ge1}(\mathcal{M}_k u(s) + g_k(s), v)_H dw_k(s) \tag{2.4}$$

holds on Ω' for all $0 \le t \le T$.

Existence and uniqueness of the traditional solution for (2.1) can be established when the equation is parabolic.

Definition 2.3. *Equation (2.1) is called* ***strongly parabolic*** *if there exists a positive number* ε *and a real number* C_0 *so that, for all* $v \in V$ *and* $t \in [0,T]$*,*

$$2\langle \mathcal{A}(t)v, v\rangle + \sum_{k\geq 1} \|\mathcal{M}(t)_k v\|_H^2 + \varepsilon \|v\|_V^2 \leq C_0 \|v\|_H^2. \tag{2.5}$$

Equation (2.1) is called ***weakly parabolic*** *(or degenerate parabolic) if condition (2.5) holds with* $\varepsilon = 0$*.*

Theorem 2.4. *If (2.3) and (2.5) hold, then there exists a unique traditional solution of (2.1). The solution process* u *is an element of the space*

$$L_2(\mathbb{F}; L_2((0,T);V)) \bigcap L_2(\mathbb{F}; \mathbf{C}((0,T),H))$$

and satisfies

$$\begin{aligned} &\mathbb{E}\left(\sup_{0<t<T} \|u(t)\|_H^2 + \int_0^T \|u(t)\|_V^2 dt\right) \\ &\leq C(C_0,\delta,T)\mathbb{E}\left(\|u_0\|_H^2 + \int_0^T \|f(t)\|_{V'}^2 dt + \sum_{k\geq 1}\int_0^T \|g_k(t)\|_H^2 dt\right). \end{aligned} \tag{2.6}$$

Proof. This follows, for example, from Theorem 3.1.4 in [42]. □

A somewhat different solvability result holds for weakly parabolic equations [42, Section 3.2].

As an application of Theorem 2.4, consider equation

$$\begin{aligned} du(t,x) = (&a_{ij}(t,x)D_iD_ju(t,x) + b_i(t,x)D_iu(t,x) + c(t,x)u(t,x) + f(t,x))dt \\ &+ (\sigma_{ik}(t,x)D_iu(t,x) + \nu_k(t,x)u(t,x) + g_k(t,x))dw_k(t) \end{aligned} \tag{2.7}$$

with $0 < t \leq T,\ x \in \mathbb{R}^d$, and initial condition $u(0,x) = u_0(x)$. Assume that

- (CL1) The functions a_{ij} are bounded and Lipschitz continuous, the functions b_i, c, σ_{ik}, and ν are bounded measurable.
- (CL2) There exists a positive number $\varepsilon > 0$ so that
$$(2a_{ij}(x) - \sigma_{ik}(x)\sigma_{jk}(x))y_iy_j \geq \varepsilon|y|^2,\ x,y \in \mathbb{R}^d,\ t \in [0,T].$$
- (CL3) There exists a positive number K so that, for all $x \in \mathbb{R}^d$, $\sum_{k\geq 1}|\nu_k(x)|^2 \leq K$.
- (CL4) The initial condition $u_0 \in L_2(\Omega; L_2(\mathbb{R}^d))$ is $\mathcal{F}_0$-measurable, the processes $f \in L_2(\Omega \times [0,T]; H_2^{-1}(\mathbb{R}^d))$ and $g_k \in L_2(\Omega \times [0,T]; L_2(\mathbb{R}^d))$ are $\mathcal{F}_t$-adapted, and $\sum_{k\geq 1}\int_0^T \mathbb{E}\|g_k\|_{L_2(\mathbb{R}^d)}^2(t)dt < \infty$.

Theorem 2.5. *Under assumptions (CL1)–(CL4), equation (2.7) has a unique traditional solution*

$$u \in L_2(\mathbb{F}; L_2((0,T); H_2^1(\mathbb{R}^d))) \bigcap L_2(\mathbb{F}; \mathbf{C}((0,T), L_2(\mathbb{R}^d))),$$

and the solution satisfies

$$\begin{aligned} &\mathbb{E}\left(\sup_{0<t<T} \|u\|_{L_2(\mathbb{R}^d)}^2(t) + \int_0^T \|u\|_{H_2^1(\mathbb{R}^d)}^2(t)dt\right) \\ &\leq C(K,\varepsilon,T)\mathbb{E}\left(\|u_0\|_{L_2(\mathbb{R}^d)}^2 + \int_0^T \|f\|_{H_2^{-1}(\mathbb{R}^d)}^2(t)dt + \sum_{k\geq 1}\int_0^T \|g_k\|_{L_2(\mathbb{R}^d)}^2(t)dt\right). \end{aligned} \tag{2.8}$$

Proof. Apply Theorem 2.4 in the normal triple $(H_2^1(\mathbb{R}^d), L_2(\mathbb{R}^d), H_2^{-1}(\mathbb{R}^d))$; condition (2.5) in this case is equivalent to assumption (CL2). The details of the proof are in [42, Section 4.1]. □

Condition (2.5) essentially means that the deterministic part of the equation dominates the stochastic part. Accordingly, there are two main ways to violate (2.5):

(1) The order of the operator $\mathcal{M}$ is more than half the order of the operator $\mathcal{A}$. Equation $du = u_x dw(t)$ is an example.
(2) The value of $\sum_k \|\mathcal{M}_k(t)v\|_H^2$ is too large. This value can be either finite, as in equation $du(t,x) = u_{xx}(t,x)dt + 5u_x(t,x)dw(t)$ or infinite, as in equation

$$du(t,x) = \Delta u(t,x)dt + \sigma_k(x)udw_k,\ \sigma_k - \text{ CONS in } L_2(\mathbb{R}^d),\ d \geq 2. \tag{2.9}$$

Indeed, it is shown in [38] that, for equation (2.9), we have

$$\sum_{k\geq 1} \|\mathcal{M}_k(t)v\|_H^2 = \infty$$

in every Sobolev space H^γ.

Without condition (2.5), analysis of equation (2.1) requires new technical tools and a different notion of solution. The white noise theory provides one possible collection of such tools.

3. White Noise Solutions of Stochastic Parabolic Equations

The central part of the white noise theory is the mathematical model for the derivative of the Brownian motion. In particular, the Itô integral $\int_0^t f(s)dw(s)$ is replaced with the integral $\int_0^t f(s) \diamond \dot{W}(s)ds$, where $\dot{W}$ is the white noise process and $\diamond$ is the Wick product. The white noise formulation is very different from the Hilbert space approach of the previous section, and requires several new constructions. The book [10] is a general reference about the white noise theory, while [12] presents the white noise analysis of stochastic partial differential equations. Below is the summary of the main definitions and results.

Denote by $\mathcal{S} = \mathcal{S}(\mathbb{R}^\ell)$ the Schwartz space of rapidly decreasing functions and by $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^\ell)$, the Schwartz space of tempered distributions. For the properties of the spaces $\mathcal{S}$ and $\mathcal{S}'$ see [43].

Definition 3.1. *The white noise probability space is the triple*

$$\mathbb{S} = (\mathcal{S}', \mathcal{B}(\mathcal{S}'), \mu),$$

where $\mathcal{B}(\mathcal{S}')$ is the Borel sigma-algebra of subsets of $\mathcal{S}'$, and μ is the normalized Gaussian measure on $\mathcal{B}(\mathcal{S}')$.

The measure μ is characterized by the property

$$\int_{\mathcal{S}'} e^{\sqrt{-1}\langle\omega,\varphi\rangle} d\mu(\omega) = e^{-\frac{1}{2}\|\varphi\|^2_{L_2(\mathbb{R}^d)}},$$

where $\langle\omega,\varphi\rangle$, $\omega \in \mathcal{S}'$, $\varphi \in \mathcal{S}$, is the duality between $\mathcal{S}$ and $\mathcal{S}'$. Existence of this measure follows from the Bochner-Minlos theorem [12, Appendix A].

Let $\{\eta_k, k \geq 1\}$ be the Hermite basis in $L_2(\mathbb{R}^\ell)$, consisting of the normalized eigenfunctions of the operator

$$\Lambda = -\Delta + |x|^2,\ x \in \mathbb{R}^\ell. \tag{3.1}$$

Each η_k is an element of $\mathcal{S}$ [12, Section 2.2].

Consider the collection of multi-indices

$$\mathcal{J}_1 = \Big\{\alpha = (\alpha_i,\ i \geq 1),\ \alpha_i \in \{0, 1, 2, \ldots\},\ \sum_i \alpha_i < \infty\Big\}.$$

The set $\mathcal{J}_1$ is countable, and, for every $\alpha \in \mathcal{J}$, only finitely many of α_i are not equal to zero. For $\alpha \in \mathcal{J}_1$, write $\alpha! = \prod_i \alpha_i!$ and define

$$\xi_\alpha(\omega) = \frac{1}{\sqrt{\alpha!}} \prod_i H_{\alpha_i}(\langle \omega, \eta_i \rangle),\ \omega \in \mathcal{S}', \tag{3.2}$$

where $\langle \cdot, \cdot \rangle$ is the duality between $\mathcal{S}$ and $\mathcal{S}'$, and

$$H_n(t) = (-1)^n e^{t^2/2} \frac{d^n}{dt^n} e^{-t^2/2} \tag{3.3}$$

is n^{th} Hermite polynomial. In particular, $H_1(t) = 1$, $H_1(t) = t$, $H_2(t) = t^2 - 1$. If, for example, $\alpha = (0, 2, 0, 1, 3, 0, 0, \ldots)$ has three non-zero entries, then

$$\xi_\alpha(\omega) = \frac{H_2(\langle \omega, \eta_2 \rangle)}{2!} \cdot \langle \omega, \eta_4 \rangle \cdot \frac{H_3(\langle \omega, \eta_5 \rangle)}{3!}.$$

Theorem 3.2. *The collection $\{\xi_\alpha,\ \alpha \in \mathcal{J}_1\}$ is an orthonormal basis in $L_2(\mathbb{S})$.*

Proof. This is a version of the classical result of Cameron and Martin [3]. In this particular form, the result is stated and proved in [12, Theorem 2.2.3]. □

By Theorem 3.2, every element φ of $L_2(\mathbb{S})$ is represented as a Fourier series $\varphi = \sum_\alpha \varphi_\alpha \xi_\alpha$, where $\varphi_\alpha = \int_{\mathcal{S}'} \varphi(\omega) \xi_\alpha(\omega) d\mu$, and $\|\varphi\|^2_{L_2(\mathbb{S})} = \sum_{\alpha \in \mathcal{J}_1} |\varphi_\alpha|^2$.

For $\alpha \in \mathcal{J}_1$ and $q \in \mathbb{R}$, we write

$$(2\mathbb{N})^{q\alpha} = \prod_j (2j)^{q\alpha_j}.$$

Definition 3.3. *For $\rho \in [0, 1]$ and $q \geq 0$,*

1. *the space $(\mathcal{S})_{\rho,q}$ is the collection of elements φ from $L_2(\mathbb{S})$ so that*
$$\|\varphi\|^2_{\rho,q} = \sum_{\alpha \in \mathcal{J}_1} (\alpha!)^\rho (2\mathbb{N})^{q\alpha} |\varphi_\alpha|^2 < \infty;$$
2. *the space $(\mathcal{S})_{-\rho,-q}$ is the closure of $L_2(\mathbb{S})$ relative to the norm*
$$\|\varphi\|^2_{-\rho,-q} = \sum_{\alpha \in \mathcal{J}_1} (\alpha!)^{-\rho} (2\mathbb{N})^{-q\alpha} |\varphi_\alpha|^2; \tag{3.4}$$
3. *the space $(\mathcal{S})_\rho$ is the projective limit of $(\mathcal{S})_{\rho,q}$ as q changes over all non-negative integers;*
4. *the space $(\mathcal{S})_{-\rho}$ is the inductive limit of $(\mathcal{S})_{-\rho,-q}$ as q changes over all non-negative integers.*

It follows that

- For each $\rho \in [0,1]$ and $q \geq 0$, $((\mathcal{S})_{\rho,q}, L_2(\mathbb{S}), (\mathcal{S})_{-\rho,-q})$ is a normal triple of Hilbert spaces.
- The space $(\mathcal{S})_\rho$ is a Frechet space with topology generated by the countable family of norms $\|\cdot\|_{\rho,n}$, $n = 0,1,2,\ldots$, and $\varphi \in (\mathcal{S})_\rho$ if and only if $\varphi \in (\mathcal{S})_{\rho,q}$ for every $q \geq 0$.
- The space $(\mathcal{S})_{-\rho}$ is the dual of $(\mathcal{S})_\rho$ and $\varphi \in (\mathcal{S})_{-\rho}$ if and only if $\varphi \in (\mathcal{S})_{-\rho,-q}$ for some $q \geq 0$. Every element φ from $(\mathcal{S})_\rho$ is identified with a formal sum $\sum_{\alpha\in\mathcal{J}_1} \varphi_\alpha \xi_\alpha$ so that (3.4) holds for some $q \geq 0$.
- For $0 < \rho < 1$,
$$(\mathcal{S})_1 \subset (\mathcal{S})_\rho \subset (\mathcal{S})_0 \subset L_2(\mathbb{S}) \subset (\mathcal{S})_{-0} \subset (\mathcal{S})_{-\rho} \subset (\mathcal{S})_{-1},$$
with all inclusions strict.

The spaces $(\mathcal{S})_0$ and $(\mathcal{S})_1$ are known as the spaces of Hida and Kondratiev test functions. The spaces $(\mathcal{S})_{-0}$ and $(\mathcal{S})_{-1}$ are known as the spaces of Hida and Kondratiev distributions. Sometimes, the spaces $(\mathcal{S})_\rho$ and $(\mathcal{S})_{-\rho}$, $0 < \rho \leq 1$, go under the name of Kondratiev test functions and Kondratiev distributions, respectively.

Let $h \in \mathcal{S}$ and $h_k = \int_{\mathbb{R}^\ell} h(x)\eta_k(x)dx$. Since the asymptotics of n^{th} eigenvalue of the operator Λ in (3.1) is $n^{1/d}$ [11, Chapter 21] and $\Lambda^k h \in \mathcal{S}$ for every positive integer k, it follows that

$$\sum_{k\geq 1} |h_k|^2 k^q < \infty \tag{3.5}$$

for every $q \in \mathbb{R}$.

For $\alpha \in \mathcal{J}_1$ and h_k as above, write $h^\alpha = \prod_j (h_j)^{\alpha_j}$, and define the stochastic exponential

$$\mathcal{E}(h) = \sum_{\alpha\in\mathcal{J}_1} \frac{h^\alpha}{\sqrt{\alpha!}}\xi_\alpha \tag{3.6}$$

Lemma 3.4. *The stochastic exponential $\mathcal{E} = \mathcal{E}(h)$, $h \in \mathcal{S}$, has the following properties:*

- $\mathcal{E}(h) \in (\mathcal{S})_\rho$, $0 < \rho < 1$;
- *For every $q > 0$, there exists a $\delta > 0$ so that $\mathcal{E}(h) \in (\mathcal{S})_{1,q}$ as long as $\sum_{k\geq 1} |h_k|^2 < \delta$.*

Proof. Both properties are verified by direct calculation [12, Chapter 2]. □

Definition 3.5. *The S-transform $S\varphi(h)$ of an element $\varphi = \sum_{\alpha\in\mathcal{J}} \varphi_\alpha \xi_\alpha$ from $(\mathcal{S})_{-\rho}$ is the number*

$$S\varphi(h) = \sum_{\alpha\in\mathcal{J}_1} \frac{h^\alpha}{\sqrt{\alpha!}}\varphi_\alpha, \tag{3.7}$$

where $h = \sum_{k\geq 1} h_k \eta_k \in \mathcal{S}$ and $h^\alpha = \prod_j (h_j)^{\alpha_j}$.

The definition implies that if $\varphi \in (\mathcal{S})_{-\rho,-q}$ for some $q \geq 0$, then $S\varphi(h) = \langle \varphi, \mathcal{E}(h)\rangle$, where $\langle\cdot,\cdot\rangle$ is the duality between $(\mathcal{S})_{\rho,q}$ and $(\mathcal{S})_{-\rho,-q}$ for suitable q. Therefore, if $\rho < 1$, then $S\varphi(h)$ is well-defined for all $h \in \mathcal{S}$, and, if $\rho = 1$, the $S\varphi(h)$ is well-defined for h with sufficiently small $L_2(\mathbb{R}^\ell)$ norm. To give a complete characterization of the S-transform, one additional construction is necessary.

Let $\mathcal{U}^\rho$, $0 \leq \rho < 1$, be the collection of mappings F from $\mathcal{S}$ to the complex numbers so that

1. For every $h_1, h_2 \in \mathcal{S}$, the function $F(h_1+zh_2)$ is an analytic function of the complex variable z.
2. There exist positive numbers K_1, K_2 and an integer number n so that, for all $h \in \mathcal{S}$ and all complex number z,

$$|F(zh)| \le K_1 \exp\left(K_2 \|\Lambda^n h\|_{L_2(\mathbb{R}^d)}^{\frac{2}{1-\rho}} |z|^{\frac{2}{1-\rho}}\right).$$

For $\rho = 1$, let $\mathcal{U}^1$ be the collection of mappings F from $\mathcal{S}$ to the complex numbers so that

1′. There exist $\varepsilon > 0$ and a positive integer n so that, for all $h_1, h_2 \in \mathcal{S}$ with $\|\Lambda^n h_1\|_{L_2(\mathbb{R}^\ell)} < \varepsilon$, the function of a complex variable $z \mapsto F(h_1 + h_2 z)$ is analytic at zero, and
2′. There exists a positive number K so that, for all $h \in \mathcal{S}$ with $\|\Lambda^n h\|_{L_2(\mathbb{R}^\ell)} < \varepsilon$, $|F(h)| \le K$.

Two mappings F, G with properties $1'$ and $2'$ are identified with the same element of $\mathcal{U}^1$ if $F = G$ on an open neighborhood of zero in $\mathcal{S}$.

The following result holds.

Theorem 3.6. *For every $\rho \in [0,1]$, the S-transform is a bijection from $(\mathcal{S})_{-\rho}$ to $\mathcal{U}^\rho$.*

In other words, for every $\varphi \in (\mathcal{S})_{-\rho}$, the S-transform $S\varphi$ is an element of $\mathcal{U}^\rho$, and, for every $F \in \mathcal{U}^\rho$, there exists a unique $\varphi \in (\mathcal{S})_{-\rho}$ so that $S\varphi = F$. This result is proved in [10] when $\rho = 0$, and in [17] when $\rho = 1$.

Definition 3.7. *For φ and ψ from $(\mathcal{S})_{-\rho}$, $\rho \in [0,1]$, the Wick product $\varphi \diamond \psi$ is the unique element of $(\mathcal{S})_{-\rho}$ whose S-transform is $S\varphi \cdot S\psi$.*

If S^{-1} is the inverse S-transform, then

$$\varphi \diamond \psi = S^{-1}(S\varphi \cdot S\psi),$$

Note that, by Theorem 3.6, the Wick product is well defined, because the space $\mathcal{U}^\rho$, $\rho \in [0,1]$ is closed under the point-wise multiplication. Theorem 3.6 also ensures the correctness of the following definition of the white noise.

Definition 3.8. *The white noise $\dot{W}$ on $\mathbb{R}^\ell$ is the unique element of $(\mathcal{S})_0$ whose S transform satisfies $S\dot{W}(h) = h$.*

Remark 3.9. *If $g \in L_p(\mathbb{S})$, $p > 1$, then $g \in (\mathcal{S})_{-0}$* [12, Corollary 2.3.8]*, and the Fourier transform*

$$\hat{g}(h) = \int_{\mathcal{S}'} \exp\left(\sqrt{-1}\langle \omega, h\rangle\right) g(\omega) d\mu(\omega)$$

is defined. Direct calculations [12, Section 2.9] *show that, for those g,*

$$Sg(\sqrt{-1}\, h) = \hat{g}(h)\, e^{\frac{1}{2}\|h\|^2_{L_2(\mathbb{R}^\ell)}}.$$

As a result, the Wick product can be interpreted as a convolution on the infinite-dimensional space $(\mathcal{S})_{-\rho}$.

In the study of stochastic parabolic equations, $\ell = d + 1$ so that the generic point from $\mathbb{R}^{d+1}$ is written as (t, x), $t \in \mathbb{R}$, $x \in \mathbb{R}^d$. As was mentioned earlier, the terms of the type $f dW(t)$ become $f \diamond \dot{W} dt$. The precise connection between the Itô integral and Wick product is discussed, for example, in [12, Section 2.5].

As an example, consider the following equation:

$$u_t(t,x) = a(x)u_{xx}(t,x) + b(x)u_x(t,x) + u_x(t,x) \diamond \dot{W}(t,x),\ 0 < t < T,\ x \in \mathbb{R}, \tag{3.8}$$

with initial condition $u(0,x) = u_0(x)$. In (3.8),

(WN1) $\dot{W}$ is the white noise process on $\mathbb{R}^2$.
(WN2) The initial condition u_0 and the coefficients a, b are bounded and have continuous bounded derivatives up to second order.
(WN3) There exists a positive number ε so that $a(x) \geq \varepsilon$, $x \in \mathbb{R}$.
(WN4) The second-order derivative of a is uniformly Hölder continuous.

The equivalent Itô formulation of (3.8) is

$$du(t,x) = (a(x)u_{xx}(t,x) + b(x)u_x(t,x))dt + e_k(x)u_x(t,x)dw_k(x), \tag{3.9}$$

where $\{e_k,\ k \geq 1\}$ is the Hermite basis in $L_2(\mathbb{R})$.

With $\mathcal{M}_k v = e_k v_x$, we see that condition (2.2) does not hold in any Sobolev space $H_2^\gamma(\mathbb{R})$. In fact, no traditional solution exists in any normal triple of Sobolev space. On the other hand, with a suitable definition of solution, equation (3.8) is solvable in the space $(\mathcal{S})_{-0}$ of Hida distributions.

Definition 3.10. *A mapping $u : \mathbb{R}^d \to (S)_{-\rho}$ is called weakly differentiable with respect to x_i at a point $x^* \in \mathbb{R}^\ell$ if and only if there exists a $U_i(x^*) \in (S)_{-\rho}$ so that, for all $\varphi \in (S)_\rho$, $D_i\langle u(x), \varphi\rangle|_{x=x^*} = \langle U_i(x^*), \varphi\rangle$. In that case, we write $U_i(x^*) = D_i u(x^*)$.*

Definition 3.11. *A mapping u from $[0,T] \times \mathbb{R}$ to $(\mathcal{S})_{-0}$ is called a white noise solution of (3.8) if and only if*

(1) *The weak derivatives u_t, u_x, and u_{xx} exist, in the sense of Definition 3.10, for all $(t,x) \in (0,T) \times \mathbb{R}$.*
(2) *Equality (3.8) holds for all $(t,x) \in (0,T) \times \mathbb{R}^d$.*
(3) $\lim_{t\downarrow 0} u(t,x) = u_0(x)$ *in the topology of $(\mathcal{S})_{-0}$.*

Theorem 3.12. *Under assumptions (WN1)–(WN4), there exists a white noise solution of (3.8). This solution is unique in the class of weakly measurable mappings v from $(0,T) \times \mathbb{R}$ to $(\mathcal{S})_{-0}$, for which there exists a non-negative integer q and a positive number K so that*

$$\int_0^T \int_{\mathbb{R}} \|v(t,x)\|_{-0,-q} e^{-Kx^2} dxdt < \infty.$$

Proof. Consider the S-transformed equation

$$F_t(t,x;h) = a(x)F_{xx}(t,x;h) + b(x)F_x(t,x;h) + F_x(t,x;h)h, \tag{3.10}$$

$0 < t < T,\ x \in \mathbb{R},\ h \in \mathcal{S}(\mathbb{R})$, with initial condition $F(0,x;h) = u_0(x)$. This a deterministic parabolic equation, and one can show, using the probabilistic representation of F, that F, F_t, F_x, and F_{xx} belong to $\mathcal{U}^0$. Then the inverse S-transform of F is a solution of (3.8), and the uniqueness follows from the uniqueness for equation (3.10). The details of the proof are in [40], where a similar equation is considered for $x \in \mathbb{R}^d$. □

Even though the initial condition in (3.8) is deterministic, there are no measurability restrictions on u_0 for the white noise solution to exist; see [12] for more details.

With appropriate modifications, the white noise solution can be defined for equations more general than (3.8). The solution $F = F(t,x;h)$ of the corresponding S-transformed equation determines the regularity of the white noise solution [12, Section 4.1].

Two main advantages of the white noise approach over the Hilbert space approach are

(1) no need for parabolicity condition;
(2) no measurability restrictions on the input data.

Still, there are substantial limitations:

(1) There seems to be little or no connection between the white noise solution and the traditional solution. While white noise solution can, in principle, be constructed for equation (2.7), this solution will be very different from the traditional solution.
(2) There are no clear ways of computing the solution numerically, even with available representations of the Feynmann-Kac type [12, Chapter 4].
(3) The white noise solution, being constructed on a special white noise probability space, is weak in the probabilistic sense. Path-wise uniqueness does not apply to such solutions because of the "averaging" nature of the solution spaces.

4. Generalized Functions on the Wiener Chaos Space

The objective of this section is to introduce the space of generalized random elements on an arbitrary stochastic basis.

Let $\mathbb{F} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a stochastic basis with the usual assumptions and Y, a separable Hilbert space with inner product $(\cdot,\cdot)_Y$ and an orthonormal basis $\{y_k,\ k \geq 1\}$. On $\mathbb{F}$ and Y, consider a cylindrical Brownian motion W, that is, a family of continuous $\mathcal{F}_t$-adapted Gaussian martingales $W_y(t)$, $y \in Y$, so that $W_y(0) = 0$ and $\mathbb{E}(W_{y_1}(t)W_{y_2}(s)) = \min(t,s)(y_1,y_2)_Y$. In particular,

$$w_k(t) = W_{y_k}(t),\ k \geq 1,\ t \geq 0, \tag{4.1}$$

are independent standard Wiener processes on $\mathbb{F}$.

Equivalently, instead of the process W, the starting point can be a system of independent standard Wiener processes $\{w_k,\ k \geq 1\}$ on $\mathbb{F}$. Then, given a separable Hilbert space Y with an orthonormal basis $\{y_k,\ k \geq 1\}$, the corresponding cylindrical Brownian motion W is defined by

$$W_y(t) = \sum_{k\geq 1} (y, y_k)_Y\, w_k(t). \tag{4.2}$$

Fix a non-random $T \in (0,\infty)$ and denote by $\mathcal{F}^W_T$ the sigma-algebra generated by $w_k(t),\ k \geq 1,\ 0 < t < T$. Denote by $L_2(\mathbb{W})$ the collection of $\mathcal{F}^W_T$-measurable square integrable random variables.

We now review construction of the Cameron-Martin basis in the Hilbert space $L_2(\mathbb{W})$.

Let $\mathfrak{m} = \{m_k,\ k \geq 1\}$ be an orthonormal basis in $L_2((0,T))$ so that each m_k belongs to $L_\infty((0,T))$. Define the independent standard Gaussian random variables

$$\xi_{ik} = \int_0^T m_i(s)dw_k(s).$$

Consider the collection of multi-indices

$$\mathcal{J} = \Big\{ \alpha = (\alpha_i^k,\ i,k \geq 1),\ \alpha_i^k \in \{0,1,2,\dots\},\ \sum_{i,k} \alpha_i^k < \infty \Big\}.$$

The set $\mathcal{J}$ is countable, and, for every $\alpha \in \mathcal{J}$, only finitely many of α_i^k are not equal to zero. The upper and lower indices in α_i^k represent, respectively, the space and time components of the noise process W. For $\alpha \in \mathcal{J}$, define

$$|\alpha| = \sum_{i,k} \alpha_i^k,\ \ \alpha! = \prod_{i,k} \alpha_i^k!,$$

and

$$\xi_\alpha = \frac{1}{\sqrt{\alpha!}} \prod_{i,k} H_{\alpha_i^k}(\xi_{ik}), \tag{4.3}$$

where H_n is n^{th} Hermite polynomial. For example, if

$$\alpha = \begin{pmatrix} 0 & 1 & 0 & 3 & 0 & 0 & \cdots \\ 2 & 0 & 0 & 0 & 4 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix}$$

with four non-zero entries $\alpha_2^1 = 1;\ \alpha_4^1 = 3;\ \alpha_1^2 = 2;\ \alpha_5^2 = 4$, then

$$\xi_\alpha = \xi_{2,1} \cdot \frac{H_3(\xi_{4,1})}{\sqrt{3!}} \cdot \frac{H_2(\xi_{1,2})}{\sqrt{2!}} \cdot \frac{H_4(\xi_{5,2})}{\sqrt{4!}}.$$

There are two main differences between (3.2) and (4.3):

(1) The basis (4.3) is constructed on an arbitrary probability space.
(2) In (4.3), there is a clear separation of the time and space components of the noise, and explicit presence of the time-dependent functions m_i facilitates the analysis of evolution equations.

Definition 4.1. *The space $L_2(\mathbb{W})$ is called the Wiener Chaos space. The N-th Wiener Chaos is the linear subspace of $L_2(\mathbb{W})$, generated by $\xi_\alpha,\ |\alpha| = N$.*

The following is another version of the classical results of Cameron and Martin [3].

Theorem 4.2. *The collection $\Xi = \{\xi_\alpha,\ \alpha \in \mathcal{J}\}$ is an orthonormal basis in $L_2(\mathbb{W})$.*

We refer to Ξ as the Cameron-Martin basis in $L_2(\mathbb{W})$. By Theorem 4.2, every element v of $L_2(\mathbb{W})$ can be written as

$$v = \sum_{\alpha \in \mathcal{J}} v_\alpha \xi_\alpha,$$

where $v_\alpha = \mathbb{E}(v\xi_\alpha)$.

We now define the space $\mathcal{D}(L_2(\mathbb{W}))$ of test functions and the space $\mathcal{D}'(L_2(\mathbb{W}); X)$ of X-valued generalized random elements.

Definition 4.3.

(1) The space $\mathcal{D}(L_2(\mathbb{W}))$ is the collection of elements from $L_2(\mathbb{W})$ that can be written in the form

$$v = \sum_{\alpha \in \mathcal{J}_v} v_\alpha \xi_\alpha$$

for some $v_\alpha \in \mathbb{R}$ *and a finite subset* $\mathcal{J}_v$ *of* $\mathcal{J}$.
(2) A sequence v_n *converges to* v *in* $\mathcal{D}(L_2(\mathbb{W}))$ *if and only if* $\mathcal{J}_{v_n} \subseteq \mathcal{J}_v$ *for all* n *and* $\lim_{n\to\infty} |v_{n,\alpha} - v_\alpha| = 0$ *for all* α.

Definition 4.4. *For a linear topological space* X *define the space* $\mathcal{D}'(L_2(\mathbb{W}); X)$ *of* X*-valued generalized random elements as the collection of continuous linear maps from the linear topological space* $\mathcal{D}(L_2(\mathbb{W}))$ *to* X. *Similarly, the elements of* $\mathcal{D}'(L_2(\mathbb{W}); L_1((0,T); X))$ *are called* X*-valued generalized random processes.*

The element u of $\mathcal{D}'(L_2(\mathbb{W}); X)$ can be identified with a formal Fourier series

$$u = \sum_{\alpha\in\mathcal{J}} u_\alpha \xi_\alpha,$$

where $u_\alpha \in X$ are the *generalized Fourier coefficients* of u. For such a series and for $v \in \mathcal{D}(L_2(\mathbb{W}))$, we have

$$u(v) = \sum_{\alpha\in\mathcal{J}_v} v_\alpha u_\alpha.$$

Conversely, for $u \in \mathcal{D}'(L_2(\mathbb{W}); X)$, we define the formal Fourier series of u by setting $u_\alpha = u(\xi_\alpha)$. If $u \in L_2(\mathbb{W})$, then $u \in \mathcal{D}'(L_2(\mathbb{W}); \mathbb{R})$ and $u(v) = \mathbb{E}(uv)$.

By Definition 4.4, a sequence $\{u_n,\ n \geq 1\}$ converges to u in $\mathcal{D}'(L_2(\mathbb{W}); X)$ if and only if $u_n(v)$ converges to $u(v)$ in the topology of X for every $v \in \mathcal{D}(\mathbb{W})$. In terms of generalized Fourier coefficients, this is equivalent to $\lim_{n\to\infty} u_{n,\alpha} = u_\alpha$ in the topology of X for every $\alpha \in \mathcal{J}$.

The construction of the space $\mathcal{D}'(L_2(\mathbb{W}); X)$ can be extended to Hilbert spaces other than $L_2(\mathbb{W})$. Let H be a real separable Hilbert space with an orthonormal basis $\{e_k,\ k \geq 1\}$. Define the space

$$\mathcal{D}(H) = \Big\{ v \in H : v = \sum_{k\in\mathcal{J}_v} v_k e_k,\ v_k \in \mathbb{R},\ \mathcal{J}_v \ -\ \text{a finite subset of } \{1, 2, \ldots\} \Big\}.$$

By definition, v_n converges to v in $\mathcal{D}(H)$ as $n \to \infty$ if and only if $\mathcal{J}_{v_n} \subseteq \mathcal{J}_v$ for all n and $\lim_{n\to\infty} |v_{n,k} - v_k| = 0$ for all k.

For a linear topological space X, $\mathcal{D}'(H; X)$ is the space of continuous linear maps from $\mathcal{D}(H)$ to X. An element g of $\mathcal{D}'(H; X)$ can be identified with a formal series $\sum_{k\geq1} g_k \otimes e_k$ so that $g_k = g(e_k) \in X$ and, for $v \in \mathcal{D}(H)$, $g(v) = \sum_{k\in\mathcal{J}_v} g_k v_k$. If $X = \mathbb{R}$ and $\sum_{k\geq1} g_k^2 < \infty$, then $g = \sum_{k\geq1} g_k e_k \in H$ and $g(v) = (g, v)_H$, the inner product in H. The space X is naturally imbedded into $\mathcal{D}'(H; X)$: if $u \in X$, then $\sum_{k\geq1} u \otimes e_k \in \mathcal{D}'(H; X)$.

A sequence $g_n = \sum_{k\geq1} g_{n,k} \otimes e_k,\ n \geq 1$, converges to $g = \sum_{k\geq1} g_k \otimes e_k$ in $\mathcal{D}'(H; X)$ if and only if, for every $k \geq 1$, $\lim_{n\to\infty} g_{n,k} = g_k$ in the topology of X.

A collection $\{\mathcal{L}_k,\ k \geq 1\}$ of linear operators from X_1 to X_2 naturally defines a linear operator $\mathcal{L}$ from $\mathcal{D}'(H; X_1)$ to $\mathcal{D}'(H; X_2)$:

$$\mathcal{L}\left(\sum_{k\geq1} g_k \otimes e_k \right) = \sum_{k\geq1} \mathcal{L}_k(g_k) \otimes e_k.$$

Similarly, a linear operator $\mathcal{L} : \mathcal{D}'(H; X_1) \to \mathcal{D}'(H; X_2)$ can be identified with a collection $\{\mathcal{L}_k,\ k \geq 1\}$ of linear operators from X_1 to X_2 by setting $\mathcal{L}_k(u) = \mathcal{L}(u \otimes e_k)$. Introduction of spaces $\mathcal{D}'(H;X)$ and the corresponding operators makes it possible to avoid conditions of the type (2.2).

5. The Malliavin Derivative and its Adjoint

In this section, we define an analog of the Itô stochastic integral for generalized random processes.

All notations from the previous section will remain in force. In particular, Y is a separable Hilbert space with a fixed orthonormal basis $\{y_k,\ k \geq 1\}$, and $\Xi = \{\xi_\alpha,\ \alpha \in \mathcal{J}\}$, the Cameron-Martin basis in $L_2(\mathbb{W})$ defined in (4.3).

We start with a brief review of the Malliavin calculus [37].

The *Malliavin derivative* $\mathbb{D}$ is a continuous linear operator from

$$L_2^1(\mathbb{W}) = \Big\{u \in L_2(\mathbb{W}) : \sum_{\alpha\in\mathcal{J}} |\alpha| u_\alpha^2 < \infty\Big\} \tag{5.1}$$

to $L_2\left(\mathbb{W}; (L_2((0,T)) \times Y)\right)$. In particular,

$$(\mathbb{D}\xi_\alpha)(t) = \sum_{i,k} \sqrt{\alpha_i^k}\,\xi_{\alpha^-(i,k)} m_i(t) y_k, \tag{5.2}$$

where $\alpha^-(i,k)$ is the multi-index with the components

$$\Big(\alpha^-(i,k)\Big)_j^l = \begin{cases} \max(\alpha_i^k - 1, 0), & \text{if } i = j \text{ and } k = l, \\ \alpha_j^l, & \text{otherwise.} \end{cases}$$

Note that, for each $t \in [0,T]$, $\mathbb{D}\xi_\alpha(t) \in \mathcal{D}(L_2(\mathbb{W}) \times Y)$. Using (5.2), we extend the operator $\mathbb{D}$ by linearity to the space $\mathcal{D}'(L_2(\mathbb{W}))$:

$$\mathbb{D}\left(\sum_{\alpha\in\mathcal{J}} u_\alpha \xi_\alpha\right) = \sum_{\alpha\in\mathcal{J}} \left(u_\alpha \sum_{i,k} \sqrt{\alpha_i^k}\,\xi_{\alpha^-(i,k)} m_i(t) y_k\right).$$

For the sake of completeness and to justify further definitions, let us establish connection between the Malliavin derivative and the stochastic Itô integral.

If u is an $\mathcal{F}_t^W$-adapted process from $L_2\left(\mathbb{W}; L_2((0,T);Y)\right)$, then $u(t) = \sum_{k\geq 1} u_k(t) y_k$, where the random variable $u_k(t)$ is $\mathcal{F}_t^W$-measurable for each t and k, and

$$\sum_{k\geq 1} \int_0^T \mathbb{E}|u_k(t)|^2 dt < \infty.$$

We define the stochastic Itô integral

$$U(t) = \int_0^t (u(s), dW(s))_Y = \sum_{k\geq 1} \int_0^t u_k(s) dw_k(s). \tag{5.3}$$

Note that $U(t)$ is $\mathcal{F}_t^W$-measurable and $\mathbb{E}|U(t)|^2 = \sum_{k\geq 1} \int_0^t \mathbb{E}|u_k(s)|^2 ds$.

The next result establishes a connection between the Malliavin derivative and the stochastic Itô integral.

Lemma 5.1. *Suppose that u is an $\mathcal{F}_t^W$-adapted process from* $L_2\left(\mathbb{W}; L_2((0,T);Y)\right)$, *and define the process U according to (5.3). Then, for every $0 < t \leq T$ and $\alpha \in \mathcal{J}$,*

$$\mathbb{E}(U(t)\xi_\alpha) = \mathbb{E}\int_0^t (u(s), (\mathbb{D}\xi_\alpha)(s))_Y ds. \tag{5.4}$$

Proof. Define $\xi_\alpha(t) = \mathbb{E}(\xi_\alpha|\mathcal{F}_t^W)$. It is known (see [33] or Remark 8.3 below) that

$$d\xi_\alpha(t) = \sum_{i,k} \sqrt{\alpha_i^k}\xi_{\alpha^-(i,k)}(t)m_i(t)dw_k(t). \tag{5.5}$$

Due to $\mathcal{F}_t^W$-measurability of $u_k(t)$, we have

$$u_{k,\alpha}(t) = \mathbb{E}\Big(u_k(t)\mathbb{E}(\xi_\alpha|\mathcal{F}_t^W)\Big) = \mathbb{E}(u_k(t)\xi_\alpha(t)). \tag{5.6}$$

The definition of U implies $dU(t) = \sum_{k\geq 1} u_k(t)dw_k(t)$, so that, by (5.5), (5.6), and the Itô formula,

$$U_\alpha(t) = \mathbb{E}(U(t)\xi_\alpha) = \int_0^t \sum_{i,k} \sqrt{\alpha_i^k}u_{k,\alpha^-(i,k)}(s)m_i(s)ds. \tag{5.7}$$

Together with (5.2), the last equality implies (5.4). Lemma 5.1 is proved. □

Note that the coefficients $u_{k,\alpha}$ of $u \in L_2(\mathbb{W}; L_2((0,T);H))$ belong to $L_2((0,T))$. We therefore define $u_{k,\alpha,i} = \int_0^T u_{k,\alpha}(t)m_i(t)dt$. Then, by (5.7),

$$U_\alpha(T) = \sum_{i,k} \sqrt{\alpha_i^k}u_{k,\alpha^-(i,k),i}. \tag{5.8}$$

Since $U(T) = \sum_{\alpha\in\mathcal{J}} U_\alpha(T)\xi_\alpha$, we shift the summation index in (5.8) and conclude that

$$U(T) = \sum_{\alpha\in\mathcal{J}}\sum_{i,k} \sqrt{\alpha_i^k + 1}u_{k,\alpha,i}\xi_{\alpha^+(i,k)}, \tag{5.9}$$

where

$$\Big(\alpha^+(i,k)\Big)_j^l = \begin{cases} \alpha_i^k + 1, & \text{if } i = j \text{ and } k = l, \\ \alpha_j^l, & \text{otherwise.} \end{cases} \tag{5.10}$$

As a result, $U(T) = \delta(u)$, where δ is the adjoint of the Malliavin derivative, also known as the Skorokhod integral; see [37] or [38] for details.

Lemma 5.1 suggests the following definition. For an $\mathcal{F}_t^W$-adapted process u from $L_2\left(\mathbb{W}; L_2((0,T))\right)$, let $\mathbb{D}_k^* u$ be the $\mathcal{F}_t^W$-adapted process from $L_2\left(\mathbb{W}; L_2((0,T))\right)$ so that

$$(\mathbb{D}_k^* u)_\alpha(t) = \int_0^t \sum_i \sqrt{\alpha_i^k}u_{\alpha^-(i,k)}(s)m_i(s)ds. \tag{5.11}$$

If $u \in L_2\left(\mathbb{W}; L_2((0,T);Y)\right)$ is $\mathcal{F}_t^W$-adapted, then u is in the domain of the operator δ and $\delta(uI(s<t)) = \sum_{k\geq 1}(\mathbb{D}_k^* u_k)(t)$.

We now extend the operators $\mathbb{D}_k^*$ to the generalized random processes. Let X be a Banach space with norm $\|\cdot\|_X$.

Definition 5.2. *If u is an X-valued generalized random process, then $\mathbb{D}_k^*u$ is the X-valued generalized random process so that*

$$(\mathbb{D}_k^*u)_\alpha(t) = \sum_i \int_0^t u_{\alpha^-(i,k)}(s)\sqrt{\alpha_i^k}m_i(s)ds. \tag{5.12}$$

If $g \in \mathcal{D}'\Big(Y;\mathcal{D}'\left(L_2(\mathbb{W});L_1((0,T);X)\right)\Big)$, then $\mathbb{D}^*g$ is the X-valued generalized random process so that, for $g=\sum_{k\geq 1} g_k\otimes y_k,\ g_k\in\mathcal{D}'(L_2(\mathbb{W});L_1((0,T);X))$,

$$(\mathbb{D}^*g)_\alpha(t) = \sum_k (\mathbb{D}_k^*g_k)_\alpha(t) = \sum_{i,k}\int_0^t g_{k,\alpha^-(i,k)}(s)\sqrt{\alpha_i^k}m_i(s)ds. \tag{5.13}$$

Using (5.2), we get a generalization of equality (5.4):

$$(\mathbb{D}^*g)_\alpha(t) = \int_0^t g(\mathbb{D}\xi_\alpha(s))(s)ds. \tag{5.14}$$

Indeed, by linearity,

$$g_k\left(\sqrt{\alpha_i^k}m_i(s)\xi_{\alpha^-(i,k)}\right)(s) = \sqrt{\alpha_i^k}m_i(s)g_{k,\alpha^-(i,k)})(s).$$

Theorem 5.3. *If $T<\infty$, then $\mathbb{D}_k^*$ and $\mathbb{D}^*$ are continuous linear operators.*

Proof. It is enough to show that, if $u,u_n\in\mathcal{D}'\left(L_2(\mathcal{F}_T^W);L_1((0,T);X)\right)$ and $\lim_{n\to\infty}\|u_\alpha-u_{n,\alpha}\|_{L_1((0,T);X)}=0$ for every $\alpha\in\mathcal{J}$, then, for every $k\geq 1$ and $\alpha\in\mathcal{J}$, $\lim_{n\to\infty}\|(\mathbb{D}_k^*u)_\alpha-(\mathbb{D}_k^*u_n)_\alpha\|_{L_1((0,T);X)}=0$.

Using (5.12), we find

$$\|(\mathbb{D}_k^*u)_\alpha-(\mathbb{D}_k^*u_n)_\alpha\|_X(t)\leq\sum_i\int_0^T\sqrt{\alpha_i^k}\|u_{\alpha^-(i,k)}-u_{n,\alpha^-(i,k)}\|_X(s)|m_i(s)|ds.$$

Note that the sum contains finitely many terms. By assumption, $|m_i(t)|\leq C_i$, and so

$$\|(\mathbb{D}_k^*u)_\alpha-(\mathbb{D}_k^*u_n)_\alpha\|_{L_1((0,T);X)}\leq C(\alpha)\sum_i\sqrt{\alpha_i^k}\|u_{\alpha^-(i,k)}-u_{n,\alpha^-(i,k)}\|_{L_1((0,T);X)}.$$

Theorem 5.3 is proved. □

6. The Wiener Chaos Solution and the Propagator

In this section we build on the ideas from [25] to introduce the Wiener Chaos solution and the corresponding propagator for a general stochastic evolution equation. The notations from Sections 4 and 5 will remain in force. It will be convenient to interpret the cylindrical Brownian motion W as a collection $\{w_k,\ k\geq 1\}$ of independent standard Wiener processes. As before, $T\in(0,\infty)$ is fixed and non-random. Introduce the following objects:

- The Banach spaces A, X, and U so that $U\subseteq X$.
- Linear operators
$$\mathcal{A}: L_1((0,T);A)\to L_1((0,T);X) \text{ and}$$
$$\mathcal{M}_k: L_1((0,T);A)\to L_1((0,T);X).$$
- Generalized random processes $f\in\mathcal{D}'\left(L_2(\mathbb{W});L_1((0,T);X)\right)$ and $g_k\in\mathcal{D}'\left(L_2(\mathbb{W});L_1((0,T);X)\right)$.

- The initial condition $u_0 \in \mathcal{D}'\left(L_2(\mathbb{W}); U\right)$.

Consider the deterministic equation

$$v(t) = v_0 + \int_0^t (\mathcal{A}v)(s)ds + \int_0^t \varphi(s)ds, \tag{6.1}$$

where $v_0 \in U$ and $\varphi \in L_1((0,T); X)$.

Definition 6.1. *A function v is called a $w(A,X)$ solution of (6.1) if and only if $v \in L_1((0,T); A)$ and equality (6.1) holds in the space $L_1((0,T); A)$.*

Definition 6.2. *An A-valued generalized random process u is called a $w(A,X)$* **Wiener Chaos solution** *of the stochastic differential equation*

$$du(t) = (\mathcal{A}u(t) + f(t))dt + (\mathcal{M}_k u(t) + g_k(t))dw_k(t),\ 0 < t \leq T,\ u|_{t=0} = u_0, \tag{6.2}$$

if and only if the equality

$$u(t) = u_0 + \int_0^t (\mathcal{A}u + f)(s)ds + \sum_{k \geq 1} (\mathbb{D}_k^*(\mathcal{M}_k u + g_k))(t) \tag{6.3}$$

holds in $\mathcal{D}'\left(L_2(\mathbb{W}); L_1((0,T); X)\right)$.

Sometimes, to stress the dependence of the Wiener Chaos solution on the terminal time T, the notation $w_T(A,X)$ will be used.

Equalities (6.3) (5.13) mean that, for every $\alpha \in \mathcal{J}$, the generalized Fourier coefficient u_α of u satisfies

$$u_\alpha(t) = u_{0,\alpha} + \int_0^t (\mathcal{A}u + f)_\alpha(s)ds + \int_0^t \sum_{i,k} \sqrt{\alpha_i^k}(\mathcal{M}_k u + g_k)_{\alpha^-(i,k)}(s) m_i(s)ds. \tag{6.4}$$

Definition 6.3. *System (6.4) is called the propagator for equation (6.2).*

The propagator is a lower triangular system. Indeed, If $\alpha = (0)$, that is, $|\alpha| = 0$, then the corresponding equation in (6.4) becomes

$$u_{(0)}(t) = u_{0,(0)} + \int_0^t (\mathcal{A}u_{(0)}(s) + f_{(0)}(s))ds. \tag{6.5}$$

If $\alpha = (j\ell)$, that is, $\alpha_j^\ell = 1$ for some fixed j and ℓ and $\alpha_i^k = 0$ for all other $i,k \geq 1$, then the corresponding equation in (6.4) becomes

$$\begin{aligned} u_{(j\ell)}(t) = u_{0,(j\ell)} &+ \int_0^t (\mathcal{A}u_{(j\ell)}(s) + f_{(j\ell)}(s))ds \\ &+ \int_0^t (\mathcal{M}_k u_{(0)}(s) + g_{\ell,(0)}(s)) m_j(s)ds. \end{aligned} \tag{6.6}$$

Continuing in this way, we conclude that (6.4) can be solved by induction on $|\alpha|$ as long as the corresponding deterministic equation (6.1) is solvable. The precise result is as follows.

Theorem 6.4. *If, for every $v_0 \in U$ and $\varphi \in L_1((0,T); X)$, equation (6.1) has a unique $w(A,X)$ solution $v(t) = V(t, v_0, \varphi)$, then equation (6.2) has a unique $w(A,X)$ Wiener Chaos* *solution so that*

$$
u_\alpha(t) = V(t, u_{0,\alpha}, f_\alpha) + \sum_{i,k} \sqrt{\alpha_i^k} V(t, 0, m_i \mathcal{M}_k u_{\alpha^-(i,k)}) + \sum_{i,k} \sqrt{\alpha_i^k} V(t, 0, m_i g_{k,\alpha^-(i,k)}). \tag{6.7}
$$

Proof. Using the assumptions of the theorem and linearity, we conclude that (6.7) is the unique solution of (6.4). □

To derive a more explicit formula for u_α, we need some additional constructions. For every multi-index α with $|\alpha| = n$, define the **characteristic set** K_α of α so that

$$
K_\alpha = \{(i_1^\alpha, k_1^\alpha), \ldots, (i_n^\alpha, k_n^\alpha)\},
$$

$i_1^\alpha \leq i_2^\alpha \leq \ldots \leq i_n^\alpha$, and if $i_j^\alpha = i_{j+1}^\alpha$, then $k_j^\alpha \leq k_{j+1}^\alpha$. The first pair (i_1^α, k_1^α) in K_α is the position numbers of the first nonzero element of α. The second pair is the same as the first if the first nonzero element of α is greater than one; otherwise, the second pair is the position numbers of the second nonzero element of α and so on. As a result, if $\alpha_i^k > 0$, then exactly α_i^k pairs in K_α are equal to (i, k). For example, if

$$
\alpha = \begin{pmatrix} 0 & 1 & 0 & 2 & 3 & 0 & 0 & \cdots \\ 1 & 2 & 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix}
$$

with nonzero elements

$$
\alpha_1^2 = \alpha_2^1 = \alpha_1^6 = 1,\ \alpha_2^2 = \alpha_4^1 = 2,\ \alpha_5^1 = 3,
$$

then the characteristic set is

$$
K_\alpha = \{(1,2),\ (2,1),\ (2,2),\ (2,2),\ (4,1),\ (4,1),\ (5,1),\ (5,1),\ (5,1),\ (6,2)\}.
$$

Theorem 6.5. *Assume that*

1. *for every $v_0 \in U$ and $\varphi \in L_1((0,T); X)$, equation (6.1) has a unique $w(A, X)$ solution $v(t) = V(t, v_0, \varphi)$,*
2. *the input data in (6.4) satisfy $g_k = 0$ and $f_\alpha = u_{0,\alpha} = 0$ if $|\alpha| > 0$.*

Let $u_{(0)}(t) = V(t, u_0, 0)$ be the solution of (6.4) for $|\alpha| = 0$. For $\alpha \in \mathcal{J}$ with $|\alpha| = n \geq 1$ and the characteristic set K_α, define functions $F^n = F^n(t; \alpha)$ by induction as follows:

$$
\begin{aligned}
F^1(t;\alpha) &= V(t, 0, m_i \mathcal{M}_k u_{(0)}) \text{ if } K_\alpha = \{(i,k)\}; \\
F^n(t;\alpha) &= \sum_{j=1}^{n} V(t, 0, m_{i_j} \mathcal{M}_{k_j} F^{n-1}(\cdot; \alpha^-(i_j, k_j))) \\
&\qquad \text{if } K_\alpha = \{(i_1, k_1), \ldots, (i_n, k_n)\}.
\end{aligned} \tag{6.8}
$$

Then

$$
u_\alpha(t) = \frac{1}{\sqrt{\alpha!}} F^n(t; \alpha). \tag{6.9}
$$

Proof. If $|\alpha| = 1$, then representation (6.9) follows from (6.6). For $|\alpha| > 1$, observe that

- If $\bar{u}_\alpha(t) = \sqrt{\alpha!}u_\alpha$ and $|\alpha| \geq 1$, then (6.4) implies
$$\bar{u}(t) = \int_0^t \mathcal{A}\bar{u}_\alpha(s)ds + \sum_{i,k}\int_0^t \alpha_i^k m_i(s)\mathcal{M}_k\bar{u}_{\alpha^-(i,k)}(s)ds.$$
- If $K_\alpha = \{(i_1,k_1),\dots,(i_n,k_n)\}$, then, for every $j=1,\dots,n$, the characteristic set $K_{\alpha^-(i_j,k_j)}$ of $\alpha^-(i_j,k_j)$ is obtained from K_α by removing the pair (i_j,k_j).
- By the definition of the characteristic set,
$$\sum_{i,k}\alpha_i^k m_i(s)\mathcal{M}_k\bar{u}_{\alpha^-(i,k)}(s) = \sum_{j=1}^n m_{i_j}(s)\mathcal{M}_{k_j}\bar{u}_{\alpha^-(i_j,k_j)}(s).$$

As a result, representation (6.9) follows by induction on $|\alpha|$ using (6.7):
if $|\alpha| = n > 1$, then

$$\begin{aligned}\bar{u}_\alpha(t) &= \sum_{j=1}^n V(t,0,m_{i_j}\mathcal{M}_{k_j}\bar{u}_{\alpha^-(i_j,k_j)})\\ &= \sum_{j=1}^n V(t,0,m_{i_j}\mathcal{M}_{k_j}F^{(n-1)}(\cdot;\alpha^-(i_j,k_j)) = F^n(t;\alpha).\end{aligned}\tag{6.10}$$

Theorem 6.5 is proved. □

Corollary 6.6. *Assume that the operator $\mathcal{A}$ is a generator of a strongly continuous semigroup $\Phi = \Phi_{t,s}$, $t \geq s \geq 0$, in some Hilbert space H so that $A \subset H$, each $\mathcal{M}_k$ is a bounded operator from A to H, and the solution $V(t,0,\varphi)$ of equation (6.1) is written as*

$$V(t,0,\varphi) = \int_0^T \Phi_{t,s}\varphi(s)ds, \quad \varphi \in L_2((0,T);H)).\tag{6.11}$$

Denote by $\mathcal{P}^n$ the permutation group of $\{1,\dots,n\}$. If $u_{(0)} \in L_2((0,T);H))$, then, for $|\alpha| = n > 1$ with the characteristic set $K_\alpha = \{(i_1,k_1),\dots,(i_n,k_n)\}$, representation (6.9) becomes

$$\begin{aligned}u_\alpha(t) = \frac{1}{\sqrt{\alpha!}}\sum_{\sigma\in\mathcal{P}^n}\int_0^t\int_0^{s_n}\dots\int_0^{s_2}\\ \Phi_{t,s_n}\mathcal{M}_{k_{\sigma(n)}}\cdots\Phi_{s_2,s_1}\mathcal{M}_{k_{\sigma(1)}}u_{(0)}(s_1)m_{i_{\sigma(n)}}(s_n)\cdots m_{i_{\sigma(1)}}(s_1)ds_1\dots ds_n.\end{aligned}\tag{6.12}$$

Also,

$$\begin{aligned}\sum_{|\alpha|=n}u_\alpha(t)\xi_\alpha = \sum_{k_1,\dots,k_n\geq 1}\int_0^t\int_0^{s_n}\dots\int_0^{s_2}\\ \Phi_{t,s_n}\mathcal{M}_{k_n}\cdots\Phi_{s_2,s_1}\left(\mathcal{M}_{k_1}u_{(0)} + g_{k_1}(s_1)\right)dw_{k_1}(s_1)\cdots dw_{k_n}(s_n),\ n\geq 1,\end{aligned}\tag{6.13}$$

and, for every Hilbert space X, the following energy equality holds:

$$\begin{aligned}\sum_{|\alpha|=n}\|u_\alpha(t)\|_X^2 = \sum_{k_1,\dots,k_n=1}^\infty\int_0^t\int_0^{s_n}\dots\int_0^{s_2}\\ \|\Phi_{t,s_n}\mathcal{M}_{k_n}\cdots\Phi_{s_2,s_1}\mathcal{M}_{k_1}u_{(0)}(s_1)\|_X^2 ds_1\dots ds_n;\end{aligned}\tag{6.14}$$

both sides in the last equality can be infinite. For $n=1$, formulas (6.12) and (6.14) become

$$u_{(ik)}(t) = \int_0^t \Phi_{t,s}\mathcal{M}_k u_{(0)}(s)\ m_i(s)ds;\tag{6.15}$$

$$\sum_{|\alpha|=1} \|u_\alpha(t)\|_X^2 = \sum_{k=1}^{\infty} \int_0^t \|\Phi_{t,s}\mathcal{M}_k u_{(0)}(s)\|_X^2 ds. \tag{6.16}$$

Proof. Using the semi-group representation (6.11), we conclude that (6.12) is just an expanded version of (6.9).

Since $\{m_i,\ i \geq 1\}$ is an orthonormal basis in $L_2(0,T)$, equality (6.16) follows from (6.15) and the Parcevall identity. Similarly, equality (6.14) will follow from (6.12) after an application of an appropriate Parcevall's identity.

To carry out the necessary arguments when $|\alpha| > 1$, denote by $\mathcal{J}_1$ the collection of one-dimensional multi-indices $\beta = (\beta_1, \beta_2, \ldots)$ so that each β_i is a non-negative integer and $|\beta| = \sum_{i\geq 1} \beta_i < \infty$. Given a $\beta \in \mathcal{J}_1$ with $|\beta| = n$, we define $K_\beta = \{i_1, \ldots, i_n\}$, the characteristic set of β and the function

$$E_\beta(s_1, \ldots, s_n) = \frac{1}{\sqrt{\beta! n!}} \sum_{\sigma \in \mathcal{P}^n} m_{i_1}(s_{\sigma(1)}) \cdots m_{i_n}(s_{\sigma(n)}). \tag{6.17}$$

By construction, the collection $\{E_\beta, \beta \in \mathcal{J}_1, |\beta| = n\}$ is an orthonormal basis in the subspace of symmetric functions in $L_2((0,T)^n; X)$.

Next, we re-write (6.12) in a symmetrized form. To make the notations shorter, denote by $s^{(n)}$ the ordered set $(s_1, \ldots, s_n)$ and write $ds^n = ds_1 \ldots ds_n$. Fix $t \in (0,T]$ and the set $k^{(n)} = \{k_1, \ldots, k_n\}$ of the second components of the characteristic set K_α. Define the symmetric function

$$\begin{aligned} &G(t, k^{(n)}; s^{(n)}) \\ &= \frac{1}{\sqrt{n!}} \sum_{\sigma \in \mathcal{P}^n} \Phi_{t, s_{\sigma(n)}} \mathcal{M}_{k_n} \cdots \Phi_{s_{\sigma(2)}, s_{\sigma(1)}} \mathcal{M}_{k_1} u_{(0)}(s_{\sigma(1)}) 1_{s_{\sigma(1)} < \cdots < s_{\sigma(n)} < t}(s^{(n)}). \end{aligned} \tag{6.18}$$

Then (6.12) becomes

$$u_\alpha(t) = \int_{[0,T]^n} G(t, k^{(n)}; s^{(n)}) E_{\beta(\alpha)}(s^{(n)}) ds^n, \tag{6.19}$$

where the multi-indices α and $\beta(\alpha)$ are related via their characteristic sets: if

$$K_\alpha = \{(i_1, k_1), \ldots, (i_n, k_n)\},$$

then

$$K_{\beta(\alpha)} = \{i_1, \ldots, i_n\}.$$

Equality (6.19) means that, for fixed $k^{(n)}$, the function u_α is a Fourier coefficient of the symmetric function $G(t, k^{(n)}; s^{(n)})$ in the space $L_2((0,T)^n; X)$. Parcevall's identity and summation over all possible $k^{(n)}$ yield

$$\sum_{|\alpha|=n} \|u_\alpha(t)\|_X^2 = \frac{1}{n!} \sum_{k_1, \ldots, k_n = 1}^{\infty} \int_{[0,T]^n} \|G(t, k^{(n)}; s^{(n)})\|_X^2 ds^n,$$

which, due to (6.18), is the same as (6.14).

To prove equality (6.13), relating the Cameron-Martin and multiple Itô integral expansions of the solution, we use the following result [13, Theorem 3.1]:

$$\xi_\alpha = \frac{1}{\sqrt{\alpha!}} \int_0^T \int_0^{s_n} \cdots \int_0^{s_2} E_{\beta(\alpha)}(s^{(n)}) dw_{k_1}(s_1) \cdots dw_{k_n}(s_n);$$

see also [37, pp. 12–13]. Since the collection of all E_β is an orthonormal basis, equality (6.13) follows from (6.19) after summation over al $k_1, \dots, k_n$.

Corollary 6.6 is proved. □

We now present several examples to illustrate the general results.

Example 6.7. Consider the following equation:

$$du(t,x) = (au_{xx}(t,x) + f(t,x))dt + (\sigma u_x(t,x) + g(t,x))dw(t),\ t>0,\ x\in\mathbb{R}, \tag{6.20}$$

where $a>0$, $\sigma\in\mathbb{R}$, $f\in L_2((0,T);H_2^{-1}(\mathbb{R}))$, $g\in L_2((0,T);L_2(\mathbb{R}))$, and $u|_{t=0}=u_0\in L_2(\mathbb{R})$. By Theorem 2.5, if $\sigma^2<2a$, then equation (6.20) has a unique traditional solution $u\in L_2\left(\mathbb{W}; L_2((0,T);H_2^1(\mathbb{R}))\right)$.

By $\mathcal{F}_t^W$-measurability of $u(t)$, we have

$$\mathbb{E}(u(t)\xi_\alpha) = \mathbb{E}(u(t)\mathbb{E}(\xi_\alpha|\mathcal{F}_t^W)).$$

Using the relation (5.5) and the Itô formula, we find that u_α satisfy

$$du_\alpha = a(u_\alpha)_{xx}dt + \sum_i \sqrt{\alpha_i}\sigma(u_{\alpha^-(i)})_x m_i(t)dt,$$

which is precisely the propagator for equation (6.20). In other words, if $2a>\sigma^2$, then the traditional solution of (6.20) coincides with the Wiener Chaos solution.

On the other hand, the heat equation

$$v(t,x) = v_0(x) + \int_0^t v_{xx}(s,x)ds + \int_0^t \varphi(s,x)ds,\ v_0\in L_2(\mathbb{R})$$

with $\varphi\in L_2((0,T);H_2^{-1}(\mathbb{R}))$ has a unique $w(H_2^1(\mathbb{R}),H_2^{-1}(\mathbb{R}))$ solution. Therefore, by Theorem 6.4, the unique $w(H_2^1(\mathbb{R}),H_2^{-1}(\mathbb{R}))$ Wiener Chaos solution of (6.20) exists for all $\sigma\in\mathbb{R}$.

In the next example, the equation, although not parabolic, can be solved explicitly.

Example 6.8. Consider the following equation:

$$du(t,x) = u_x(t,x)dw(t),\ t>0,\ x\in\mathbb{R};\quad u(0,x)=x. \tag{6.21}$$

Clearly, $u(t,x)=x+w(t)$ satisfies (6.21).

To find the Wiener Chaos solution of (6.21), note that, with one-dimensional Wiener process, $\alpha_i^k=\alpha_i$, and the propagator in this case becomes

$$u_\alpha(t,x) = xI(|\alpha|=0) + \int_0^t \sum_i \sqrt{\alpha_i}(u_{\alpha^-(i)}(s,x))_x m_i(s)ds.$$

Then $u_\alpha=0$ if $|\alpha|>1$, and

$$u(t,x) = x + \sum_{i\ge1}\xi_i\int_0^t m_i(s)ds = x+w(t). \tag{6.22}$$

Even though Theorem 6.4 does not apply, the above arguments show that $u(t,x)=x+w(t)$ is the unique $w(A,X)$ Wiener Chaos solution of (6.21) for suitable spaces A and X, for example,

$$X = \left\{f : \int_{\mathbb{R}}(1+x^2)^{-2}f^2(x)dx<\infty\right\} \text{ and } A=\{f: f,f'\in X\}.$$

Section 14 provides a more detailed analysis of equation (6.21).

If equation (6.2) is anticipating, that is, the initial condition is not deterministic and/or the free terms f, g are not $\mathcal{F}_t^W$-adapted, then the Wiener Chaos solution generalizes the Skorohod integral interpretation of the equation.

Example 6.9. Consider the equation

$$du(t,x) = \frac{1}{2}u_{xx}(t,x)dt + u_x(t,x)dw(t),\ t \in (0,T],\ x \in \mathbb{R}, \tag{6.23}$$

with initial condition $u(0,x) = x^2 w(T)$. Since $w(T) = \sqrt{T}\xi_1$, we find

$$(u_\alpha)_t(t,x) = \frac{1}{2}(u_\alpha)_{xx}(t,x) + \sum_i \sqrt{\alpha_i} m_i(t)(u_{\alpha^-(i)})_x(t,x) \tag{6.24}$$

with initial condition $u_\alpha(0,x) = \sqrt{T}x^2 I(|\alpha| = 1, \alpha_1 = 1)$. By Theorem 6.4, there exists a unique $w(A,X)$ Wiener Chaos solution of (6.23) for suitable spaces A and X. For example, we can take

$$X = \left\{ f : \int_{\mathbb{R}} (1+x^2)^{-8} f^2(x)dx < \infty \right\} \text{ and } A = \{f : f, f', f'' \in X\}.$$

System (6.24) can be solved explicitly. Indeed, $u_\alpha \equiv 0$ if $|\alpha| = 0$ or $|\alpha| > 3$ or if $\alpha_1 = 0$. Otherwise, writing $M_i(t) = \int_0^t m_i(s)ds$, we find:

$$\begin{aligned}
u_\alpha(t,x) &= (t+x^2)\sqrt{T},\ \text{if } |\alpha| = 1,\ \alpha_1 = 1;\\
u_\alpha(t,x) &= 2\sqrt{2}\, xt,\ \text{if } |\alpha| = 2,\ \alpha_1 = 2;\\
u_\alpha(t,x) &= 2\sqrt{T}\, xM_i(t),\ \text{if } |\alpha| = 2,\ \alpha_1 = \alpha_i = 1,\ 1 < i;\\
u_\alpha(t,x) &= \sqrt{\frac{6}{T}}\, t^2,\ \text{if } |\alpha| = 3,\ \alpha_1 = 3;\\
u_\alpha(t,x) &= 2\sqrt{2T}\, M_1(t)M_i(t),\ \text{if } |\alpha| = 3,\ \alpha_1 = 2,\ \alpha_i = 1,\ 1 < i;\\
u_\alpha(t,x) &= \sqrt{2T}\, M_i^2(t),\ \text{if } |\alpha| = 3,\ \alpha_1 = 1,\ \alpha_i = 2,\ 1 < i;\\
u_\alpha(t,x) &= 2\sqrt{T}\, M_i(t)M_j(t),\ \text{if } |\alpha| = 3,\ \alpha_1 = \alpha_i = \alpha_j = 1,\ 1 < i < j.
\end{aligned}$$

Then

$$u(t,x) = \sum_{\alpha \in \mathcal{J}} u_\alpha \xi_\alpha = w(T)w^2(t) - 2tw(t) + 2(W(T)w(t) - t)x + x^2 w(T) \tag{6.25}$$

is the Wiener Chaos solution of (6.23). It can be verified using the properties of the Skorohod integral [37] that the function u defined by (6.25) satisfies

$$u(t,x) = x^2 w(T) + \frac{1}{2}\int_0^t u_{xx}(s,x)ds + \int_0^t u_x(s,x)dw(s),\ t \in [0,T],\ x \in \mathbb{R},$$

where the stochastic integral is in the sense of Skorohod.

7. Weighted Wiener Chaos Spaces and S-Transform

The space $\mathcal{D}'(L_2(\mathbb{W}); X)$ is too big to provide any reasonable information about regularity of the Wiener Chaos solution. Introduction of weighted Wiener chaos spaces makes it possible to resolve this difficulty.

As before, let $\Xi = \{\xi_\alpha,\ \alpha \in \mathcal{J}\}$ be the Cameron-Martin basis in $L_2(\mathbb{W})$, and $\mathcal{D}(L_2(\mathbb{W}); X)$, the collection of finite linear combinations of ξ_α with coefficients in a Banach space X.

Definition 7.1. *Given a collection* $\{r_\alpha,\ \alpha \in \mathcal{J}\}$ *of positive numbers, the space* $\mathcal{R}L_2(\mathbb{W}; X)$ *is the closure of* $\mathcal{D}(L_2(\mathbb{W}); X)$ *with respect to the norm*

$$\|v\|^2_{\mathcal{R}L_2(\mathbb{W};X)} := \sum_{\alpha\in\mathcal{J}} r_\alpha^2 \|v_\alpha\|^2_X.$$

The operator $\mathcal{R}$ defined by $(\mathcal{R}v)_\alpha := r_\alpha v_\alpha$ is a linear homeomorphism from $\mathcal{R}L_2(\mathbb{W}; X)$ to $L_2(\mathbb{W}; X)$.

There are several special choices of the weight sequence $\mathcal{R} = \{r_\alpha,\ \alpha \in \mathcal{J}\}$ and special notations for the corresponding weighted Wiener chaos spaces.

- If $Q = \{q_1, q_2, \ldots\}$ is a sequence of positive numbers, define
$$q^\alpha = \prod_{i,k} q_k^{\alpha_i^k}.$$
The operator $\mathcal{R}$, corresponding to $r_\alpha = q^\alpha$, is denotes by $\mathcal{Q}$. The space $\mathcal{Q}L_2(\mathbb{W}; X)$ is denoted by $L_{2,Q}(\mathbb{W}; X)$ and is called a *Q-weighted Wiener chaos space.* The significance of this choice of weights will be explained shortly (see, in particular, Proposition 7.4).
- If
$$r_\alpha^2 = (\alpha!)^\rho \prod_{i,k} (2ik)^{\gamma\alpha_i^k},\ \ \rho, \gamma \in \mathbb{R},$$
then the corresponding space $\mathcal{R}L_2(\mathbb{W}; X)$ is denoted by $(\mathcal{S})_{\rho,\gamma}(X)$. As always, the argument X will be omitted if $X = \mathbb{R}$. Note the analogy with Definition 3.3.

The structure of weights in the spaces $L_{2,Q}$ and $(\mathcal{S})_{\rho,\gamma}$ is different, and in general these two classes of spaces are not related. There exist generalized random elements that belong to some $L_{2,Q}(\mathbb{W}; X)$, but do not belong to any $(\mathcal{S})_{\rho,\gamma}(X)$. For example, $u = \sum_{k\geq1} e^{k^2}\xi_{1,k}$ belongs to $L_{2,Q}(\mathbb{W})$ with $q_k = e^{-2k^2}$, but to no $(\mathcal{S})_{\rho,\gamma}$, because the sum $\sum_{k\geq1} e^{2k^2}(k!)^\rho(2k)^\gamma$ diverges for every $\rho, \gamma \in \mathbb{R}$. Similarly, there exist generalized random elements that belong to some $(\mathcal{S})_{\rho,\gamma}(X)$, but to no $L_{2,Q}(\mathbb{W}; X)$. For example, $u = \sum_{n\geq1} \sqrt{n!}\xi_{(n)}$, where (n) is the multi-index with $\alpha_1^1 = n$ and $\alpha_i^k = 0$ elsewhere, belongs to $(S)_{-1,-1}$, but does not belong to any $L_{2,Q}(\mathbb{W})$, because the sum $\sum_{n\geq1} q^n n!$ diverges for every $q > 0$.

The next result is the space-time analog of Proposition 2.3.3 in [12].

Proposition 7.2. *The sum*

$$\sum_{\alpha\in\mathcal{J}} \prod_{i,k\geq1} (2ik)^{-\gamma\alpha_i^k}$$

converges if and only if $\gamma > 1$.

Proof. Note that

$$\sum_{\alpha\in\mathcal{J}} \prod_{i,k\geq1} (2ik)^{-\gamma\alpha_i^k} = \prod_{i,k\geq1}\left(\sum_{n\geq0}((2ik)^{-\gamma})^n\right) = \prod_{i,k} \frac{1}{(1-(2ik)^{-\gamma})},\ \ \gamma > 0 \tag{7.1}$$

The infinite product on the right of (7.1) converges if and only if each of the sums $\sum_{i\geq1} i^{-\gamma}$, $\sum_{k\geq1} k^{-\gamma}$ converges, that is, if an only if $\gamma > 1$. □

Corollary 7.3. *For every* $u \in \mathcal{D}'(\mathbb{W}; X)$*, there exists an operator* $\mathcal{R}$ *so that* $\mathcal{R}u \in L_2(\mathbb{W}; X)$.

Proof. Define

$$r_\alpha^2 = \frac{1}{1+\|u_\alpha\|_X^2} \prod_{i,k\geq 1} (2ik)^{-2\alpha_i^k}.$$

Then

$$\|\mathcal{R}u\|^2_{L_2(\mathbb{W};X)} = \sum_{\alpha\in\mathcal{J}} \frac{\|u_\alpha\|_X^2}{1+\|u_\alpha\|_X^2} \prod_{i,k\geq 1} (2ik)^{-2\alpha_i^k} \leq \sum_{\alpha\in\mathcal{J}} \prod_{i,k\geq 1} (2ik)^{-2\alpha_i^k} < \infty.$$

□

The importance of the operator $\mathcal{Q}$ in the study of stochastic equations is due to the fact that the operator $\mathcal{R}$ maps a Wiener Chaos solution to a Wiener Chaos solution if and only $\mathcal{R} = \mathcal{Q}$ for some sequence Q. Indeed, direct calculations show that the functions $u_\alpha, \alpha \in \mathcal{J}$, satisfy the propagator (6.4) if and only if $v_\alpha = (\mathcal{R}u)_\alpha$ satisfy

$$\begin{aligned} v_\alpha(t) &= (\mathcal{R}u_0)_\alpha + \int_0^t (\mathcal{A}v + \mathcal{R}f)_\alpha(s)ds \\ &+ \int_0^t \sum_{i,k} \sqrt{\alpha_i^k} \frac{\rho_\alpha}{\rho_{\alpha^-(i,k)}} (\mathcal{M}_k\mathcal{R}u + \mathcal{R}g_k)_{\alpha^-(i,k)}(s) m_i(s) ds. \end{aligned} \tag{7.2}$$

Therefore, the operator $\mathcal{R}$ preserves the structure of the propagator if and only if

$$\frac{\rho_\alpha}{\rho_{\alpha^-(i,k)}} = q_k,$$

that is, $\rho_\alpha = q^\alpha$ for some sequence Q.

Below is the summary of the main properties of the operator $\mathcal{Q}$.

Proposition 7.4.

(1) *If $q_k \leq q < 1$ for all $k \geq 1$, then $L_{2,Q}(\mathbb{W}) \subset (\mathcal{S})_{0,-\gamma}$ for some $\gamma > 0$.*
(2) *If $q_k \geq q > 1$ for all k, then $L_{2,Q}(\mathbb{W}) \subset L_2^n(\mathbb{W})$ for all $n \geq 1$, that is, the elements of $L_{2,Q}(\mathbb{W})$ are infinitely differentiable in the Malliavin sense.*
(3) *If $u \in L_{2,Q}(\mathbb{W}; X)$ with generalized Fourier coefficients u_α satisfying the propagator (6.4), and $v = \mathcal{Q}u$, then the corresponding system for the generalized Fourier coefficients of v is*

$$\begin{aligned} v_\alpha(t) &= (\mathcal{Q}u_0)_\alpha + \int_0^t (\mathcal{A}v + \mathcal{Q}f)_\alpha(s)ds \\ &+ \int_0^t \sum_{i,k} \sqrt{\alpha_i^k} (\mathcal{M}_k v + \mathcal{Q}g_k)_{\alpha^-(i,k)}(s) q_k m_i(s) ds. \end{aligned} \tag{7.3}$$

(4) *The function u is a Wiener Chaos solution of*

$$u(t) = u_0 + \int_0^t (\mathcal{A}u(s) + f(s))dt + \int_0^t (\mathcal{M}u(s) + g(s), dW(s))_Y \tag{7.4}$$

if and only if $v = \mathcal{Q}u$ is a Wiener Chaos solution of

$$v(t) = (\mathcal{Q}u)_0 + \int_0^t (\mathcal{A}v(s) + \mathcal{Q}f(s))dt + \int_0^t (\mathcal{M}v(s) + \mathcal{Q}g(s), dW^Q(s))_Y, \tag{7.5}$$

where, for $h \in Y$, $W_h^Q(t) = \sum_{k\geq 1}(h, y_k)_Y q_k w_k(t)$.

The following examples demonstrate how the operator $\mathcal{Q}$ helps with the analysis of various stochastic evolution equations.

Example 7.5. Consider the $w(H_2^1(\mathbb{R}), H_2^{-1}(\mathbb{R}))$ Wiener Chaos solution u of equation

$$du(t,x) = (au_{xx}(t,x) + f(t,x))dt + \sigma u_x(t,x)dw(t),\ 0 < t \leq T,\ x \in \mathbb{R}, \tag{7.6}$$

with $f \in L_2(\Omega \times (0,T); H_2^{-1}(\mathbb{R}))$, $g \in L_2(\Omega \times (0,T); L_2(\mathbb{R}))$, and $u|_{t=0} = u_0 \in L_2(\mathbb{R})$. Assume that $\sigma > 0$ and define the sequence Q so that $q_k = q$ for all $k \geq 1$ and $q < \sqrt{2a}/\sigma$. By Theorem 2.5, equation

$$dv = (av_{xx} + f)dt + (q\sigma u_x + g)dw$$

with $v|_{t=0} = u_0$, has a unique traditional solution

$$v \in L_2\left(\mathbb{W}; L_2((0,T); H_2^1(\mathbb{R}))\right) \bigcap L_2\left(\mathbb{W}; \mathbf{C}((0,T); L_2(\mathbb{R}))\right).$$

By Proposition 7.4, the $w(H_2^1(\mathbb{R}), H_2^{-1}(\mathbb{R}))$ Wiener Chaos solution u of equation (7.6) satisfies $u = \mathcal{Q}^{-1}v$ and

$$u \in L_{2,Q}\left(\mathbb{W}; L_2((0,T); H_2^1(\mathbb{R}))\right) \bigcap L_{2,Q}\left(\mathbb{W}; \mathbf{C}((0,T); L_2(\mathbb{R}))\right).$$

Note that if equation (7.6) is strongly parabolic, that is, $2a > \sigma^2$, then the weight q can be taken bigger than one, and, according to the first statement of Proposition 7.4, regularity of the solution is better than the one guaranteed by Theorem 2.5.

Example 7.6. The Wiener Chaos solutions can be constructed for stochastic ordinary differential equations. Consider, for example,

$$u(t) = 1 + \int_0^t \sum_{k\geq 1} u(s)dw_k(s), \tag{7.7}$$

which clearly does not have a traditional solution. On the other hand, the unique $w(\mathbb{R},\mathbb{R})$ Wiener Chaos solution of this equation belongs to $L_{2,Q}\left(\mathbb{W}; L_2((0,T)\right)$ for every Q satisfying $\sum_k q_k^2 < \infty$. Indeed, for (7.7), equation (7.5) becomes

$$v(t) = 1 + \int_0^t \sum_k v(s)q_k dw_k(s).$$

If $\sum_k q_k^2 < \infty$, then the traditional solution of this equation exists and belongs to $L_2\left(\mathbb{W}; L_2((0,T))\right)$.

There exist equations for which the Wiener Chaos solution does not belong to any weighted Wiener chaos space $L_{2,Q}$. An example is given below in Section 14.

To define the S-transform, consider the following analog of the stochastic exponential (3.6).

Lemma 7.7. *If* $h \in \mathcal{D}\left(L_2((0,T);Y)\right)$ *and*

$$\mathcal{E}(h) = \exp\left(\int_0^T (h(t), dW(t))_Y - \frac{1}{2}\int_0^T \|h(t)\|_Y^2 dt\right),$$

then

- $\mathcal{E}(h) \in L_{2,Q}(\mathbb{W})$ *for every sequence* Q.
- $\mathcal{E}(h) \in (\mathcal{S})_{\rho,\gamma}$ *for* $0 \leq \rho < 1$ *and* $\gamma \geq 0$.
- $\mathcal{E}(h) \in (\mathcal{S})_{1,\gamma}$, $\gamma \geq 0$, *as long as* $\|h\|^2_{L_2((0,T);Y)}$ *is sufficiently small.*

Proof. Recall that, if $h \in \mathcal{D}(L_2((0,T);Y))$, then $h(t) = \sum_{i,k\in I_h} h_{k,i} m_i(t) y_k$, where I_h is a finite set. Direct computations show that

$$\mathcal{E}(h) = \prod_{i,k} \left(\sum_{n\geq 0} \frac{H_n(\xi_{ik})}{n!} (h_{k,i})^n \right) = \sum_{\alpha\in\mathcal{J}} \frac{h^\alpha}{\sqrt{\alpha!}} \xi_\alpha$$

where $h^\alpha = \prod_{i,k} h_{k,i}^{\alpha_i^k}$. In particular,

$$(\mathcal{E}(h))_\alpha = \frac{h^\alpha}{\sqrt{\alpha!}}. \tag{7.8}$$

Consequently, for every sequence Q of positive numbers,

$$\|\mathcal{E}(h)\|^2_{L_{2,Q}(\mathbb{W})} = \exp\left(\sum_{i,k\in I_h} h_{k,i}^2 q_k^2 \right) < \infty. \tag{7.9}$$

Similarly, for $0 \leq \rho < 1$ and $\gamma \geq 0$,

$$\|\mathcal{E}(h)\|^2_{(\mathcal{S})_{\rho,\gamma}} = \sum_{\alpha\in\mathcal{J}} \prod_{i,k} \frac{((2ik)^\gamma h_{k,i})^{2\alpha_i^k}}{(\alpha_i^k!)^{1-\rho}} = \prod_{i,k\in I_h} \left(\sum_{n\geq 0} \frac{((2ik)^\gamma h_{k,i})^{2n}}{(n!)^{1-\rho}} \right) < \infty, \tag{7.10}$$

and, for $\rho = 1$,

$$\|\mathcal{E}(h)\|^2_{(\mathcal{S})_{1,\gamma}} = \sum_{\alpha\in\mathcal{J}} \prod_{i,k} ((2ik)^\gamma h_{k,i})^{2\alpha_i^k} = \prod_{i,k\in I_h} \left(\sum_{n\geq 0} ((2ik)^\gamma h_{k,i})^{2n} \right) < \infty, \tag{7.11}$$

if $2\left(\max_{(m,n)\in I_h} (mn)^\gamma\right) \sum_{i,k} h_{k,i}^2 < 1$. Lemma 7.7 is proved. □

Remark 7.8. *It is well-known (see, for example,* [24, Proof of Theorem 5.5]) *that the family* $\{\mathcal{E}(h), h \in \mathcal{D}(L_2((0,T);Y))\}$ *is dense in* $L_2(\mathbb{W})$ *and consequently in every* $L_{2,Q}(\mathbb{W})$ *and every* $(\mathcal{S})_{\rho,\gamma}$, $-1 < \rho \leq 1$, $\gamma \in \mathbb{R}$.

Definition 7.9. *If* $u \in L_{2,Q}(\mathbb{W};X)$ *for some* Q, *or if* $u \in \bigcup_{q\geq 0} (\mathcal{S})_{-\rho,-\gamma}(X)$, $0 \leq \rho \leq 1$, *then the deterministic function*

$$Su(h) = \sum_{\alpha\in\mathcal{J}} \frac{u_\alpha h^\alpha}{\sqrt{\alpha!}} \in X \tag{7.12}$$

is called the **S-transform** *of* u. *Similarly, for* $g \in \mathcal{D}'(Y; L_{2,Q}(\mathbb{W};X))$ *the S-transform* $Sg(h) \in \mathcal{D}'(Y;X)$ *is defined by setting* $(Sg(h))_k = (Sg_k)(h)$.

Note that if $u \in L_2(\mathbb{W};X)$, then $Su(h) = \mathbb{E}(u\mathcal{E}(h))$. If u belongs to $L_{2,Q}(\mathbb{W};X)$ or to $\bigcup_{q\geq 0} (\mathcal{S})_{-\rho,-\gamma}(X)$, $0 \leq \rho < 1$, then $Su(h)$ is defined for all $h \in \mathcal{D}(L_2((0,T);Y))$. If $u \in \bigcup_{\gamma\geq 0} (\mathcal{S})_{-1,-\gamma}(X)$, then $Su(h)$ is defined only for h sufficiently close to zero.

By Remark 7.8, an element u from $L_{2,Q}(\mathbb{W};X)$ or $\bigcup_{\gamma\geq 0} (\mathcal{S})_{-\rho,-\gamma}(X)$, $0 \leq \rho < 1$, is uniquely determined by the collection of deterministic functions $Su(h)$, $h \in \mathcal{D}(L_2((0,T);Y))$. Since $\mathcal{E}(h) > 0$ for all $h \in \mathcal{D}(L_2((0,T);Y))$, Remark 7.8 also suggests the following definition.

Definition 7.10. *An element* u *from* $L_{2,Q}(\mathbb{W})$ *or* $\bigcup_{\gamma\geq 0} (\mathcal{S})_{-\rho,-\gamma}$, $0 \leq \rho < 1$ *is called non-negative* ($u \geq 0$) *if and only if* $Su(h) \geq 0$ *for all* $h \in \mathcal{D}(L_2((0,T);Y))$.

The definition of the operator $\mathcal{Q}$ and Definition 7.10 imply the following result.

Proposition 7.11. *A generalized random element u from $L_{2,Q}(\mathbb{W})$ is non-negative if and only if $\mathcal{Q}u \geq 0$.*

For example, the solution of equation (7.7) is non-negative because

$$\mathcal{Q}u(t) = \exp\left(\sum_{k\geq 1}(q_k w_k(t) - (1/2)q_k^2)\right).$$

We conclude this section with one technical remark.

Definition 7.9 expresses the S-transform in terms of the generalized Fourier coefficients. The following results makes it possible to recover generalized Fourier coefficients from the corresponding S-transform.

Proposition 7.12. *If u belongs to some $L_{2,Q}(\mathbb{W};X)$ or $\bigcup_{\gamma\geq 0}(\mathcal{S})_{-\rho,-\gamma}(X)$, $0 \leq \rho \leq 1$, then*

$$u_\alpha = \frac{1}{\sqrt{\alpha!}}\left(\prod_{i,k}\frac{\partial^{\alpha_i^k} Su(h)}{\partial h_{k,i}^{\alpha_i^k}}\right)\Bigg|_{h=0}. \tag{7.13}$$

Proof. For each $\alpha \in \mathcal{J}$ with K non-zero entries, equality (7.12) and Lemma 7.7 imply that the function $Su(h)$, as a function of K variables $h_{k,i}$, is analytic in some neighborhood of zero. Then (7.13) follows after differentiation of the series (7.12). □

8. General Properties of the Wiener Chaos Solutions

Using notations and assumptions from Section 6, consider the linear evolution equation

$$du(t) = (\mathcal{A}u(t) + f(t))dt + (\mathcal{M}u(t) + g(t), dW(t))_Y,\ 0 < t \leq T,\ u|_{t=0} = u_0. \tag{8.1}$$

The objective of this section is to study how the Wiener Chaos compares with the traditional and white noise solutions.

To make the presentation shorter, call an X-valued generalized random element *S-admissible* if and only if it belongs to $L_{2,Q}(\mathcal{F}^W;X)$ for some Q or to $(\mathcal{S})_{\rho,q}(X)$ for some $\rho \in [-1,1]$ and $q \in \mathbb{R}$. It was shown in Section 7 that, for every S-admissible u, the S-transform $Su(h)$ is defined when $h = \sum_{i,k} h_{k,i}m_i y_k \in \mathcal{D}(L_2((0,T);Y))$ and is an analytic function of $h_{k,i}$ in some neighborhood of $h = 0$.

The next result describes the S-transform of the Wiener Chaos solution.

Theorem 8.1. *Assume that*

1. *there exists a unique $w(A,X)$ Wiener Chaos solution u of (8.1) and u is S-admissible;*
2. *For each $t \in [0,T]$, the linear operators $\mathcal{A}(t), \mathcal{M}_k(t)$ are bounded from A to X;*
3. *the generalized random elements u_0, f, g_k are S-admissible.*

Then, for every $h \in \mathcal{D}(L_2((0,T);Y))$ with $\|h\|^2_{L_2((0,T);Y)}$ sufficiently small, the function $v = Su(h)$ is a $w(A,X)$ solution of the deterministic equation

$$v(t) = Su_0(h) + \int_0^t \Big(\mathcal{A}v + Sf(h) + (\mathcal{M}_k v + Sg_k(h))h_k\Big)(s)ds. \tag{8.2}$$

Proof. By assumption, $Su(h)$ exists for suitable functions h. Then the S-transformed equation (8.2) follows from the definition of the S-transform (7.12) and the propagator equation (6.4) satisfied by the generalized Fourier coefficients of u. Indeed, continuity of operator $\mathcal{A}$ implies

$$S(\mathcal{A}u)(h) = \sum_{\alpha} \frac{h^\alpha}{\sqrt{\alpha!}} \mathcal{A}u_\alpha = \mathcal{A} \sum_{\alpha} \frac{h^\alpha}{\sqrt{\alpha!}} u_\alpha = \mathcal{A}(Su(h)).$$

Similarly,

$$\begin{aligned} \sum_{\alpha} \frac{h^\alpha}{\sqrt{\alpha!}} \sum_{i,k} \sqrt{\alpha_i^k} \mathcal{M}_k u_{\alpha^-(i,k)} m_i &= \sum_{\alpha} \sum_{i,k} \frac{h^{\alpha^-(i,k)}}{\sqrt{\alpha^-(i,k)!}} \mathcal{M}_k u_{\alpha^-(i,k)} m_i h_{k,i} \\ &= \sum_{i,k} \left(\sum_{\alpha} \frac{h^\alpha}{\sqrt{\alpha}} \mathcal{M}_k u_\alpha \right) m_i h_{k,i} = \mathcal{M}_k (Su(h)) h_k. \end{aligned}$$

Computations for the other terms are similar. Theorem 8.1 is proved. □

Remark 8.2. *If $h \in \mathcal{D}(L_2((0,T);Y))$ and*

$$\mathcal{E}_t(h) = \exp\left(\int_0^t (h(s), dW(s))_Y - \frac{1}{2} \int_0^t \|h(t)\|_Y^2 dt \right), \tag{8.3}$$

then, by the Itô formula,

$$d\mathcal{E}_t(h) = \mathcal{E}_t(h)(h(t), dW(t))_Y. \tag{8.4}$$

If u_0 is deterministic, f and g_k are $\mathcal{F}_t^W$-adapted, and u is a square-integrable solution of (8.1), then equality (8.2) is obtained by multiplying equations (8.4) and (8.1) according to the Itô formula and taking the expectation.

Remark 8.3. *Rewriting (8.4) as*

$$d\mathcal{E}_t(h) = \mathcal{E}_t(h) h_{k,i} m_i(t) dw_k(t)$$

and using the relations

$$\mathcal{E}_t(h) = \mathbb{E}(\mathcal{E}_T(h)|\mathcal{F}_t^W), \ \xi_\alpha = \frac{1}{\sqrt{\alpha!}} \left(\prod_{i,k} \frac{\partial^{\alpha_i^k} \mathcal{E}_T(h)}{\partial h_{k,i}^{\alpha_i^k}} \right) \Bigg|_{h=0},$$

we arrive at representation (5.5) for $\mathbb{E}(\xi_\alpha|\mathcal{F}_t^W)$.

A partial converse of Theorem 8.1 is that, under some regularity conditions, the Wiener Chaos solution can be recovered from the solution of the S-transformed equation (8.2).

Theorem 8.4. *Assume that the linear operators $\mathcal{A}(t)$, $\mathcal{M}_k(t)$, $t \in [0,T]$, are bounded from A to X, the input data u_0, f, g_k are S-admissible, and, for every $h \in \mathcal{D}(L_2((0,T);Y))$ with $\|h\|^2_{L_2((0,T);Y)}$ sufficiently small, there exists a $w(A,X)$ solution $v = v(t;h)$ of equation (8.2). We write $h = h_{k,i} m_i y_k$ and consider v as a function of the variables $h_{k,i}$. Assume that all the derivatives of v at the point $h = 0$ exists, and, for $\alpha \in \mathcal{J}$, define*

$$u_\alpha(t) = \frac{1}{\sqrt{\alpha!}} \left(\prod_{i,k} \frac{\partial^{\alpha_i^k} v(t;h)}{\partial h_{k,i}^{\alpha_i^k}} \right) \Bigg|_{h=0}. \tag{8.5}$$

Then the generalized random process $u(t) = \sum_{\alpha \in \mathcal{J}} u_\alpha(t) \xi_\alpha$ is a $w(A,X)$ Wiener Chaos solution of (8.1).

Proof. Differentiation of (8.2) and application of Proposition 7.12 show that the functions u_α satisfy the propagator (6.4). □

Remark 8.5. *The central part in the construction of the white noise solution of (8.1) is proving that the solution of (8.2) is an S-transform of a suitable generalized random process. For many particular cases of equation (8.1), the corresponding analysis is carried out in* [10, 12, 33, 40]. *The consequence of Theorems 8.1 and 8.4 is that a white noise solution of (8.1), if exists, must coincide with the Wiener Chaos solution.*

The next theorem establishes the connection between the Wiener Chaos solution and the traditional solution. Recall that the traditional, or square-integrable, solution of (8.1) was introduced in Definition 2.2. Accordingly, the notations from Section 2 will be used.

Theorem 8.6. *Let (V, H, V') be a normal triple of Hilbert spaces. Take a deterministic function u_0 and $\mathcal{F}_t^W$-adapted random processes function, f and g_k so that (2.3) holds. Under these assumptions we have the following two statements.*

1. *An $\mathcal{F}_t^W$-adapted traditional solution of (8.1) is also a Wiener Chaos solution.*
2. *If u is a $w(V, V')$ Wiener Chaos solution of (8.1) so that*

$$\sum_{\alpha\in\mathcal{J}}\left(\int_0^T \|u_\alpha(t)\|_V^2 dt + \sup_{0\le t\le T}\|u_\alpha(t)\|_H^2\right) < \infty, \tag{8.6}$$

then u is an $\mathcal{F}_t^W$-adapted traditional solution of (8.1).

Proof. (1) If $u = u(t)$ is an $\mathcal{F}_t^W$-adapted traditional solution, then

$$u_\alpha(t) = \mathbb{E}(u(t)\xi_\alpha) = \mathbb{E}\left(u(t)\mathbb{E}(\xi_\alpha|\mathcal{F}_t^W)\right) = \mathbb{E}(u(t)\xi_\alpha(t)).$$

Then the propagator (6.4) for u_α follows after applying the Itô formula to the product $u(t)\xi_\alpha(t)$ and using (5.5).

(2) Assumption (8.6) implies

$$u \in L_2(\Omega\times(0,T);V)\bigcap L_2(\Omega;\mathbf{C}((0,T);H)).$$

Then, by Theorem 8.1, for every $\varphi \in V$ and $h \in \mathcal{D}((0,T);Y)$, the S-transform u_h of u satisfies

$$\begin{aligned}(u_h(t),\varphi)_H = (u_0,\varphi)_H &+ \int_0^t \langle \mathcal{A}u_h(s),\varphi\rangle ds + \int_0^t \langle f(s),\varphi\rangle ds\\ &+ \sum_{\alpha\in\mathcal{J}}\frac{h^\alpha}{\alpha!}\sum_{i,k}\int_0^t \sqrt{\alpha_i^k}m_i(s)\big((\mathcal{M}_k u_{\alpha^-(i,k)}(s),\varphi)_H\\ &+ (g_k(s),\varphi)_H I(|\alpha|=1)\big)ds.\end{aligned}$$

If $I(t) = \int_0^t(\mathcal{M}_k u(s),\varphi)_H dw_k(s)$, then

$$\mathbb{E}(I(t)\xi_\alpha(t)) = \int_0^t \sum_{i,k}\sqrt{\alpha_i^k}m_i(s)(\mathcal{M}_k u_{\alpha^-(i,k)}(s),\varphi)_H ds. \tag{8.7}$$

Similarly,

$$\mathbb{E}\left(\xi_\alpha(t)\int_0^t (g_k(s),\varphi)_H dw_k(s)\right) = \sum_{i,k}\int_0^t\sqrt{\alpha_i^k}m_i(s)(g_k(s),\varphi)_H I(|\alpha|=1)ds.$$

Therefore,

$$\sum_{\alpha\in\mathcal{J}}\frac{h^\alpha}{\alpha!}\sum_{i,k}\int_0^t\sqrt{\alpha_i^k}m_i(s)(\mathcal{M}_k u_{\alpha^-(i,k)}(s),\varphi)_H ds$$
$$=\mathbb{E}\left(\mathcal{E}(h)\int_0^t\left((\mathcal{M}_k u(s),\varphi)_H+(g_k(s),\varphi)_H\right)dw_k(s)\right).$$

As a result,

$$\begin{aligned}\mathbb{E}\left(\mathcal{E}(h)(u(t),\varphi)_H\right)&=\mathbb{E}\left(\mathcal{E}(h)(u_0,\varphi)_H\right)\\&+\mathbb{E}\left(\mathcal{E}(h)\int_0^t\langle\mathcal{A}u(s),\varphi\rangle ds\right)+\mathbb{E}\left(\mathcal{E}(h)\int_0^t\langle f(s),\varphi\rangle ds\right)\\&+\mathbb{E}\left(\mathcal{E}(h)\int_0^t\left((\mathcal{M}_k u(s),\varphi)_H+(g_k(s),\varphi)_H\right)dw_k(s)\right).\end{aligned}\tag{8.8}$$

Equality (8.8) and Remark 7.8 imply that, for each t and each φ, (2.4) holds with probability one. Continuity of u implies that, for each φ, a single probability-one set can be chosen for all $t\in[0,T]$. Theorem 9.6 is proved. □

9. Regularity of the Wiener Chaos Solution

Let $\mathbb{F}=(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$ be a stochastic basis with the usual assumptions and $w_k=w_k(t),\ k\geq 1,\ t\geq 0$, a collection of standard Wiener processes on $\mathbb{F}$. As in Section 2, let (V,H,V') be a normal triple of Hilbert spaces and $\mathcal{A}(t):V\to V'$, $\mathcal{M}_k(t):V\to H$, linear bounded operators; $t\in[0,T]$.

In this section we study the linear equation

$$u(t)=u_0+\int_0^t(\mathcal{A}u(s)+f(s))ds+\int_0^t(\mathcal{M}_k u(s)+g_k(s))dw_k,\ 0\leq t\leq T,\tag{9.1}$$

under the following **assumptions**:

A1 There exist positive numbers C_1 and δ so that

$$\langle\mathcal{A}(t)v,v\rangle+\delta\|v\|_V^2\leq C_1\|v\|_H^2,\ v\in V,\ t\in[0,T].\tag{9.2}$$

A2 There exists a real number C_2 so that

$$2\langle\mathcal{A}(t)v,v\rangle+\sum_{k\geq 1}\|\mathcal{M}_k(t)v\|_H^2\leq C_2\|v\|_H^2,\ v\in V,\ t\in[0,T].\tag{9.3}$$

A3 The initial condition u_0 is non-random and belongs to H; the process $f=f(t)$ is deterministic and $\int_0^T\|f(t)\|_{V'}^2dt<\infty$; each $g_k=g_k(t)$ is a deterministic processes and $\sum_{k\geq 1}\int_0^T\|g_k(t)\|_H^2dt<\infty$.

Note that condition (9.3) is weaker than (2.5). Traditional analysis of equation (9.1) under (9.3) requires additional regularity assumptions on the input data and additional Hilbert space constructions beyond the normal triple [42, Section 3.2]. In particular, no existence of a traditional solution is known under assumptions **A1**-**A3**, and the Wiener chaos approach provides new existence and regularity results for equation (9.1). A different version of the following theorem is presented in [29].

Theorem 9.1. *Under assumptions **A1**–**A3**, for every $T>0$, equation (9.1) has a unique $w(V,V')$ Wiener Chaos solution. This solution $u=u(t)$ has the following properties:*

(1) *There exists a weight sequence Q so that*

$$u \in L_{2,Q}(\mathbb{W}; L_2((0,T);V)) \bigcap L_{2,Q}(\mathbb{W}; \mathbf{C}((0,T);H)).$$

(2) *For every $0 \le t \le T$, $u(t) \in L_2(\Omega; H)$ and*

$$\mathbb{E}\|u(t)\|_H^2 \le 3e^{C_2 t}\left(\|u_0\|_H^2 + C_f\int_0^t \|f(s)\|_{V'}^2 ds + \sum_{k\ge1}\int_0^t \|g_k(s)\|_H^2 ds\right), \tag{9.4}$$

where the number C_2 is from (9.3) and the positive number C_f depends only on δ and C_1 from (9.2).

(3) *For every $0 \le t \le T$,*

$$\begin{aligned} u(t) = u_{(0)} + \sum_{n\ge1}\sum_{k_1,\ldots,k_n\ge1}\int_0^t\int_0^{s_n}\cdots\int_0^{s_2} \\ \Phi_{t,s_n}\mathcal{M}_{k_n}\cdots\Phi_{s_2,s_1}\left(\mathcal{M}_{k_1}u_{(0)} + g_{k_1}(s_1)\right)dw_{k_1}(s_1)\cdots dw_{k_n}(s_n), \end{aligned} \tag{9.5}$$

where $\Phi_{t,s}$ is the semi-group of the operator $\mathcal{A}$.

Proof. Assumption **A2** and the properties of the normal triple imply that there exists a positive number C^* so that

$$\sum_{k\ge1}\|\mathcal{M}_k(t)v\|_H^2 \le C^*\|v\|_V^2,\ v\in V,\ t\in[0,T]. \tag{9.6}$$

Define the sequence Q so that

$$q_k = \left(\frac{\mu\delta}{C^*}\right)^{1/2} := q,\ k\ge1, \tag{9.7}$$

where $\mu \in (0,2)$ and δ is from Assumption **A1**. Then, by Assumption **A2**,

$$2\langle Av, v\rangle + \sum_{k\ge1} q^2\|\mathcal{M}_k v\|_H^2 \le -(2-\mu)\delta\|v\|_V^2 + C_1\|v\|_H^2. \tag{9.8}$$

It follows from Theorem 2.4 that equation

$$v(t) = u_0 + \int_0^t(\mathcal{A}v + f)(s)ds + \sum_{k\ge1}\int_0^t q(\mathcal{M}_k v + g_k)(s)dw_k(s) \tag{9.9}$$

has a unique solution

$$v \in L_2(\mathbb{W}; L_2((0,T);V)) \bigcap L_2(\mathbb{W}; \mathbf{C}((0,T);H)).$$

Comparison of the propagators for equations (9.1) and (9.9) shows that $u = \mathcal{Q}^{-1}v$ is the unique $w(V,V')$ solution of (9.1) and

$$u \in L_{2,Q}(\mathbb{W}; L_2((0,T);V)) \bigcap L_{2,Q}(\mathbb{W}; \mathbf{C}((0,T);H)). \tag{9.10}$$

If $C^* < 2\delta$, then equation (9.1) is strongly parabolic and $q > 1$ is an admissible choice of the weight. As a result, for strongly parabolic equations, the result (9.10) is stronger than the conclusion of Theorem 2.4.

The proof of (9.4) is based on the analysis of the propagator

$$
\begin{aligned}
u_\alpha(t) &= u_0 I(|\alpha|=0) + \int_0^t \Big(\mathcal{A}u_\alpha(s) + f(s)I(|\alpha|=0)\Big)ds \\
&+ \int_0^t \sum_{i,k} \sqrt{\alpha_i^k}(\mathcal{M}_k u_{\alpha^-(i,k)}(s) + g_k(s)I(|\alpha|=1))m_i(s)ds.
\end{aligned} \tag{9.11}
$$

We consider three particular cases: (1) $f = g_k = 0$ (the homogeneous equation); (2) $u_0 = g_k = 0$; (3) $u_0 = f = 0$. The general case will then follow by linearity and the triangle inequality.

Denote by $(\Phi_{t,s},\ t \geq s \geq 0)$ the semi-group generated by the operator $\mathcal{A}(t)$; $\Phi_t := \Phi_{t,0}$. One of the consequence of Theorem 2.4 is that, under Assumption **A1**, this semi-group exists and is strongly continuous in H.

Consider the homogeneous equation: $f = g_k = 0$. By Corollary 6.6,

$$
\sum_{|\alpha|=n} \|u_\alpha(t)\|_H^2 = \sum_{k_1,\ldots,k_n\geq 1} \int_0^t \int_0^{s_n} \cdots \int_0^{s_2} \|\Phi_{t,s_n}\mathcal{M}_{k_n}\cdots\Phi_{s_2,s_1}\mathcal{M}_{k_1}\Phi_{s_1}u_0\|_H^2 ds^n, \tag{9.12}
$$

where $ds^n = ds_1 \ldots ds_n$. Define $F_n(t) = \sum_{|\alpha|=n} \|u_\alpha(t)\|_H^2$, $n \geq 0$. Direct application of (9.3) shows that

$$
\frac{d}{dt}F_0(t) \leq C_2 F_0(t) - \sum_{k\geq 1} \|\mathcal{M}_k \Phi_t u_0\|_H^2. \tag{9.13}
$$

For $n \geq 1$, equality (9.12) implies

$$
\begin{aligned}
\frac{d}{dt}F_n(t) &= \sum_{k_1,\ldots,k_n\geq 1} \int_0^t \int_0^{s_{n-1}} \cdots \int_0^{s_2} \|\mathcal{M}_{k_n}\Phi_{t,s_{n-1}}\cdots\mathcal{M}_{k_1}\Phi_{s_1}u_0\|_H^2 ds^{n-1} \\
&+ \sum_{k_1,\ldots,k_n\geq 1} \int_0^t \int_0^{s_n} \ldots \int_0^{s_2} \langle \mathcal{A}\Phi_{t,s_n}\mathcal{M}_{k_n}\ldots\Phi_{s_1}u_0, \Phi_{t,s_n}\mathcal{M}_{k_n}\ldots\Phi_{s_1}u_0\rangle ds^n.
\end{aligned} \tag{9.14}
$$

By (9.3),

$$
\begin{aligned}
&\sum_{k_1,\ldots,k_n\geq 1} \int_0^t \int_0^{s_n} \ldots \int_0^{s_2} \langle \mathcal{A}\Phi_{t,s_n}\mathcal{M}_{k_n}\ldots\Phi_{s_1}u_0, \Phi_{t,s_n}\mathcal{M}_{k_n}\ldots\Phi_{s_1}u_0\rangle ds^n \\
&\leq - \sum_{k_1,\ldots,k_{n+1}\geq 1} \int_0^t \int_0^{s_n} \ldots \int_0^{s_2} \|\mathcal{M}_{k_{n+1}}\Phi_{t,s_n}\mathcal{M}_{k_n}\ldots\mathcal{M}_{k_1}\Phi_{s_1}u_0\|_H^2 ds^n \\
&\quad + C_2 \sum_{k_1,\ldots,k_n\geq 1} \int_0^t \int_0^{s_n} \ldots \int_0^{s_2} \|\Phi_{t,s_n}\mathcal{M}_{k_n}\ldots\mathcal{M}_{k_1}\Phi_{s_1}u_0\|_H^2 ds^n.
\end{aligned} \tag{9.15}
$$

As a result, for $n \geq 1$,

$$
\begin{aligned}
&\frac{d}{dt}F_n(t) \leq C_2 F_n(t) \\
&+ \sum_{k_1,\ldots,k_n\geq 1} \int_0^t \int_0^{s_{n-1}} \ldots \int_0^{s_2} \|\mathcal{M}_{k_n}\Phi_{t,s_{n-1}}\mathcal{M}_{k_{n-1}}\ldots\mathcal{M}_{k_1}\Phi_{s_1}u_0\|_H^2 ds^{n-1} \\
&- \sum_{k_1,\ldots,k_{n+1}\geq 1} \int_0^t \int_0^{s_n} \ldots \int_0^{s_2} \|\mathcal{M}_{k_{n+1}}\Phi_{t,s_n}\mathcal{M}_{k_n}\ldots\mathcal{M}_{k_1}\Phi_{s_1}u_0\|_H^2 ds^n.
\end{aligned} \tag{9.16}
$$

Consequently,

$$\frac{d}{dt}\sum_{n=0}^{N}\sum_{|\alpha|=n}\|u_\alpha(t)\|_H^2 \le C_2\sum_{n=0}^{N}\sum_{|\alpha|=n}\|u_\alpha(t)\|_H^2, \tag{9.17}$$

so that, by the Gronwall inequality,

$$\sum_{n=0}^{N}\sum_{|\alpha|=n}\|u_\alpha(t)\|_H^2 \le e^{C_2 t}\|u_0\|_H^2 \tag{9.18}$$

or

$$\mathbb{E}\|u(t)\|_H^2 \le e^{C_2 t}\|u_0\|_H^2. \tag{9.19}$$

Next, let us assume that $u_0 = g_k = 0$. Then the propagator (9.11) becomes

$$u_\alpha(t) = \int_0^t (\mathcal{A}u_\alpha(s) + f(s)I(|\alpha|=0))ds + \int_0^t \sum_{i,k}\sqrt{\alpha_i^k}\mathcal{M}_k u_{\alpha^-(i,k)}(s)m_i(s)ds. \tag{9.20}$$

Denote by $u_{(0)}(t)$ the solution corresponding to $\alpha = 0$. Note that

$$\begin{aligned}\|u_{(0)}(t)\|_H^2 &= 2\int_0^t \langle \mathcal{A}u_{(0)}(s), u_{(0)}(s)\rangle ds + 2\int_0^t \langle f(s), u_{(0)}(s)\rangle ds\\ &\le C_2\int_0^t \|u_{(0)}(s)\|_H^2 ds - \int_0^t \sum_{k\ge1}\|\mathcal{M}_k u_{(0)}(s)\|_H^2 ds + C_f\int_0^t \|f(s)\|_{V'}^2 ds.\end{aligned}$$

By Corollary 6.6,

$$\sum_{|\alpha|=n}\|u_\alpha(t)\|_H^2 = \sum_{k_1,\ldots,k_n\ge1}\int_0^t\int_0^{s_n}\ldots\int_0^{s_2}\|\Phi_{t,s_n}\mathcal{M}_{k_n}\ldots\mathcal{M}_{k_1}u_{(0)}(s_1)\|_H^2 ds^n \tag{9.21}$$

for $n \ge 1$. Then, repeating the calculations (9.14)–(9.16), we conclude that

$$\sum_{n=1}^{N}\sum_{|\alpha|=n}\|u_\alpha(t)\|_H^2 \le C_f\int_0^t \|f(s)\|_{V'}^2 ds + C_2\int_0^t\sum_{n=1}^{N}\sum_{|\alpha|=n}\|u_\alpha(s)\|_H^2 ds, \tag{9.22}$$

and, by the Gronwal inequality,

$$\mathbb{E}\|u(t)\|_H^2 \le C_f e^{C_2 t}\int_0^t \|f(s)\|_{V'}^2 ds. \tag{9.23}$$

Finally, let us assume that $u_0 = f = 0$. Then the propagator (9.11) becomes

$$\begin{aligned}u_\alpha(t) &= \int_0^t \mathcal{A}u_\alpha(s)ds\\ &+ \int_0^t\left(\sum_{i,k}\sqrt{\alpha_i^k}\mathcal{M}_k u_{\alpha^-(i,k)}(s) + g_k(s)I(|\alpha|=1)\right)m_i(s)ds.\end{aligned} \tag{9.24}$$

Even though $u_\alpha(t) = 0$ if $\alpha = 0$, we have

$$u_{(ik)} = \int_0^t \Phi_{t,s}g_k(s)m_i(s)ds, \tag{9.25}$$

and then the arguments from the proof of Corollary 6.6 apply, resulting in

$$\sum_{|\alpha|=n} \|u_\alpha(t)\|_H^2 = \sum_{k_1,\ldots,k_n\geq 1} \int_0^t \int_0^{s_n} \ldots \int_0^{s_2} \|\Phi_{t,s_n}\mathcal{M}_{k_n} \ldots \Phi_{s_2,s_1} g_{k_1}(s_1)\|_H^2 ds^n$$

for $n \geq 1$. Note that

$$\sum_{|\alpha|=1} \|u_\alpha(t)\|_H^2 = \sum_{k\geq 1} \int_0^t \|g_k(s)\|_H^2 ds + 2\sum_{k\geq 1} \int_0^t \langle \mathcal{A}\Phi_{t,s} g_k(s), \Phi_{t,s} g_k(s)\rangle ds.$$

Then, repeating the calculations (9.14)–(9.16), we conclude that

$$\sum_{n=1}^N \sum_{|\alpha|=n} \|u_\alpha(t)\|_H^2 \leq \sum_{k\geq 1} \int_0^t \|g_k(s)\|_H^2 ds + C_2 \int_0^t \sum_{n=1}^N \sum_{|\alpha|=n} \|u_\alpha(s)\|_H^2 ds, \tag{9.26}$$

and, by the Gronwal inequality,

$$\mathbb{E}\|u(t)\|_H^2 \leq e^{C_2 t} \sum_{k\geq 1} \int_0^t \|g_k(s)\|_H^2 ds. \tag{9.27}$$

To derive (9.4), it remains to combine (9.19), (9.23), and (9.27) with the inequality $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$.

Representation (9.5) of the Wiener chaos solution as a sum of iterated Itô integrals now follows from Corollary 6.6. Theorem 9.1 is proved. □

Corollary 9.2. *If* $\sum_{\alpha\in\mathcal{J}} \int_0^T \|u_\alpha(s)\|_V^2 ds < \infty$, *then* $\sum_{\alpha\in\mathcal{J}} \sup_{0\leq t\leq T} \|u_\alpha(t)\|_H^2 < \infty$.

Proof. The proof of Theorem 9.1 shows that it is enough to consider the homogeneous equation. Then by inequalities (9.15)–(9.16),

$$\begin{aligned} &\sum_{\ell=n+1}^{n_1} \sum_{|\alpha|=\ell} \|u_\alpha(t)\|_H^2 = \sum_{\ell=n+1}^{n_1} F_\ell(t) \\ &\leq e^{C_2 T} \sum_{k_1,\ldots,k_{n+1}\geq 1} \int_0^T \int_0^t \int_0^{s_n} \ldots \int_0^{s_2} \|\mathcal{M}_{k_{n+1}}\Phi_{t,s_n}\mathcal{M}_{k_n} \ldots \Phi_{s_1} u_0\|_H^2 ds^n dt. \end{aligned} \tag{9.28}$$

By Corollary 6.6,

$$\begin{aligned} &\int_0^T \|u_\alpha(s)\|_V^2 ds \\ &= \sum_{n\geq 1} \sum_{k_1,\ldots,k_n\geq 1} \int_0^T \int_0^t \int_0^{s_n} \ldots \int_0^{s_2} \|\mathcal{M}_{k_n}\Phi_{t,s_n}\mathcal{M}_{k_n} \ldots \Phi_{s_1} u_0\|_V^2 ds^n dt < \infty. \end{aligned} \tag{9.29}$$

As a result, (9.6) and (9.29) imply

$$\lim_{n\to\infty} \int_0^T \int_0^t \int_0^{s_n} \ldots \int_0^{s_2} \|\mathcal{M}_{k_{n+1}}\Phi_{t,s_n}\mathcal{M}_{k_n} \ldots \mathcal{M}_{k_1}\Phi_{s_1} u_0\|_H^2 ds^n dt = 0,$$

which, by (9.28), implies uniform, with respect to t, convergence of the series $\sum_{\alpha\in\mathcal{J}} \|u_\alpha(t)\|_H^2$. Corollary 9.2 is proved. □

Corollary 9.3. *Let $a_{ij}, b_i, c, \sigma_{ik}, \nu_k$ be deterministic measurable functions of (t,x) so that*

$$|a_{ij}(t,x)| + |b_i(t,x)| + |c(t,x)| + |\sigma_{ik}(t,x)| + |\nu_k(t,x)| \leq K,$$

$i,j = 1,\dots,d,\ k \geq 1,\ x \in \mathbb{R}^d,\ 0 \leq t \leq T;$

$$\left(a_{ij}(t,x) - \frac{1}{2}\sigma_{ik}(t,x)\sigma_{jk}(t,x)\right) y_i y_j \geq 0,$$

$x, y \in \mathbb{R}^d,\ 0 \leq t \leq T;$ *and*

$$\sum_{k\geq 1} |\nu_k(t,x)|^2 \leq C_\nu < \infty,$$

$x \in \mathbb{R}^d,\ 0 \leq t \leq T$. *Consider the equation*

$$du = (D_i(a_{ij}D_j u) + b_i D_i u + c\,u + f)dt + (\sigma_{ik}D_i u + \nu_k u + g_k)dw_k. \tag{9.30}$$

Assume that the input data satisfy $u_0 \in L_2(\mathbb{R}^d)$, $f \in L_2((0,T); H_2^{-1}(\mathbb{R}^d))$, $\sum_{k\geq 1} \|g_k\|^2_{L_2((0,T)\times\mathbb{R}^d)} < \infty$, and there exists an $\varepsilon > 0$ so that

$$a_{ij}(t,x) y_i y_j \geq \varepsilon |y|^2,\ x, y \in \mathbb{R}^d,\ 0 \leq t \leq T.$$

Then there exists a unique Wiener Chaos solution $u = u(t,x)$ of (9.30). The solution has the following regularity:

$$u(t,\cdot) \in L_2(\mathbb{W}; L_2(\mathbb{R}^d)),\ 0 \leq t \leq T, \tag{9.31}$$

and

$$\begin{aligned} \mathbb{E}\|u\|^2_{L_2(\mathbb{R}^d)}(t) \leq C^* \Big(&\|u_0\|^2_{L_2(\mathbb{R}^d)} + \|f\|^2_{L_2((0,T);H_2^{-1}(\mathbb{R}^d))} \\ &+ \sum_{k\geq 1} \|g_k\|^2_{L_2((0,T)\times\mathbb{R}^d)}\Big), \end{aligned} \tag{9.32}$$

where the positive number C^* depends only on C_ν, K, T, and ε.

Remark 9.4.
(1) If (2.5) holds instead of (9.3), then the proof of Theorem 9.1, in particular, (9.15)–(9.16), shows that the term $\mathbb{E}\|u(t)\|^2_H$ in the left-hand-side of inequality (9.4) can be replaced with

$$\mathbb{E}\left(\|u(t)\|^2_H + \varepsilon \int_0^t \|u(s)\|^2_V ds \right).$$

(2) If $f = g_k = 0$ and the equation is fully degenerate, that is, $2\langle \mathcal{A}(t)v, v\rangle + \sum_{k\geq 1} \|\mathcal{M}_k(t)v\|^2_H = 0$, $t \in [0,T]$, then it is natural to expect conservation of energy. Once again, analysis of (9.15)–(9.16) shows that equality

$$\mathbb{E}\|u(t)\|^2_H = \|u_0\|^2_H$$

holds if and only if

$$\lim_{n\to\infty} \int_0^T \int_0^t \int_0^{s_n} \dots \int_0^{s_2} \|\mathcal{M}_{k_{n+1}} \Phi_{t,s_n} \mathcal{M}_{k_n} \dots \mathcal{M}_{k_1} \Phi_{s_1} u_0\|^2_H ds^n dt = 0.$$

The proof of Corollary 9.2 shows that a sufficient condition for the conservation of energy in a fully degenerate homogeneous equation is $\mathbb{E}\int_0^T \|u(t)\|^2_V dt < \infty$.

One of applications of the Wiener Chaos solution is new numerical methods for solving the evolution equations. Indeed, an approximation of the solution is obtained by truncating the sum $\sum_{\alpha\in\mathcal{J}} u_\alpha(t)\xi_\alpha$. For the Zakai filtering equation, these numerical methods were studied in [25, 26, 27]; see also Section 11 below. The main question in the analysis is the rate of convergence, in n, of the series $\sum_{n\geq 1}\sum_{|\alpha|=n}\|u(t)\|_H^2$. In general, this convergence can be arbitrarily slow. For example, consider the equation

$$du=\frac{1}{2}u_{xx}dt+u_x dw(t),\ t>0,\ x\in\mathbb{R},$$

in the normal triple $(H_2^1(\mathbb{R}), L_2(\mathbb{R}), H_2^{-1}(\mathbb{R}))$, with initial condition $u|_{t=0}=u_0\in L_2(\mathbb{R})$. It follows from (9.12) that

$$F_n(t)=\sum_{|\alpha|=n}\|u\|^2_{L_2(\mathbb{R})}(t)=\frac{t^n}{n!}\int_{\mathbb{R}}|y|^{2n}e^{-y^2t}|\hat{u}_0|^2dy,$$

where $\hat{u}_0$ is the Fourier transform of u_0. If

$$|\hat{u}_0(y)|^2=\frac{1}{(1+|y|^2)^{\gamma}},\ \gamma>1/2,$$

then the rate of decay of $F_n(t)$ is close to $n^{-(1+2\gamma)/2}$. Note that, in this example, $\mathbb{E}\|u\|^2_{L_2(\mathbb{R})}(t)=\|u_0\|^2_{L_2(\mathbb{R})}$.

An exponential convergence rate that is uniform in $\|u_0\|_H^2$ is achieved under strong parabolicity condition (2.5). An even faster factorial rate is achieved when the operators $\mathcal{M}_k$ are bounded on H.

Theorem 9.5. *Assume that the there exist a positive number ε and a real number C_0 so that*

$$2\langle\mathcal{A}(t)v,v\rangle+\sum_{k\geq 1}\|\mathcal{M}_k(t)v\|_H^2+\varepsilon\|v\|_V^2\leq C_0\|v\|_H^2,\ t\in[0,T],\ v\in V.$$

Then there exists a positive number b so that, for all $t\in[0,T]$,

$$\sum_{|\alpha|=n}\|u_\alpha(t)\|_H^2\leq\frac{\|u_0\|_H^2}{(1+b)^n}. \tag{9.33}$$

If, in addition, $\sum_{k\geq 1}\|\mathcal{M}_k(t)\varphi\|_H^2\leq C_3\|\varphi\|_H^2$, then

$$\sum_{|\alpha|=n}\|u_\alpha(t)\|_H^2\leq\frac{(C_3t)^n}{n!}e^{C_1t}\|u_0\|_H^2. \tag{9.34}$$

Proof. If C^* is from (9.6) and $b=\varepsilon/C^*$, then the operators $\sqrt{1+b}\mathcal{M}_k$ satisfy

$$2\langle\mathcal{A}(t)v,v\rangle+(1+b)\sum_{k\geq 1}\|\mathcal{M}_k(t)\|_H^2\leq C_0\|v\|_H^2.$$

By Theorem 9.1,

$$(1+b)^n\sum_{k_1,\ldots,k_n\geq 1}\int_0^t\int_0^{s_n}\ldots\int_0^{s_2}\|\Phi_{t,s_n}\mathcal{M}_{k_n}\ldots\mathcal{M}_{k_1}\Phi_{s_1}u_0\|_H^2ds^n\leq\|u_0\|_H^2,$$

and (9.33) follows.

To establish (9.34), note that, by (9.2),

$$\|\Phi_t f\|_H^2\leq e^{C_1t}\|f\|_H^2,$$

and therefore the result follows from (9.12). Theorem 9.5 is proved. □

The Wiener Chaos solution of (9.1) is not, in general, a solution of the equation in the sense of Definition 2.2. Indeed, if $u\notin L_2(\Omega\times(0,T);V)$, then the expressions $\langle\mathcal{A}u(s),\varphi\rangle$

and $(\mathcal{M}_k u(s), \varphi)_H$ are not defined. On the other hand, if there is a possibility to move the operators $\mathcal{A}$ and $\mathcal{M}$ from the solution process u to the test function φ, then equation (9.1) admits a natural analog of the traditional weak formulation (2.4).

Theorem 9.6. *In addition to* ***A1–A3****, assume that there exist operators $\mathcal{A}^*(t)$, $\mathcal{M}_k^*(t)$ and a dense subset V_0 of the space V so that*

(1) $\mathcal{A}^*(t)(V_0) \subseteq H$, $\mathcal{M}_k^*(t)(V_0) \subseteq H$, $t \in [0, T]$;
(2) *for every* $v \in V$, $\varphi \in V_0$, *and* $t \in [0, T]$, $\langle \mathcal{A}(t)v, \varphi\rangle = (v, \mathcal{A}^*(t)\varphi)_H$, $(\mathcal{M}_k(t)v, \varphi)_H = (v, \mathcal{M}_k^*(t)\varphi)_H$.

If $u = u(t)$ is the Wiener Chaos solution of (9.1), then, for every $\varphi \in V_0$ and every $t \in [0, T]$, the equality

$$(u(t), \varphi)_H = (u_0, \varphi)_H + \int_0^t (u(s), \mathcal{A}^*(s)\varphi)_H ds + \int_0^t \langle f(s), \varphi\rangle ds + \int_0^t (u(s), \mathcal{M}_k^*(s)\varphi)_H dw_k(s) + \int_0^t (g_k(s), \varphi)_H dw_k(s) \tag{9.35}$$

holds in $L_2(\mathbb{W})$.

Proof. The arguments are identical to the proof of Theorem 8.6(2). □

As was mentioned earlier, the Wiener Chaos solution can be constructed for anticipating equations, that is, equations with $\mathcal{F}_T^W$-measurable input data. With obvious modifications, inequality (9.4) holds if each of the input functions u_0, f, and g_k in (9.1) is a *finite* linear combination of the basis elements ξ_α. The following example demonstrates that inequality (9.4) is impossible for general anticipating equation.

Example 9.7. Let $u = u(t, x)$ be a Wiener Chaos solution of an ordinary differential equation

$$du = u dw(t), 0 < t \leq 1, \tag{9.36}$$

with $u_0 = \sum_{\alpha \in \mathcal{J}} a_\alpha \xi_\alpha$. For $n \geq 0$, denote by (n) the multi-index with $\alpha_1 = n$ and $\alpha_i = 0$, $i \geq 2$, and assume that $a_{(n)} > 0$, $n \geq 0$. Then

$$\mathbb{E}u^2(1) \geq C \sum_{n \geq 0} e^{\sqrt{n}} a_{(n)}^2. \tag{9.37}$$

Indeed, the first column of propagator for $\alpha = (n)$ is $u_{(0)}(t) = a_{(0)}$ and

$$u_{(n)}(t) = a_{(n)} + \sqrt{n} \int_0^t u_{(n-1)}(s) ds,$$

so that

$$u_{(n)}(t) = \sum_{k=0}^{n} \frac{\sqrt{n!}}{\sqrt{(n-k)!k!}} \frac{a_{(n-k)}}{\sqrt{k!}} t^k.$$

Then $u_{(n)}^2(1) \geq \sum_{k=0}^n \binom{n}{k} \frac{a_{(n-k)}^2}{k!}$ and

$$\sum_{n \geq 0} u_{(n)}^2(1) \geq \sum_{n \geq 0} \left(\sum_{k \geq 0} \frac{1}{k!} \binom{n+k}{n} \right) a_{(n)}^2.$$

Since

$$\sum_{k\geq 0}\frac{1}{k!}\binom{n+k}{n} \geq \sum_{k\geq 0}\frac{n^k}{(k!)^2} \geq Ce^{\sqrt{n}},$$

the result follows.

The consequence of Example 9.7 is that it is possible, in (9.1), to have $u_0 \in L_2^n(\mathbb{W}; H)$ for every n, and still get $\mathbb{E}\|u(t)\|_H^2 = +\infty$ for all $t > 0$. More generally, the solution operator for (9.1) is not bounded on any $L_{2,Q}$ or $(\mathcal{S})_{-\rho,-\gamma}$. On the other hand, the following result holds.

Theorem 9.8. *In addition to Assumptions* ***A1****,* ***A2****, let u_0 be an element of $\mathcal{D}'(\mathbb{W}; H)$, f, an element of $\mathcal{D}'(\mathbb{W}; L_2((0,T), V'))$, and each g_k, an element of $\mathcal{D}'(\mathbb{W}; L_2((0,T), H))$. Then the Wiener Chaos solution of equation (9.1) satisfies*

$$\sqrt{\sum_{\alpha\in\mathcal{J}}\frac{\|u_\alpha(t)\|_H^2}{\alpha!}} \leq C\sum_{\alpha\in\mathcal{J}}\frac{1}{\sqrt{\alpha!}}\Bigg(\|u_{0\alpha}\|_H + \left(\int_0^t \|f_\alpha(s)\|_{V'}^2 ds\right)^{1/2} + \left(\sum_{k\geq 1}\int_0^t \|g_{k,\alpha}(s)\|_H^2 ds\right)^{1/2}\Bigg), \tag{9.38}$$

where $C > 0$ depends only on T and the numbers δ, C_1, and C_2 from (9.2) and (9.3).

Proof. To simplify the presentation, assume that $f = g_k = 0$. For fixed $\gamma \in \mathcal{J}$, denote by $u(t;\varphi;\gamma)$ the Wiener Chaos solution of the equation (9.1) with initial condition $u(0;\varphi;\gamma) = \varphi\xi_\gamma$. Denote by (0) the zero multi-index. The structure of the propagator implies the following relation:

$$\frac{u_{\alpha+\gamma}(t;\varphi;\gamma)}{\sqrt{(\alpha+\gamma)!}} = \frac{u_\alpha\left(t; \frac{\varphi}{\sqrt{\gamma!}}; (0)\right)}{\sqrt{\alpha!}}. \tag{9.39}$$

Clearly, $u_\alpha(t;\varphi;\gamma) = 0$ if $|\alpha| < |\gamma|$. If

$$\|v(t)\|_{(\mathcal{S})_{-1,0}(H)}^2 = \sum_{\alpha\in\mathcal{J}}\frac{\|v_\alpha(t)\|_H^2}{\alpha!},$$

then, by linearity and triangle inequality,

$$\|u(t)\|_{(\mathcal{S})_{-1,0}(H)} \leq \sum_{\gamma\in\mathcal{J}}\|u(t;u_{0\gamma};\gamma)\|_{(\mathcal{S})_{-1,0}(H)}.$$

We also have by (9.39) and Theorem 9.1

$$\begin{aligned}\|u(t;u_{0\gamma};\gamma)\|_{(\mathcal{S})_{-1,0}(H)}^2 &= \left\|u\left(t;\frac{u_{0\gamma}}{\sqrt{\gamma!}};(0)\right)\right\|_{(\mathcal{S})_{-1,0}(H)}^2 \\ &\leq \mathbb{E}\left\|u\left(t;\frac{u_{0\gamma}}{\sqrt{\gamma!}};(0)\right)\right\|_H^2 \leq e^{C_2 t}\frac{\|u_{0\gamma}\|_H^2}{\gamma!}.\end{aligned}$$

Inequality (9.38) then follows. Theorem 9.8 is proved. □

Remark 9.9. *Using Proposition 7.2 and the Cauchy-Schwartz inequality, (9.38) can be re-written in a slightly weaker form to reveal continuity of the solution operator for equation (9.1) from* $(\mathcal{S})_{-1,\gamma}$ *to* $(\mathcal{S})_{-1,0}$ *for every* $\gamma > 1$*:*

$$\|u(t)\|^2_{(\mathcal{S})_{-1,0}(H)} \leq C\Bigg(\|u_0\|^2_{(\mathcal{S})_{-1,\gamma}(H)} + \int_0^t \|f(s)\|^2_{(\mathcal{S})_{-1,\gamma}(V')} ds + \sum_{k\geq 1} \int_0^t \|g_k(s)\|^2_{(\mathcal{S})_{-1,\gamma}(H)} ds \Bigg).$$

10. Probabilistic Representation of Wiener Chaos Solutions

The general discussion so far has been dealing with the abstract evolution equation

$$du = (\mathcal{A}u + f)dt + \sum_{k\geq 1} (\mathcal{M}_k u + g_k) dw_k.$$

By further specifying the operators $\mathcal{A}$ and $\mathcal{M}_k$, as well as the input data u_0, f, and g_k, it is possible to get additional information about the Wiener Chaos solution of the equation.

Definition 10.1. *For* $r \in \mathbb{R}$*, the space* $L_{2,(r)} = L_{2,(r)}(\mathbb{R}^d)$ *is the collection of real-valued measurable functions so that* $f \in L_{2,(r)}$ *if and only if* $\int_{\mathbb{R}^d} |f(x)|^2 (1+|x|^2)^r dx < \infty$*. The space* $H^1_{2,(r)} = H^1_{2,(r)}(\mathbb{R}^d)$ *is the collection of real-valued measurable functions so that* $f \in H^1_{2,(r)}$ *if and only if* f *and all the first-order generalized derivatives* $D_i f$ *of* f *belong to* $L_{2,(r)}$*.*

It is known, for example, from Theorem 3.4.7 in [42], that $L_{2,(r)}$ is a Hilbert space with norm

$$\|f\|^2_{0,(r)} = \int_{\mathbb{R}^d} |f(x)|^2 (1+|x|^2)^r dx,$$

and $H^1_{2,(r)}$ is a Hilbert space with norm

$$\|f\|_{1,(r)} = \|f\|_{0,(r)} + \sum_{i=1}^d \|D_i f\|_{0,(r)}.$$

Denote by $H^{-1}_{2,(r)}$ the dual of $H^1_{2,(r)}$ with respect to the inner product in $L_{2,(r)}$. Then $(H^1_{2,(r)}, L_{2,(r)}, H^{-1}_{2,(r)})$ is a normal triple of Hilbert spaces.

Let $\mathbb{F} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a stochastic basis with the usual assumptions and $w_k = w_k(t),\ k \geq 1,\ t \geq 0$, a collection of standard Wiener processes on $\mathbb{F}$. Consider the linear equation

$$du = (a_{ij} D_i D_j u + b_i D_i u + cu + f)dt + (\sigma_{ik} D_i u + \nu_k u + g_k) dw_k \tag{10.1}$$

under the following **assumptions:**

- **B0** All coefficients, free terms, and the initial condition are non-random.
- **B1** The functions $a_{ij} = a_{ij}(t,x)$ and their first-order derivatives with respect to x are uniformly bounded in (t,x), and the matrix (a_{ij}) is uniformly positive definite, that is, there exists a $\delta > 0$ so that, for all vectors $y \in \mathbb{R}^d$ and all (t,x), $a_{ij} y_i y_j \geq \delta |y|^2$.
- **B2** The functions $b_i = b_i(t,x)$, $c = c(t,x)$, and $\nu_k = \nu_k(t,x)$ are measurable and bounded in (t,x).
- **B3** The functions $\sigma_{ik} = \sigma_{ik}(t,x)$ are continuous and bounded in (t,x).

B4 The functions $f = f(t,x)$ and $g_k = g_k(t,x)$ belong to $L_2((0,T);L_{2,(r)})$ for some $r \in \mathbb{R}$.

B5 The initial condition $u_0 = u_0(x)$ belongs to $L_{2,(r)}$.

Under Assumptions **B2**–**B4**, there exists a sequence $Q = \{q_k, k \geq 1\}$ of positive numbers with the following properties:

P1 The matrix A with $A_{ij} = a_{ij} - (1/2)\sum_{k\geq 1} q_k \sigma_{ik}\sigma_{jk}$ satisfies

$$A_{ij}(t,x)y_i y_j \geq 0,$$

$x, y \in \mathbb{R}^d$, $0 \leq t \leq T$.

P2 There exists a number $C > 0$ so that

$$\sum_{k\geq 1}\left(\sup_{t,x}|q_k \nu_k(t,x)|^2 + \int_0^T \|q_k g_k\|^p_{0,(r)}(t)dt\right) \leq C.$$

For the matrix A and each t,x, we have $A_{ij}(t,x) = \tilde{\sigma}_{ik}(t,x)\tilde{\sigma}_{jk}(t,x)$, where the functions $\tilde{\sigma}_{ik}$ are bounded. This representation might not be unique; see, for example, [7, Theorem III.2.2] or [44, Lemma 5.2.1]. Given any such representation of A, consider the following backward Itô equation

$$\begin{aligned} X_{t,x,i}(s) = x_i + \int_s^t B_i(\tau, X_{t,x}(\tau))\,d\tau + \sum_{k\geq 1} q_k \sigma_{ik}(\tau, X_{t,x}(\tau))\,\overleftarrow{dw_k}(\tau) \\ + \int_s^t \tilde{\sigma}_{ik}(\tau, X_{t,x}(\tau))\,\overleftarrow{d\tilde{w}_k}(\tau);\ s \in (0,t),\ t \in (0,T],\ t - \text{fixed}, \end{aligned} \tag{10.2}$$

where $B_i = b_i - \sum_{k\geq 1} q_k^2 \sigma_{ik}\nu_k$ and $\tilde{w}_k,\ k \geq 1$, are independent standard Wiener processes on $\mathbb{F}$ that are independent of $w_k,\ k \geq 1$. This equation might not have a strong solution, but does have weak, or martingale, solutions due to Assumptions **B1**–**B3** and properties **P1** and **P2** of the sequence Q; this weak solution is unique in the sense of probability law [44, Theorem 7.2.1].

The following result is a variation of Theorem 4.1 in [29].

Theorem 10.2. *Under assumptions **B0**–**B5** equation (10.1) has a unique $w(H^1_{2,(r)}, H^{-1}_{2,(r)})$ Wiener Chaos solution. If Q is a sequence with properties **P1** and **P2**, then the solution of (10.1) belongs to*

$$L_{2,Q}\left(\mathbb{W}; L_2((0,T);H^1_{2,(r)})\right) \bigcap L_{2,Q}\left(\mathbb{W}; \mathbf{C}((0,T);L_{2,(r)})\right)$$

and has the following representation:

$$\begin{aligned} u(t,x) = \mathcal{Q}^{-1}\mathbb{E}\Bigg(\int_0^t f(s, X_{t,x}(s))\gamma(t,s,x)ds \\ + \sum_{k\geq 1}\int_0^t q_k g_k(s, X_{t,x}(s))\gamma(t,s,x)\overleftarrow{dw_k}(s) + u_0(X_{t,x}(0))\gamma(t,0,x)\Big|\mathcal{F}^W_t\Bigg),\ t \leq T, \end{aligned} \tag{10.3}$$

where $X_{t,x}(s)$ is a weak solution of (10.2), and

$$\gamma(t,s,x) = \exp\left(\int_s^t c(\tau, X_{t,x}(\tau))d\tau + \sum_{k\geq 1}\int_s^t q_k\nu_k(\tau, X_{t,x}(\tau))\overleftarrow{dw_k}(\tau) - \frac{1}{2}\int_s^t \sum_{k\geq 1} q_k^2|\nu_k(\tau, X_{t,x}(\tau))|^2 d\tau\right). \tag{10.4}$$

Proof. It is enough to establish (10.3) when $t = T$. Consider the equation

$$dU = (a_{ij}D_iD_jU + b_iD_iU + cU + f)dt + \sum_{k\geq 1}(\sigma_{ik}D_iU + \nu_kU + g_k)q_kdw_k \tag{10.5}$$

with initial condition $U(0,x) = u_0(x)$. Applying Theorem 2.4 in the normal triple $(H^1_{2,(r)}, L_{2,(r)}, H^{-1}_{2,(r)})$, we conclude that there is a unique solution

$$U \in L_2\left(\mathbb{W}; L_2((0,T); H^1_{2,(r)})\right) \bigcap L_2\left(\mathbb{W}; \mathbf{C}((0,T); L_{2,(r)})\right)$$

of this equation. By Proposition 7.4, the process $u = \mathcal{Q}^{-1}U$ is the corresponding Wiener Chaos solution of (10.1). To establish representation (10.3), consider the S-transform U_h of U. According to Theorem 8.1, the function U_h is the unique $w(H^1_{2,(r)}, H^{-1}_{2,(r)})$ solution of the equation

$$dU_h = (a_{ij}D_iD_jU_h + b_iD_iU_h + cU_h + f)dt + \sum_{k\geq 1}(\sigma_{ik}D_iU_h + \nu_kU_h + g_k)q_kh_kdt \tag{10.6}$$

with initial condition $U_h|_{t=0} = u_0$. We also define

$$Y(T,x) = \int_0^T f(s, X_{T,x}(s))\gamma(T,s,x)ds + \sum_{k\geq 1}\int_0^T g_k(s, X_{T,x}(s))\gamma(T,s)q_k\overleftarrow{dw_k}(s) + u_0(X_{T,x}(0))\gamma(T,0,x). \tag{10.7}$$

By direct computation,

$$\mathbb{E}\left(\mathbb{E}\left(\mathcal{E}(h)Y(T,x)|\mathcal{F}^W_T\right)\right) = \mathbb{E}\left(\mathcal{E}(h)Y(T,x)\right) = \mathbb{E}'Y(T,x),$$

where $\mathbb{E}'$ is the expectation with respect to the measure $d\mathbb{P}'_T = \mathcal{E}(h)d\mathbb{P}_T$ and $\mathbb{P}_T$ is the restriction of $\mathbb{P}$ to $\mathcal{F}^W_T$.

To proceed, let us first assume that the input data u_0, f, and g_k are all smooth functions with compact support. Then, applying the Feynmann-Kac formula to the solution of equation (10.6) and using the Girsanov theorem (see, for example, Theorems 3.5.1 and 5.7.6 in [15]), we conclude that $U_h(T,x) = \mathbb{E}'Y(T,x)$ or

$$\mathbb{E}\left(\mathcal{E}(h)\mathbb{E}Y(t,x)|\mathcal{F}^W_T\right) = \mathbb{E}\left(\mathcal{E}\left(h\right)U(T,x)\right).$$

By Remark 7.8, the last equality implies $U\left(T,\cdot\right) = \mathbb{E}\left(Y(T,\cdot)|\mathcal{F}^W_T\right)$ as elements of $L_2\left(\Omega; L_{2,(r)}(\mathbb{R}^d)\right)$.

To remove the additional smoothness assumption on the input data, let u_0^n, f^n, and g_k^n be sequences of smooth compactly supported functions so that

$$\lim_{n\to\infty}\Bigg(\|u_0-u_0^n\|^2_{L_{2,(r)}(\mathbb{R}^d)}+\int_0^T\|f-f^n\|^2_{L_{2,(r)}(\mathbb{R}^d)}(t)dt + \sum_{k\geq 1}\int_0^T q_k^2\|g_k-g_k^n\|^2_{L_{2,(r)}(\mathbb{R}^d)}(t)dt\Bigg)=0. \tag{10.8}$$

Denote by U^n and Y^n the corresponding objects defined by (10.5) and (10.7) respectively. By Theorem 9.1, we have

$$\lim_{n\to\infty}\mathbb{E}\|U-U^n\|^2_{L_{2,(r)}(\mathbb{R}^d)}(T)=0. \tag{10.9}$$

To complete the proof, it remains to show that

$$\lim_{n\to\infty}\mathbb{E}\left\|\mathbb{E}\left(Y(T,\cdot)-Y^n(T,\cdot)\Big|\mathcal{F}_T^W\right)\right\|^2_{L_{2,(r)}(\mathbb{R}^d)}=0. \tag{10.10}$$

To this end, introduce a new probability measure $\mathbb{P}_T''$ by

$$d\mathbb{P}_T''=\exp\Bigg(2\sum_{k\geq 1}\int_0^T\nu_k(s,X^Q_{T,x}(s))q_k\overleftarrow{dw_k}(s) - 2\int_0^T\sum_{k\geq 1}q_k^2|\nu_k(s,X^Q_{T,x}(s))|^2ds\Bigg)d\mathbb{P}_T.$$

By Girsanov's theorem, equation (10.2) can be rewritten as

$$\begin{aligned} X_{T,x,i}(s)=x_i&+\int_s^T\sum_{k\geq 1}\sigma_{ik}(\tau,X_{T,x}(\tau))\,h_k(\tau)\,q_k d\tau\\ &+\int_s^t(b_i+\sum_{k\geq 1}q_k^2\sigma_{ik}\nu_k)(\tau,X_{T,x}(\tau))\,d\tau\\ &+\int_s^t\sum_{k\geq 1}q_k\sigma_{ik}(\tau,X_{T,x}(\tau))\,\overleftarrow{dw_k''}(\tau)+\int_s^t\tilde{\sigma}_{ik}(\tau,X_{T,x}(\tau))\,\overleftarrow{d\tilde{w}''_k}(\tau),\end{aligned} \tag{10.11}$$

where w_k'' and $\tilde{w}''_k$ are independent Winer processes with respect to the measure $\mathbb{P}_T''$. Denote by $p(s,y|x)$ the corresponding distribution density of $X_{T,x}(s)$ and write $\ell(x)=(1+|x|^2)^r$. It then follows by the Hölder and Jensen inequalities that

$$\begin{aligned} &\mathbb{E}\left\|\mathbb{E}\left(\int_0^T\gamma^2(T,s,\cdot)(f-f^n)(s,X_{T,\cdot}(s))ds\Big|\mathcal{F}_T^W\right)\right\|^2_{L_{2,(r)}(\mathbb{R}^d)}\\ &\leq K_1\int_{\mathbb{R}^d}\left(\int_0^T\mathbb{E}\left(\gamma^2(T,s,x)(f-f^n)^2(s,X_{T,x}(s))\right)ds\right)\ell(x)dx\\ &\leq K_2\int_{\mathbb{R}^d}\left(\int_0^T\mathbb{E}''(f-f^n)^2(s,X_{T,x}(s))ds\right)\ell(x)dx\\ &=K_2\int_{\mathbb{R}^d}\int_0^T\int_{\mathbb{R}^d}(f(s,y)-f^n(s,y))^2p(s,y|x)dy\;ds\;\ell(x)dx,\end{aligned} \tag{10.12}$$

where the number K_1 depends only on T, and the number K_2 depends only on T and $\sup_{(t,x)}|c(t,x)|+\sum_{k\geq 1}q_k^2\sup_{(t,x)}|\nu_k(t,x)|^2$. Assumptions **B0**–**B2** imply that there exist positive numbers K_3 and K_4 so that

$$p(s,y|x) \le \frac{K_3}{(T-s)^{d/2}} \exp\left(-K_4 \frac{|x-y|^2}{T-s}\right); \tag{10.13}$$

see, for example, [6]. As a result,

$$\int_{\mathbb{R}^d} p(s,y|x)\ell(x)dx \le K_5 \ell(y),$$

and

$$\begin{aligned} &\int_{\mathbb{R}^d} \int_0^T \int_{\mathbb{R}^d} (f(s,y)-f^n(s,y))^2 p(s,y|x) dy\, ds\, \ell(x)dx \\ &\le K_5 \int_0^T \|f-f^n\|^2_{L_{2,(r)}(\mathbb{R}^d)}(s)ds \to 0,\ n\to\infty, \end{aligned} \tag{10.14}$$

where the number K_5 depends only on K_3, K_4, T, and r.

Calculations similar to (10.12)–(10.14) show that

$$\begin{aligned} &\mathbb{E}\left\|\mathbb{E}\left(\gamma^2(T,0,\cdot)(u_0-u_0^n)(X_{T,\cdot}(0))\Big|\mathbb{W}\right)\right\|^2_{L_{2,(r)}(\mathbb{R}^d)} \\ &+\mathbb{E}\left\|\mathbb{E}\left(\int_0^T \sum_{k\ge1}(g_k-g_k^n)(s,X_{T,\cdot}(s))\gamma(t,s,\cdot)q_k \overleftarrow{dw_k}(s)\Big|\mathbb{W}\right)\right\|^2_{L_{2,(r)}(\mathbb{R}^d)} \to 0 \end{aligned} \tag{10.15}$$

as $n\to\infty$. Then convergence (10.10) follows, which, together with (10.9), implies that $U(T,\cdot) = \mathbb{E}\left(U^Q(T,\cdot)|\mathcal{F}_T^W\right)$ as elements of $L_2\left(\Omega; L_{2,(r)}(\mathbb{R}^d)\right)$. It remains to note that $u = \mathcal{Q}^{-1}U$. Theorem 10.2 is proved. □

Given $f\in L_{2,(r)}$, we say that $f\ge 0$ if and only if

$$\int_{\mathbb{R}^d} f(x)\varphi(x)dx \ge 0$$

for every non-negative $\varphi\in \mathbf{C}_0^\infty(\mathbb{R}^d)$. Then Theorem 10.2 implies the following result.

Corollary 10.3. *In addition to Assumptions* ***B0–B5****, let* $u_0\ge 0$*,* $f\ge 0$*, and* $g_k=0$ *for all* $k\ge 1$*. Then* $u\ge 0$*.*

Proof. This follows from (10.3) and Proposition 7.11. □

Example 10.4. *(Krylov-Veretennikov formula)*

Consider the equation

$$du = (a_{ij}D_iD_ju + b_iD_iu)\,dt + \sum_{k=1}^d \sigma_{ik}D_iu dw_k,\ u(0,x) = u_0(x). \tag{10.16}$$

Assume **B0–B5** and suppose that $a_{ij}(t,x) = \frac{1}{2}\sigma_{ik}(t,x)\sigma_{jk}(t,x)$. By Theorem 9.1, equation (10.16) has a unique Wiener chaos solution so that

$$\mathbb{E}\|u\|^2_{L_2(\mathbb{R}^d)}(t) \le C^*\|u_0\|^2_{L_2(\mathbb{R}^d)}$$

and

$$u(t,x) = \sum_{n=1}^{\infty}\sum_{|\alpha|=n} u_\alpha(t,x)\xi_\alpha = u_0(x) + \sum_{n=1}^{\infty}\sum_{k_1,\ldots,k_n=1}^{d}\int_0^t\int_0^{s_n}\cdots\int_0^{s_2} \Phi_{t,s_n}\sigma_{jk_n}D_j\cdots\Phi_{s_2,s_1}\sigma_{ik_1}D_i\Phi_{s_1,0}u_0(x)dw_{k_1}(s_1)\cdots dw_{k_n}(s_n), \tag{10.17}$$

where $\Phi_{t,s}$ is the semi-group generated by the operator $\mathcal{A} = a_{ij}D_iD_ju + b_iD_iu$. On the other hand, in this case, Theorem 10.2 yields

$$u(t,x) = \mathbb{E}\Bigg(u_0(X_{t,x}(0))\,|\mathcal{F}_t^W\Bigg),$$

where $W = (w_1, ..., w_d)$ and

$$X_{t,x,i}(s) = x_i + \int_s^t b_i(\tau, X_{t,x}(\tau))\,d\tau + \sum_{k=1}^{d}\sigma_{ik}(\tau, X_{t,x}(\tau))\,\overleftarrow{dw_k}(\tau) \qquad s\in(0,t),\ t\in(0,T],\ t-\text{fixed}. \tag{10.18}$$

Thus, we have arrived at the Krylov-Veretennikov formula [20, Theorem 4]

$$\mathbb{E}\left(u_0(X_{t,x}(0))\,|\mathcal{F}_t^W\right) = u_0(x) + \sum_{n=1}^{\infty}\sum_{k_1,\ldots,k_n=1}^{d}\int_0^t\int_0^{s_n}\cdots\int_0^{s_2} \Phi_{t,s_n}\sigma_{jk_n}D_j\cdots\Phi_{s_2,s_1}\sigma_{ik_1}D_i\Phi_{s_1,0}u_0(x)dw_{k_1}(s_1)\cdots dw_{k_n}(s_n). \tag{10.19}$$

11. Wiener Chaos and Nonlinear Filtering

In this section, we discuss some applications of the Wiener Chaos expansion to numerical solution of the nonlinear filtering problem for diffusion processes; the presentation is essentially based on [25].

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with independent standard Wiener processes $W = W(t)$ and $V = V(t)$ of dimensions d_1 and r respectively. Let X_0 be a random variable independent of W and V. In the *diffusion filtering model*, the unobserved d - dimensional state (or signal) process $X = X(t)$ and the r-dimensional observation process $Y = Y(t)$ are defined by the stochastic ordinary differential equations

$$\begin{aligned} &dX(t) = b(X(t))dt + \sigma(X(t))dW(t) + \rho(X(t))dV(t),\\ &dY(t) = h(X(t))dt + dV(t),\ 0 < t \le T;\\ &X(0) = X_0, \quad Y(0) = 0, \end{aligned} \tag{11.1}$$

where $b(x)\in\mathbb{R}^d$, $\sigma(x)\in\mathbb{R}^{d\times d_1}$, $\rho(x)\in\mathbb{R}^{d\times r}$, $h(x)\in\mathbb{R}^r$.

Denote by $\mathbf{C}^n(\mathbb{R}^d)$ the Banach space of bounded, n times continuously differentiable functions on $\mathbb{R}^d$ with finite norm

$$\|f\|_{\mathbf{C}^n(\mathbb{R}^d)} = \sup_{x\in\mathbb{R}^d}|f(x)| + \max_{1\le k\le n}\sup_{x\in\mathbb{R}^d}|D^kf(x)|.$$

Assumption R1. The the components of the functions σ and ρ are in $\mathbf{C}^2(\mathbb{R}^d)$, the components of the functions b are in $\mathbf{C}^1(\mathbb{R})$, the components of the function h are bounded measurable, and the random variable X_0 has a density u_0.

Assumption R2. The matrix $\sigma\sigma^*$ is uniformly positive definite: there exists an $\varepsilon > 0$ so that

$$\sum_{i,j=1}^{d}\sum_{k=1}^{d_1}\sigma_{ik}(x)\sigma_{jk}(x)y_iy_j \geq \varepsilon|y|^2,\ x, y \in \mathbb{R}^d.$$

Under Assumption **R1** system (11.1) has a unique strong solution [15, Theorems 5.2.5 and 5.2.9]. Extra smoothness of the coefficients in assumption **R1** insure the existence of a convenient representation of the optimal filter.

If $f = f(x)$ is a scalar measurable function on $\mathbb{R}^d$ so that $\sup_{0\leq t\leq T}\mathbb{E}|f(X(t))|^2 < \infty$, then the *filtering problem* for (11.1) is to find the best mean square estimate $\hat{f}_t$ of $f(X(t)),\ t \leq T$, given the observations $Y(s),\ 0 < s \leq t$.

Denote by $\mathcal{F}_t^Y$ the σ-algebra generated by $Y(s),\ 0 \leq s \leq t$. Then the properties of the conditional expectation imply that the solution of the filtering problem is

$$\hat{f}_t = \mathbb{E}\left(f(X(t))|\mathcal{F}_t^Y\right).$$

To derive an alternative representation of $\hat{f}_t$, some additional constructions will be necessary.

Define a new probability measure $\widetilde{\mathbb{P}}$ on $(\Omega, \mathcal{F})$ as follows: for $A \in \mathcal{F}$,

$$\widetilde{\mathbb{P}}(A) = \int_A Z_T^{-1}d\mathbb{P},$$

where

$$Z_t = \exp\left\{\int_0^t h^*(X(s))dY(s) - \frac{1}{2}\int_0^t |h(X(s))|^2 ds\right\}$$

(here and below, if $\zeta \in \mathbb{R}^k$, then ζ is a *column* vector, $\zeta^* = (\zeta_1, \ldots, \zeta_k)$, and $|\zeta|^2 = \zeta^*\zeta$). If the function h is bounded, then the measures $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are equivalent. The expectation with respect to the measure $\widetilde{\mathbb{P}}$ will be denoted by $\widetilde{\mathbb{E}}$.

The following properties of the measure $\widetilde{\mathbb{P}}$ are well known [14, 42]:

P1. Under the measure $\widetilde{\mathbb{P}}$, the distributions of the Wiener process W and the random variable X_0 are unchanged, the observation process Y is a standard Wiener process, and, for $0 < t \leq T$, the state process X satisfies

$$dX(t) = b(X(t))dt + \sigma(X(t))dW(t) + \rho(X(t))\left(dY(t) - h(X(t))dt\right),$$
$$X(0) = X_0;$$

P2. Under the measure $\widetilde{\mathbb{P}}$, the Wiener processes W and Y and the random variable X_0 are independent of one another;

P3. The optimal filter $\hat{f}_t$ satisfies

$$\hat{f}_t = \frac{\widetilde{\mathbb{E}}\left[f(X(t))Z_t|\mathcal{F}_t^Y\right]}{\widetilde{\mathbb{E}}[Z_t|\mathcal{F}_t^Y]}. \tag{11.2}$$

Because of property **P2** of the measure $\widetilde{\mathbb{P}}$ the filtering problem will be studied on the probability space $(\Omega, \mathcal{F}, \widetilde{\mathbb{P}})$. In particular, we will consider the stochastic basis $\widetilde{\mathbb{F}} = \{\Omega, \mathcal{F}, \{\mathcal{F}_t^Y\}_{0\leq t\leq T}, \widetilde{\mathbb{P}}\}$ and the Wiener Chaos space $\widetilde{L}_2(\mathbb{Y})$ of $\mathcal{F}_T^Y$-measurable random variables η with $\widetilde{\mathbb{E}}|\eta|^2 < \infty$.

If the function h is bounded, then, by the Cauchy-Schwarz inequality,

$$\mathbb{E}|\eta| \leq C(h,T)\sqrt{\widetilde{\mathbb{E}}|\eta|^2},\ \eta \in \widetilde{L}_2(\mathbb{Y}). \tag{11.3}$$

Next, consider the partial differential operators

$$\mathcal{L}g(x) = \frac{1}{2}\sum_{i,j=1}^{d} \left((\sigma(x)\sigma^*(x))_{ij} + (\rho(x)\rho^*(x))_{ij}\right) \frac{\partial^2 g(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x)\frac{\partial g(x)}{\partial x_i};$$

$$\mathcal{M}_l g(x) = h_l(x)g(x) + \sum_{i=1}^{d} \rho_{il}(x)\frac{\partial g(x)}{\partial x_i},\ l = 1,\ldots,r;$$

and their adjoints

$$\begin{aligned}\mathcal{L}^* g(x) = &\frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial^2}{\partial x_i \partial x_j}\left((\sigma(x)\sigma^*(x))_{ij} g(x) + (\rho(x)\rho^*(x))_{ij} g(x)\right)\\ &- \sum_{i=1}^{d} \frac{\partial}{\partial x_i}\left(b_i(x)g(x)\right);\end{aligned}$$

$$\mathcal{M}_l^* g(x) = h_l(x)g(x) - \sum_{i=1}^{d} \frac{\partial}{\partial x_i}\left(\rho_{il}(x)g(x)\right),\ l = 1,\ldots,r.$$

Note that, under the assumptions **R1** and **R2**, the operators $\mathcal{L}, \mathcal{L}^*$ are bounded from $H_2^1(\mathbb{R}^d)$ to $H_2^{-1}(\mathbb{R}^d)$, operators $\mathcal{M}, \mathcal{M}^*$ are bounded from $H_2^1(\mathbb{R}^d)$ to $L_2(\mathbb{R}^d)$, and

$$2\langle \mathcal{L}^* v, v\rangle + \sum_{l=1}^{r} \|\mathcal{M}_l^* v\|^2_{L_2(\mathbb{R}^d)} + \varepsilon\|v\|^2_{H_1^2(\mathbb{R}^d)} \leq C\|v\|^2_{L_2(\mathbb{R}^d)},\ v \in H_2^1(\mathbb{R}^d), \tag{11.4}$$

where $\langle\cdot,\cdot\rangle$ is the duality between $H_2^1\mathbb{R}^d$ and $H_2^{-1}(\mathbb{R}^d)$. The following result is well known [42, Theorem 6.2.1].

Proposition 11.1. *In addition to Assumptions **R1** and **R1** suppose that the initial density u_0 belongs to $L_2(\mathbb{R}^d)$. Then there exists a random field $u = u(t,x),\ t \in [0,T],\ x \in \mathbb{R}^d$, with the following properties:*

1. $u \in \widetilde{L}_2(\mathbb{Y}; L_2((0,T); H_2^1(\mathbb{R}^d))) \cap \widetilde{L}_2(\mathbb{Y}; \mathbf{C}([0,T], L_2(\mathbb{R}^d)))$.

2. The function $u(t,x)$ is a traditional solution of the stochastic partial differential equation

$$\begin{aligned} du(t,x) &= \mathcal{L}^* u(t,x)dt + \sum_{l=1}^{r} \mathcal{M}_l^* u(t,x)dY_l(t),\ 0 < t \leq T,\ x \in \mathbb{R}^d;\\ u(0,x) &= u_0(x).\end{aligned} \tag{11.5}$$

3. The equality

$$\widetilde{\mathbb{E}}\left[f(X(t))Z_t|\mathcal{F}_t^Y\right] = \int_{\mathbb{R}^d} f(x)u(t,x)dx \tag{11.6}$$

holds for all bounded measurable functions f.

The random field $u = u(t,x)$ is called the *unnormalized filtering density* (UFD) and the random variable $\phi_t[f] = \widetilde{\mathbb{E}}\left[f(X(t))Z_t|\mathcal{F}_t^Y\right]$, the *unnormalized optimal filter.*

A number of authors studied the nonlinear filtering problem using the multiple Itô integral version of the Wiener chaos [2, 21, 39, 46, etc.]. In what follows, we construct approximations of u and $\phi_t[f]$ using the Cameron-Martin version.

By Theorem 8.6,

$$u(t,x) = \sum_{\alpha\in\mathcal{J}} u_\alpha(t,x)\xi_\alpha, \tag{11.7}$$

where

$$\xi_\alpha = \frac{1}{\sqrt{\alpha!}}\prod_{i,k} H_{\alpha_i^k}(\xi_{ik}),\ \xi_{ik} = \int_0^T m_i(t)dY_k(t),\ k=1,\dots,r; \tag{11.8}$$

as before, $H_n(\cdot)$ is the Hermite polynomial (3.3) and $m_i \in \mathfrak{m}$, an orthonormal basis in $L_2((0,T))$. The functions u_α satisfy the corresponding propagator

$$\begin{aligned}\frac{\partial}{\partial t}u_\alpha(t,x) &= \mathcal{L}^* u_\alpha(t,x)\\ &\quad + \sum_{k,i}\sqrt{\alpha_i^k}\mathcal{M}_k^* u_{\alpha^-(i,k)}(t,x)m_i(t),\ 0<t\le T,\ x\in\mathbb{R}^d;\\ u(0,x) &= u_0(x)I(|\alpha|=0).\end{aligned} \tag{11.9}$$

Writing

$$f_\alpha(t) = \int_{\mathbb{R}^d} f(x)u_\alpha(t,x)dx,$$

we also get a Wiener chaos expansion for the unnormalized optimal filter:

$$\phi_t[f] = \sum_{\alpha\in\mathcal{J}} f_\alpha(t)\xi_\alpha,\ t\in[0,T]. \tag{11.10}$$

For a positive integer N, define

$$u_N(t,x) = \sum_{|\alpha|\le N} u_\alpha(t,x)\xi_\alpha. \tag{11.11}$$

Theorem 11.2. *Under Assumptions **R1** and **R2**, there exists a positive number ν, depending only on the functions h and ρ, so that*

$$\widetilde{\mathbb{E}}\|u-u_N\|^2_{L_2(\mathbb{R}^d)}(t) \le \frac{\|u_0\|^2_{L_2(\mathbb{R}^d)}}{\nu(1+\nu)^N},\ t\in[0,T]. \tag{11.12}$$

If, in addition, $\rho=0$, then there exists a real number C, depending only on the functions b and σ, so that

$$\widetilde{\mathbb{E}}\|u-u_N\|^2_{L_2(\mathbb{R}^d)}(t) \le \frac{(4h_\infty t)^{N+1}}{(N+1)!}e^{Ct}\|u_0\|^2_{L_2(\mathbb{R}^d)},\ t\in[0,T], \tag{11.13}$$

where $h_\infty = \max_{k=1,\dots,r}\sup_x|h_k(x)|$.

For positive integers N,n, define a set of multi-indices

$$\mathcal{J}_N^n = \{\alpha = (\alpha_i^k,\ k=1,\dots,r,\ i=1,\dots,n):\ |\alpha|\le N\}.$$

and let

$$u_N^n(t,x) = \sum_{\alpha\in\mathcal{J}_N^n} u_\alpha(t,x)\xi_\alpha. \tag{11.14}$$

Unlike Theorem 11.2, to compute the approximation error in this case we need to choose a special basis $\mathfrak{m}$ — to do the error analysis for the Fourier approximation in time. We also need extra regularity of the coefficients in the state and observation equations — to have the semi-group generated by the operator $\mathcal{L}^*$ continuous not only in $L_2(\mathbb{R}^d)$ but also in $H_2^2(\mathbb{R}^d)$. The resulting error bound is presented below; the proof can be found in [25].

Theorem 11.3. *Assume that*

(1) *The basis $\mathfrak{m}$ is the Fourier cosine basis*

$$m_1(s) = \frac{1}{\sqrt{T}};\ m_k(t) = \sqrt{\frac{2}{T}} \cos\left(\frac{\pi(k-1)t}{T}\right),\ k > 1;\ 0 \le t \le T, \tag{11.15}$$

(2) *The components of the functions σ are in $\mathbf{C}^4(\mathbb{R}^d)$, the components of the functions b are in $\mathbf{C}^3(\mathbb{R})$, the components of the function h are in $\mathbf{C}^2(\mathbb{R}^d)$; $\rho = 0$; $u_0 \in H_2^2(\mathbb{R}^d)$.*

Then there exist a positive number B_1 and a real number B_2, both depending only on the functions b and σ so that

$$\widetilde{\mathbb{E}} \|u - u_N^n\|^2_{L_2(\mathbb{R}^d)}(T) \le B_1 e^{B_2 T} \left(\frac{(4h_\infty T)^{N+1}}{(N+1)!} e^{Ct} \|u_0\|^2_{L_2(\mathbb{R}^d)} + \frac{T^3}{n} \|u_0\|^2_{H_2^2(\mathbb{R}^d)} \right), \tag{11.16}$$

where $h_\infty = \max_{k=1,\dots,r} \sup_x |h_k(x)|$.

12. Passive Scalar in a Gaussian Field

This section presents the results from [29] and [28] about the stochastic transport equation.

The following viscous transport equation is used to describe time evolution of a scalar quantity θ in a given velocity field $\mathbf{v}$:

$$\dot\theta(t,x) = \nu \Delta\theta(t,x) - \mathbf{v}(t,x) \cdot \nabla\theta(t,x) + f(t,x);\ x \in \mathbb{R}^d,\ d > 1. \tag{12.1}$$

The scalar θ is called passive because it does not affect the velocity field $\mathbf{v}$.

We assume that $\mathbf{v} = \mathbf{v}(t,x) \in \mathbb{R}^d$ is an isotropic Gaussian vector field with zero mean and covariance

$$\mathbb{E}(v^i(t,x)v^j(s,y)) = \delta(t-s)C^{ij}(x-y),$$

where $C = (C^{ij}(x), i,j = 1,\dots,d)$ is a matrix-valued function so that $C(0)$ is a scalar matrix; with no loss of generality we will assume that $C(0) = I$, the identity matrix.

It is known from [22, Section 10.1] that, for an isotropic Gaussian vector field, the Fourier transform $\hat C = \hat C(z)$ of the function $C = C(x)$ is

$$\hat C(y) = \frac{A_0}{(1+|y|^2)^{(d+\alpha)/2}} \left(a\frac{yy^*}{|y|^2} + \frac{b}{d-1}\left(I - \frac{yy^T}{|y|^2} \right) \right), \tag{12.2}$$

where y^* is the row vector $(y_1,\dots,y_d)$, y is the corresponding column vector, $|y|^2 = y^*y$; $\gamma > 0$, $a \ge 0$, $b \ge 0$, $A_0 > 0$ are real numbers. Similar to [22], we assume that $0 < \gamma < 2$. This range of values of γ corresponds to a turbulent velocity field $\mathbf{v}$, also known as the generalized Kraichnan model [8]; the original Kraichnan model [18] corresponds to $a = 0$. For small x, the asymptotics of $C^{ij}(x)$ is $(\delta_{ij} - c^{ij}|x|^\gamma)$ [22, Section 10.2].

By direct computation (cf. [1]), the vector field $\mathbf{v} = (v^1,\dots,v^d)$ can be written as

$$v^i(t,x) = \sigma_k^i(x)\dot w_k(t), \tag{12.3}$$

where $\{\sigma_k,\ k \geq 1\}$ is an orthonormal basis in the space H_C, the reproducing kernel Hilbert space corresponding to the kernel function C. It is known from [22] that H_C is all or part of the Sobolev space $H^{(d+\gamma)/2}(\mathbb{R}^d;\mathbb{R}^d)$.

If $a > 0$ and $b > 0$, then the matrix $\hat{C}$ is invertible and

$$H_C = \left\{ f \in \mathbb{R}^d : \int_{\mathbb{R}^d} \hat{f}^*(y)\hat{C}^{-1}(y)\hat{f}(y)dy < \infty \right\} = H^{(d+\gamma)/2}(\mathbb{R}^d;\mathbb{R}^d),$$

because $\|\hat{C}(y)\| \sim (1+|y|^2)^{-(d+\gamma)/2}$.

If $a > 0$ and $b = 0$, then

$$H_C = \left\{ f \in \mathbb{R}^d : \int_{\mathbb{R}^d} |\hat{f}(y)|^2(1+|y|^2)^{(d+\gamma)/2}dy < \infty;\ yy^*\hat{f}(y) = |y|^2\hat{f}(y) \right\},$$

the subset of gradient fields in $H^{(d+\gamma)/2}(\mathbb{R}^d;\mathbb{R}^d)$, that is, vector fields f for which $\hat{f}(y) = y\hat{F}(y)$ for some scalar $F \in H^{(d+\gamma+2)/2}(\mathbb{R}^d)$.

If $a = 0$ and $b > 0$, then

$$H_C = \left\{ f \in \mathbb{R}^d : \int_{\mathbb{R}^d} |\hat{f}(y)|^2(1+|y|^2)^{(d+\gamma)/2}dy < \infty;\ y^*\hat{f}(y) = 0 \right\},$$

the subset of divergence-free fields in $H^{(d+\gamma)/2}(\mathbb{R}^d;\mathbb{R}^d)$.

By the embedding theorems, each σ_k^i is a bounded continuous function on $\mathbb{R}^d$; in fact, every σ_k^i is Hölder continuous of order $\gamma/2$. In addition, being an element of the corresponding space H_C, each σ_k is a gradient field if $b = 0$ and is divergence free if $a = 0$.

Equation (12.1) becomes

$$d\theta(t,x) = (\nu\Delta\theta(t,x) + f(t,x))dt - \sum_k \sigma_k(x)\cdot\nabla\theta(t,x)dw_k(t). \tag{12.4}$$

We summarize the above constructions in the following **assumptions**:

- **S1** There is a fixed stochastic basis $\mathbb{F} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ with the usual assumptions and $(w_k(t), k \geq 1, t \geq 0)$ is a collection of independent standard Wiener processes on $\mathbb{F}$.
- **S2** For each k, the vector field σ_k is an element of the Sobolev space $H_2^{(d+\gamma)/2}(\mathbb{R}^d;\mathbb{R}^d)$, $0 < \gamma < 2$, $d \geq 2$.
- **S3** For all x, y in $\mathbb{R}^d$, $\sum_k \sigma_k^i(x)\sigma_k^j(y) = C^{ij}(x-y)$ so that the matrix-valued function $C = C(x)$ satisfies (12.2) and $C(0) = I$.
- **S4** The input data θ_0, f are deterministic and satisfy

$$\theta_0 \in L_2(\mathbb{R}^d),\ f \in L_2((0,T);H_2^{-1}(\mathbb{R}^d));$$

  $\nu > 0$ is a real number.

Theorem 12.1. *Let Q be a sequence with $q_k = q < \sqrt{2\nu}$, $k \geq 1$.*

*Under assumptions **S1**–**S4**, there exits a unique $w(H_2^1(\mathbb{R}^d), H_2^{-1}(\mathbb{R}^d))$ Wiener Chaos solution of (12.4). This solution is an $\mathcal{F}_t^W$-adapted process and satisfies*

$$\|\theta\|^2_{L_{2,Q}(\mathbb{W};L_2((0,T);H_2^1(\mathbb{R}^d)))} + \|\theta\|^2_{L_{2,Q}(\mathbb{W};\mathbf{C}((0,T);L_2(\mathbb{R}^d)))} \leq C(\nu,q,T)\left(\|\theta_0\|^2_{L_2(\mathbb{R}^d)} + \|f\|^2_{L_2((0,T);H_2^{-1}(\mathbb{R}^d))}\right).$$

Theorem 12.1 provides new information about the solution of equation (12.1) for all values of $\nu > 0$. Indeed, if $\sqrt{2\nu} > 1$, then $q > 1$ is an admissible choice of the weights, and, by Proposition 7.4(1), the solution θ has Malliavin derivatives of every order. If $\sqrt{2\nu} \leq 1$, then equation (12.4) does not have a square-integrable solution.

Note that if the weight is chosen so that $q = \sqrt{2\nu}$, then equation (12.1) can still be analyzed using Theorem 9.1 in the normal triple $(H_2^1(\mathbb{R}^d), L_2(\mathbb{R}^d), H_2^{-1}(\mathbb{R}^d))$.

If $\nu = 0$, equation (12.4) must be interpreted in the sense of Stratonovich:

$$du(t,x) = f(t,x)dt - \sigma_k(x) \cdot \nabla\theta(t,x) \circ dw_k(t). \tag{12.5}$$

To simplify the presentation, we assume that $f = 0$. If (12.2) holds with $a = 0$, then each σ_k is divergence free and (12.5) has an equivalent Itô form

$$d\theta(t,x) = \frac{1}{2}\Delta\theta(t,x)dt - \sigma_k^i(x)D_i\theta(t,x)dw_k(t). \tag{12.6}$$

Equation (12.6) is a model of non-viscous turbulent transport [5]. The propagator for (12.6) is

$$\frac{\partial}{\partial t}\theta_\alpha(t,x) = \frac{1}{2}\Delta\theta_\alpha(t,x) - \sum_{i,k}\sqrt{\alpha_i^k}\sigma_k^j D_j\theta_{\alpha-(i,k)}(t,x)m_i(t),\ 0 < t \leq T, \tag{12.7}$$

with initial condition $\theta_\alpha(0,x) = \theta_0(x)I(|\alpha| = 0)$.

The following result about solvability of (12.6) is proved in [29] and, in a slightly weaker form, in [28].

Theorem 12.2. *In addition to* ***S1–S4****, assume that each σ_k is divergence free. Then there exits a unique $w(H_2^1(\mathbb{R}^d), H_2^{-1}(\mathbb{R}^d))$ Wiener Chaos solution $\theta = \theta(t,x)$ of (12.6). This solution has the following properties:*

(A) For every $\varphi \in \mathbf{C}_0^\infty(\mathbb{R}^d)$ and all $t \in [0,T]$, the equality

$$(\theta,\varphi)(t) = (\theta_0,\varphi) + \frac{1}{2}\int_0^t(\theta,\Delta\varphi)(s)ds + \int_0^t(\theta,\sigma_k^i D_i\varphi)dw_k(s) \tag{12.8}$$

holds in $L_2(\mathcal{F}_t^W)$, where $(\cdot,\cdot)$ is the inner product in $L_2(\mathbb{R}^d)$.

(B) If $X = X_{t,x}$ is a weak solution of

$$X_{t,x} = x + \int_0^t \sigma_k\left(X_{s,x}\right)dw_k\left(s\right), \tag{12.9}$$

then, for each $t \in [0,T]$,

$$\theta\left(t,x\right) = \mathbb{E}\left(\theta_0\left(X_{t,x}\right)|\mathcal{F}_t^W\right). \tag{12.10}$$

(C) For $1 \leq p < \infty$ and $r \in \mathbb{R}$, define $L_{p,(r)}(\mathbb{R}^d)$ as the Banach space of measurable functions with norm

$$\|f\|^p_{L_{p,(r)}(\mathbb{R}^d)} = \int_{\mathbb{R}^d}|f(x)|^p(1+|x|^2)^{pr/2}dx$$

is finite. Then there exits a number K depending only on p, r so that, for each $t > 0$,

$$\mathbb{E}\|\theta\|^p_{L_{p,(r)}(\mathbb{R}^d)}(t) \leq e^{Kt}\|\theta_0\|^p_{L_{p,(r)}(\mathbb{R}^d)}. \tag{12.11}$$

In particular, if $r = 0$, then $K = 0$.

It follows that, for all s, t and almost all x, y,

$$\mathbb{E}\theta\,(t,x) = \theta_\alpha\,(t,x)\,I_{|\alpha|=0}$$

and

$$\mathbb{E}\theta\,(t,x)\,\theta\,(s,y) = \sum_{\alpha\in\mathcal{J}} \theta_\alpha\,(t,x)\,\theta_\alpha\,(s,y)\,.$$

If the initial condition θ_0 belongs to $L_2(\mathbb{R}^d)\cap L_p(\mathbb{R}^d)$ for $p\geq 3$, then, by (12.11), higher order moments of θ exist. To obtain the expressions of the higher-order moments in terms of the coefficients θ_α, we need some auxiliary constructions.

For $\alpha,\ \beta\in\mathcal{J}$, define $\alpha+\beta$ as the multi-index with components $\alpha_i^k+\beta_i^k$. Similarly, we define the multi-indices $|\alpha-\beta|$ and $\alpha\wedge\beta = \min(\alpha,\beta)$. We write $\beta\leq\alpha$ if and only if $\beta_i^k\leq\alpha_i^k$ for all $i,k\geq 1$. If $\beta\leq\alpha$, we define

$$\binom{\alpha}{\beta} := \prod_{i,k}\frac{\alpha_i^k!}{\beta_i^k!(\alpha_i^k-\beta_i^k)!}.$$

Definition 12.3. *We say that a triple of multi-indices (α,β,γ) is complete and write $(\alpha,\beta,\gamma)\in\triangle$ if all the entries of the multi-index $\alpha+\beta+\gamma$ are even numbers and $|\alpha-\beta|\leq\gamma\leq\alpha+\beta$. For fixed $\alpha,\beta\in\mathcal{J}$, we write*

$$\triangle\,(\alpha) := \{\gamma,\mu\in\mathcal{J} : (\alpha,\gamma,\mu)\in\triangle\}$$

and

$$\triangle(\alpha,\beta) := \{\gamma\in\mathcal{J} : (\alpha,\beta,\gamma)\in\triangle\}.$$

For $(\alpha,\beta,\gamma)\in\triangle$, we define

$$\Psi\,(\alpha,\beta,\gamma) := \sqrt{\alpha!\beta!\gamma!}\left(\left(\frac{\alpha-\beta+\gamma}{2}\right)!\left(\frac{\beta-\alpha+\gamma}{2}\right)!\left(\frac{\alpha+\beta-\gamma}{2}\right)!\right)^{-1}. \tag{12.12}$$

Note that the triple (α,β,γ) is complete if and only if any permutation of the triple (α,β,γ) is complete. Similarly, the value of $\Psi\,(\alpha,\beta,\gamma)$ is invariant under permutation of the arguments.

We also define

$$C\,(\gamma,\beta,\mu) := \left[\binom{\gamma+\beta-2\mu}{\gamma-\mu}\binom{\gamma}{\mu}\binom{\beta}{\mu}\right]^{1/2},\ \mu\leq\gamma\wedge\beta. \tag{12.13}$$

It is readily checked that if f is a function on $\mathcal{J}$, then for $\gamma,\beta\in\mathcal{J}$,

$$\sum_{\mu\leq\gamma\wedge\beta} C\,(\gamma,\beta,p)\,f\,(\gamma+\beta-2\mu) = \sum_{\mu\in(\gamma,\beta)} f\,(\mu)\,\Phi\,(\gamma,\beta,\mu) \tag{12.14}$$

The next theorem presents the formulas for the third and fourth moments of the solution of equation (12.6) in terms of the coefficients θ_α.

Theorem 12.4. *In addition to **S1–S4**, assume that each σ_k is divergence free and the initial condition θ_0 belongs to $L_2(\mathbb{R}^d)\cap L_4(\mathbb{R}^d)$. Then*

$$\mathbb{E}\theta(t,x)\theta\left(t',x'\right)\theta(s,y) = \sum_{(\alpha,\beta,\gamma)\in\triangle}\Psi\,(\alpha,\beta,\gamma)\,\theta_\alpha(t,x)\theta_\beta(t',x')\theta_\gamma\,(s,y) \tag{12.15}$$

and

$$\begin{aligned}(12.16)\qquad &\mathbb{E}\theta(t,x)\theta(t',x')\theta\left(s,y\right)\theta\left(s',y'\right)\\ &=\sum_{\rho\in\triangle(\alpha,\beta)\cap\triangle(\gamma,\kappa)}\Psi\left(\alpha,\beta,\rho\right)\Psi\left(\rho,\gamma,\kappa\right)\theta_{\alpha}\left(t,x\right)\theta_{\beta}(t',x')\theta_{\gamma}\left(s,y\right)\theta_{\kappa}\left(s',y'\right).\end{aligned}$$

Proof. It is known [30] that

$$\xi_{\gamma}\xi_{\beta}=\sum_{\mu\leq\gamma\wedge\beta}C\left(\gamma,\beta,\mu\right)\xi_{\gamma+\beta-2\mu}. \tag{12.17}$$

Let us consider the triple product $\xi_{\alpha}\xi_{\beta}\xi_{\gamma}$. By (12.17),

$$\mathbb{E}\xi_{\alpha}\xi_{\beta}\xi_{\gamma}=\mathbb{E}\sum_{\mu\in\triangle(\alpha,\beta)}\xi_{\gamma}\xi_{\mu}\Psi\left(\alpha,\beta,\mu\right)=\begin{cases}\Psi\left(\alpha,\beta,\gamma\right), & (\alpha,\beta,\gamma)\in\triangle;\\ 0, & \text{otherwise.}\end{cases} \tag{12.18}$$

Equality (12.15) now follows.

To compute the fourth moment, note that

$$\begin{aligned}\xi_{\alpha}\xi_{\beta}\xi_{\gamma}&=\sum_{\mu\leq\alpha\wedge\beta}C\left(\alpha,\beta,\mu\right)\xi_{\alpha+\beta-2\mu}\xi_{\gamma}\\ &=\sum_{\mu\leq\alpha\wedge\beta}C\left(\alpha,\beta,\mu\right)\sum_{\rho\leq(\alpha+\beta-2\mu)\wedge\gamma}C\left(\alpha+\beta-2\mu,\gamma,\rho\right)\xi_{\alpha+\beta+\gamma-2\mu-2\rho}.\end{aligned} \tag{12.19}$$

Repeated applications of (12.14) yield

$$\begin{aligned}\xi_{\alpha}\xi_{\beta}\xi_{\gamma}&=\sum_{\mu\leq\alpha\wedge\beta}C\left(\alpha,\beta,\mu\right)\sum_{\rho\in\triangle(\alpha+\beta-2\mu,\gamma)}\xi_{\rho}\Psi\left(\alpha+\beta-2\mu,\gamma,\rho\right)\\ &=\sum_{\mu\in\triangle(\alpha,\beta)}\sum_{\rho\in\triangle(\mu,\gamma)}\Psi\left(\alpha,\beta,\mu\right)\Psi\left(\mu,\gamma,\rho\right)\xi_{\rho}\end{aligned}$$

Thus,

$$\begin{aligned}\mathbb{E}\xi_{\alpha}\xi_{\beta}\xi_{\gamma}\xi_{\kappa}&=\sum_{\mu\in\triangle(\alpha,\beta)}\sum_{\rho\in\triangle(\mu,\gamma)}\Psi\left(\alpha,\beta,\mu\right)\Psi\left(\mu,\gamma,\rho\right)I_{\{\mu=\kappa\}}\\ &=\sum_{\rho\in\triangle(\alpha,\beta)\cap\triangle(\gamma,\kappa)}\Psi\left(\alpha,\beta,\rho\right)\Psi\left(\rho,\gamma,\kappa\right).\end{aligned}$$

Equality (12.16) now follows. □

In the same way, one can get formulas for fifth- and higher-order moments.

Remark 12.5. *Expressions (12.15) and (12.16) do not depend on the structure of equation (12.6) and can be used to compute the third and fourth moments of any random field with a known Cameron-Martin expansion. The interested reader should keep in mind that the formulas for the moments of orders higher then two should be interpreted with care. In fact, they represent the pseudo-moments (for detail see* [35]*).*

We now return to the analysis of the passive scalar equation (12.4). By reducing the smoothness assumptions on σ_k, it is possible to consider velocity fields $\mathbf{v}$ that are more turbulent than in the Kraichnan model, for example,

$$v^{i}(t,x)=\sum_{k\geq 0}\sigma_{k}^{i}(x)\dot{w}_{k}(t), \tag{12.20}$$

where $\{\sigma_k,\ k \geq 1\}$ is an orthonormal basis in $L_2(\mathbb{R}^d;\mathbb{R}^d)$. With $\mathbf{v}$ as in (12.20), the passive scalar equation (12.4) becomes

$$\dot{\theta}(t,x) = \nu\Delta\theta(t,x) + f(t,x) - \nabla\theta(t,x)\cdot\dot{W}(t,x), \tag{12.21}$$

where $\dot{W} = \dot{W}(t,x)$ is a d-dimensional space-time white noise and the Itô stochastic differential is used. Previously, such equations have been studied using white noise approach in the space of Hida distributions [4, 40]. A summary of the related results can be found in [12, Section 4.3].

The Q-weighted Wiener chaos spaces allow us to state a result that is fully analogous to Theorem 12.1. The proof is derived from Theorem 9.1; see [29] for details.

Theorem 12.6. *Suppose that $\nu > 0$ is a real number, each $|\sigma_k^i(x)|$ is a bounded measurable function, and the input data are deterministic and satisfy $u_0 \in L_2(\mathbb{R}^d)$, $f \in L_2\left((0,T); H_2^{-1}(\mathbb{R}^d)\right)$.*

Fix $\varepsilon > 0$ and let $Q = \{q_k,\ k \geq 1\}$ be a sequence so that, for all $x, y \in \mathbb{R}^d$,

$$2\nu|y|^2 - \sum_{k\geq 1} q_k^2 \sigma_k^i(x)\sigma_k^j(x) y_i y_j \geq \varepsilon |y|^2.$$

Then, for every $T > 0$, there exits a unique $w(H_2^1(\mathbb{R}^d), H_2^{-1}(\mathbb{R}^d))$ Wiener Chaos solution θ of equation

$$d\theta(t,x) = (\nu\Delta\theta(t,x) + f(t,x))dt - \sigma_k(x)\cdot\nabla\theta(t,x)dw_k(t), \tag{12.22}$$

The solution is an $\mathcal{F}_t$-adapted process and satisfies

$$\begin{aligned}\|\theta\|^2_{L_{2,Q}(\mathbb{W};L_2((0,T);H_2^1(\mathbb{R}^d)))} &+ \|\theta\|^2_{L_{2,Q}(\mathbb{W};\mathbf{C}((0,T);L_2(\mathbb{R}^d)))} \\ &\leq C(\nu,q,T)\left(\|\theta_0\|^2_{L_2(\mathbb{R}^d)} + \|f\|^2_{L_2((0,T);H_2^{-1}(\mathbb{R}^d))}\right).\end{aligned}$$

If $\max_i \sup_x |\sigma_k^i(x)| \leq C_k$, $k \geq 1$, then a possible choice of Q is

$$q_k = (\delta\nu)^{1/2}/(d2^k C_k),\ 0 < \delta < 2.$$

If $\sigma_k^i(x)\sigma_k^j(x) \leq C_\sigma < +\infty$, $i,j = 1,\dots,d$, $x \in \mathbb{R}^d$, then a possible choice of Q is

$$q_k = \varepsilon\left(2\nu/(C_\sigma d)\right)^{1/2},\ 0 < \varepsilon < 1.$$

13. Stochastic Navier-Stokes Equation

In this section, we review the main facts about the stochastic Navier-Stokes equation and indicate how the Wiener Chaos approach can be used in the study of non-linear equations. Most of the results of this section come from the two papers [35] and [31].

A priori, it is not clear in what sense the motion described by Kraichnan's velocity (see Section 12) might fit into the paradigm of Newtonian mechanics. Accordingly, relating the Kraichnan velocity field $\mathbf{v}$ to classic fluid mechanics naturally leads to the question whether we can compensate $\mathbf{v}(t,x)$ by a field $\mathbf{u}(t,x)$ that is more regular with respect to the time variable, so that there is a balance of momentum for the resulting field $\mathbf{U}(t,x) = \mathbf{u}(t,x) + \mathbf{v}(t,x)$ or, equivalently, that the motion of a fluid particle in the velocity field $\mathbf{U}(t,x)$ satisfies the Second Law of Newton.

A positive answer to this question is given in [35], where it is shown that the equation for the smooth component $\mathbf{u} = (u^1, \ldots, u^d)$ of the velocity is given by

$$
(13.1) \qquad \begin{cases} du^i = [\nu\Delta u^i - u^j D_j u^i - D_i P + f_i]dt \\ \\ + \left(g_k^i - D_i\tilde{P}_k - D_j\sigma_k^j u^i\right) dw_k,\ i = 1, \ldots, d,\ 0 < t \le T; \\ \\ \operatorname{div}\mathbf{u} = 0,\ \mathbf{u}(0,x) = \mathbf{u}_0(x). \end{cases}
$$

where $w_k,\ k \ge 1$ are independent standard Wiener processes on a stochastic basis $\mathbb{F}$, the functions σ_k^j are given by (12.3), the known functions $\mathbf{f} = (f^1, \ldots, f^d)$, $\mathbf{g}_k = (g_k^i)$, $i = i, \ldots, d$, $k \ge 1$ are, respectively, the drift and the diffusion components of the free force, and the unknown functions P, $\tilde{P}_k$ are the drift and diffusion components of the pressure.

Remark 13.1. *It is useful to study equation (13.1) for more general coefficients σ_k^j. So, in the future, σ_k^j are not necessarily the same as in Section 12.*

We make the following assumptions:

NS1 The functions $\sigma_k^i = \sigma_k^i(t,x)$ are deterministic and measurable,

$$\sum_{k\ge1}\left(\sum_{i=1}^d |\sigma_k^i(t,x)|^2 + |D_i\sigma_k^i(t,x)|^2\right) \le K,$$

and there exists $\varepsilon > 0$ so that, for all $y \in \mathbb{R}^d$,

$$\nu|y|^2 - \frac{1}{2}\sigma_k^i(t,x)\sigma_k^j(t,x)y_iy_j \ge \varepsilon|y|^2,$$

$t \in [0,T]$, $x \in \mathbb{R}^d$.

NS2 The functions f^i, g_k^i are non-random and

$$\sum_{i=1}^d \left(\|f^i\|^2_{L_2((0,T);H_2^{-1}(\mathbb{R}^d))} + \sum_{k\ge1}\|g_k^i\|^2_{L_2((0,T);L_2(\mathbb{R}^d))}\right) < \infty.$$

Remark 13.2. *In **NS1,** the derivatives $D_i\sigma_k^i$ are understood as Schwartz distributions, but it is assumed that $div\sigma := \sum_{i=1}^d \partial_i\sigma^i$ is a bounded l_2-valued function. Obviously, the latter assumption holds in the important case when $\sum_{i=1}^d \partial_i\sigma^i = 0$.*

Our next step is to use the divergence-free property of $\mathbf{u}$ to eliminate the pressure P and $\tilde{P}$ from equation (13.1). For that, we need the decomposition of $L_2(\mathbb{R}^d;\mathbb{R}^d)$ into potential and solenoidal components.

Write $\mathfrak{S}(L_2(\mathbb{R}^d;\mathbb{R}^d)) = \{\mathbf{V} \in L_2(\mathbb{R}^d;\mathbb{R}^d) : \operatorname{div}\mathbf{V} = 0\}$. It is known (see e.g. [16]) that

$$L_2(\mathbb{R}^d;\mathbb{R}^d) = \mathfrak{G}(L_2(\mathbb{R}^d;\mathbb{R}^d)) \oplus \mathfrak{S}(L_2(\mathbb{R}^d;\mathbb{R}^d)),$$

where $\mathfrak{G}(L_2(\mathbb{R}^d;\mathbb{R}^d))$ is a Hilbert subspace orthogonal to $\mathfrak{S}(L_2(\mathbb{R}^d;\mathbb{R}^d))$.

The functions $\mathfrak{G}(\mathbf{V})$ and $\mathfrak{S}(\mathbf{V})$ can be defined for $\mathbf{V}$ from any Sobolev space $H_2^\gamma(\mathbb{R}^d;\mathbb{R}^d)$ and are usually referred to as the potential and the divergence free (or solenoidal), projections, respectively, of the vector field $\mathbf{V}$.

Now let $\mathbf{u}$ be a solution of equation (13.1). Since $\operatorname{div}\mathbf{u} = 0$, we have

$$D_i(\nu\Delta u^i - u^jD_ju^i - D_iP + f^i) = 0;\ D_i(\sigma_k^jD_jw^ju^i + g_k^i - D_i\tilde{P}_k) = 0,\ k \ge 1.$$

As a result,

$$D_iP = \mathfrak{G}(\nu\Delta u^i - u^j D_j u^i + f^i);\ D_i\tilde{P}_k = \mathfrak{G}(\sigma_k^j D_j u^i + g_k^i),\ i = 1,\dots,d,\ k \geq 1.$$

So, instead of equation (13.1), we can and will consider its equivalent form for the unknown vector $\mathbf{u} = (u^1, \dots, u^d)$:

$$d\mathbf{u} = \mathfrak{S}(\nu\Delta\mathbf{u} - u^j D_j\mathbf{u} + \mathbf{f})dt + \mathfrak{S}(\sigma_k^j D_j\mathbf{u} + \mathbf{g}_k)dw_k,\ 0 < t \leq T, \tag{13.2}$$

with initial condition $\mathbf{u}|_{t=0} = \mathbf{u}_0$.

Definition 13.3. *An $\mathcal{F}_t$-adapted random process $\mathbf{u}$ from the space $L_2(\Omega\times[0,T]; H_2^1(\mathbb{R}^d;\mathbb{R}^d))$ is called a solution of equation (13.2) if*

1. *With probability one, the process $\mathbf{u}$ is weakly continuous in $L_2(\mathbb{R}^d;\mathbb{R}^d)$.*
2. *For every $\varphi \in \mathbf{C}_0^\infty(\mathbb{R}^d, \mathbb{R}^d)$, with $\operatorname{div}\varphi = 0$ there exists a measurable set $\Omega' \subset \Omega$ so that, for all $t \in [0,T]$, the equality*

$$\begin{aligned}(u^i, \varphi^i)(t) = (u_0^i, \varphi^i) + \int_0^t \big((\nu D_j u^i, D_j\varphi^i)(s) + \langle f^i, \varphi^i\rangle(s)\big)ds \\ \int_0^t (\sigma_k^j D_j u^i + g^i, \varphi^i)dw_k(s)\end{aligned} \tag{13.3}$$

holds on Ω'. In (13.3), $(\cdot,\cdot)$ is the inner product in $L_2(\mathbb{R}^d)$ and $\langle\cdot,\cdot,\rangle$ is the duality between $H_2^1(\mathbb{R}^d)$ and $H_2^{-1}(\mathbb{R}^d)$.

The following existence and uniqueness result is proved in [31].

Theorem 13.4. *In addition to **NS1** and **NS2**, assume that the initial condition $\mathbf{u}_0$ is non-random and belongs to $L_2(\mathbb{R}^d;\mathbb{R}^d)$. Then there exist a stochastic basis $\mathbb{F} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ with the usual assumptions, a collection $\{w_k, k \geq 1\}$ of independent standard Wiener processes on $\mathbb{F}$, and a process $\mathbf{u}$ so that $\mathbf{u}$ is a solution of (13.2) and*

$$\mathbb{E}\left(\sup_{s\leq T}\|\mathbf{u}(s)\|^2_{L_2(\mathbb{R}^d;\mathbb{R}^d)} + \int_0^T \|\nabla\mathbf{u}(s)\|^2_{L_2(\mathbb{R}^d;\mathbb{R}^d)}\,ds\right) < \infty.$$

If, in addition, $d = 2$, then the solution of (13.2) exists on any prescribed stochastic basis, is strongly continuous in t, is $\mathcal{F}_t^W$-adapted, and is unique, both path-wise and in distribution.

When $d \geq 3$, existence of a strong solution as well as uniqueness (strong or weak) for equation (13.2) are important open problems.

By the Cameron-Martin theorem,

$$\mathbf{u}(t,x) = \sum_{\alpha\in\mathcal{J}} \mathbf{u}_\alpha(t,x)\xi_\alpha.$$

If the solution of (13.2) is $\mathcal{F}_t^W$-adapted, then, using the Itô formula together with relation (5.5) for the time evolution of $\mathbb{E}(\xi_\alpha|\mathcal{F}_t^W)$ and relation (12.17) for the product of two elements of the Cameron-Martin basis, we can derive the propagator system for coefficients $\mathbf{u}_\alpha$ [31, Theorem 3.2]:

Theorem 13.5. *In addition to **NS1** and **NS2**, assume that $\mathbf{u}_0 \in L_2(\mathbb{R}^d;\mathbb{R}^d)$ and equation (13.2) has an $\mathcal{F}_t^W$-adapted solution $\mathbf{u}$ so that*

$$\sup_{t\leq T}\mathbb{E}\|\mathbf{u}\|^2_{L_2(\mathbb{R}^d;\mathbb{R}^d)}(t) < \infty. \tag{13.4}$$

Then

$$\mathbf{u}\,(t,x) = \sum_{\alpha\in\mathcal{J}} \mathbf{u}_\alpha\,(t,x)\,\xi_\alpha, \tag{13.5}$$

and the Hermite-Fourier coefficients $\mathbf{u}_\alpha(t,x)$ *are* $L_2(\mathbb{R}^d;\mathbb{R}^d)$*-valued weakly continuous functions so that*

$$\sup_{t\le T}\sum_{\alpha\in\mathcal{J}} \|\mathbf{u}_\alpha\|^2_{L_2(\mathbb{R}^d;\mathbb{R}^d)}(t) + \int_0^T \sum_{\alpha\in\mathcal{J}} \|\nabla\mathbf{u}_\alpha\|^2_{L_2(\mathbb{R}^d;\mathbb{R}^{d\times d})}(t)\,dt < \infty. \tag{13.6}$$

The functions $\mathbf{u}_\alpha\,(t,x)\,,\alpha\in\mathcal{J}$*, satisfy the (nonlinear) propagator*

$$\begin{aligned}\frac{\partial}{\partial t}\mathbf{u}_\alpha &= \mathfrak{S}\Big(\Delta\mathbf{u}_\alpha - \sum_{\gamma,\beta\in\Delta(\alpha)}\Psi\,(\alpha,\beta,\gamma)\,(\mathbf{u}_\gamma,\nabla\mathbf{u}_\beta) + I_{\{|\alpha|=0\}}\mathbf{f}\\ &\quad+\sum_{j,k}\sqrt{\alpha_j^k}\,\Big(\Big(\sigma^k,\nabla\Big)\,\mathbf{u}_{\alpha^-(j,k)} + I_{\{|\alpha|=1\}}\mathbf{g}^k\Big)\,m_j\,(t)\Big),\ 0<t\le T;\\ \mathbf{u}_\alpha|_{t=0} &= \mathbf{u}_0 I_{\{|\alpha|=0\}};\end{aligned} \tag{13.7}$$

recall that the numbers $\Psi(\alpha,\beta,\gamma)$ *are defined in (12.12).*

One of the questions in the theory of the Navier-Stokes equation is computation of the mean value $\bar{\mathbf{u}}=\mathbb{E}\mathbf{u}$ of the solution. The traditional approach relies on the Reynolds equation for the mean

$$\partial_t\bar{\mathbf{u}} - \nu\Delta\bar{\mathbf{u}} + \overline{(\ \mathbf{u},\nabla)\ \mathbf{u}} = 0, \tag{13.8}$$

which is not really an equation with respect to $\bar{\mathbf{u}}$. Decoupling (13.8) has been an area of active research: Reynolds approximations, coupled equations for the moments, Gaussian closures, and so on (see e.g. [36], [45] and the references therein)

Another way to compute $\bar{\mathbf{u}}\,(t,x)$ is to find the distribution of $\mathbf{v}\,(t,x)$ using the infinite-dimensional Kolmogorov equation associated with (13.2). The complexity of this Kolmogorov equation is prohibitive for any realistic application, at least for now.

The propagator provides a third way: expressing the mean and other statistical moments of $\mathbf{u}$ in terms of $\mathbf{u}_\alpha$. Indeed, by Cameron-Martin Theorem,

$$\begin{aligned}\mathbb{E}\mathbf{u}(t,x) &= \mathbf{u}_0(t,x),\\ \mathbb{E}u^i(t,x)u^j(s,y) &= \sum_{\alpha\in\mathcal{J}} u^i_\alpha(t,x)u^j_\alpha(s,y)\end{aligned}$$

If exist, the third- and fourth-order moments can be computed using (12.15) and (12.16).

The next theorem, proved in [31], shows that the existence of a solution of the propagator (13.7) is not only necessary but, to some extent, sufficient for the global existence of a probabilistically strong solution of the stochastic Navier-Stokes equation (13.2).

Theorem 13.6. *Let **NS1** and **NS2** hold and* $\mathbf{u}_0\in L_2(\mathbb{R}^d;\mathbb{R}^d)$*. Assume that the propagator (13.7) has a solution* $\{\mathbf{u}_\alpha\,(t,x)\,,\ \alpha\in\mathcal{J}\}$ *on the interval* $(0,T]$ *so that, for every* α*, the process* $\mathbf{u}_\alpha$ *is weakly continuous in* $L_2(\mathbb{R}^d;\mathbb{R}^d)$ *and the inequality*

$$\sup_{t\le T}\sum_{\alpha\in\mathcal{J}} \|\mathbf{u}_\alpha\|^2_{L_2(\mathbb{R}^d;\mathbb{R}^d)}(t) + \int_0^T \sum_{\alpha\in\mathcal{J}} \|\nabla\mathbf{u}_\alpha\|^2_{L_2(\mathbb{R}^d;\mathbb{R}^{d\times d})}(t)\,dt < \infty \tag{13.9}$$

holds. If the process

$$\bar{\mathbf{U}}\,(t,x) := \sum_{\alpha\in\mathcal{J}} \mathbf{u}_\alpha\,(t,x)\,\xi_\alpha \tag{13.10}$$

is $\mathcal{F}_t^W$-adapted, then it is a solution of (13.2).

The process $\bar{\mathbf{U}}$ satisfies

$$\mathbb{E}\left(\sup_{s\le T}\|\bar{\mathbf{U}}(s)\|^2_{L_2(\mathbb{R}^d;\mathbb{R}^d)} + \int_0^T \|\nabla\bar{\mathbf{U}}(s)\|^2_{L_2(\mathbb{R}^d;\mathbb{R}^{d\times d})}\,ds\right) < \infty$$

and, for every $\mathbf{v}\in\mathbf{L_2}(\mathbb{R}^\mathbf{d};\mathbb{R}^\mathbf{d})$, $\mathbb{E}\left(\bar{\mathbf{U}},\mathbf{v}\right)$ is a continuous function of t.

Since $\bar{\mathbf{U}}$ is constructed on a prescribed stochastic basis and over a prescribed time interval $[0,T]$, this solution of (13.2) is strong in the probabilistic sense and is global in time. Being true in any space dimension d, Theorem 13.6 suggests another possible way to study equation (13.2) when $d\ge 3$. Unlike the propagator for the linear equation, the system (13.7) is not lower-triangular and not solvable by induction, so that analysis of (13.7) is an open problem.

14. First-Order Itô Equations

The objective of this section is to study equation

$$du(t,x) = u_x(t,x)dw(t),\ t>0,\ x\in\mathbb{R}, \tag{14.1}$$

and its analog for $x\in\mathbb{R}^d$.

Equation (14.1) was first encountered in Example 6.8; see also [9]. With a non-random initial condition $u(0,x)=\varphi(x)$, direct computations show that, if exists, the Fourier transform $\hat{u}=\hat{u}(t,y)$ of the solution must satisfy

$$d\hat{u}(t,y) = \sqrt{-1}y\hat{u}(t,y)dw(t),\ \text{ or } \hat{u}(t,y)=\hat{\varphi}(y)e^{\sqrt{-1}yw(t)+\frac{1}{2}y^2t}. \tag{14.2}$$

The last equality shows that the properties of the solution essentially depend on the initial condition, and, in general, the solution is not in $L_2(\mathbb{W})$.

The S-transformed equation, $v_t=h(t)v_x$, has a unique solution

$$v(t,x)=\varphi\left(x+\int_0^t h(s)ds\right),\ h(t)=\sum_{i=1}^N h_i m_i(t).$$

The results of Section 3 imply that a white noise solution of the equation can exist only if φ is a real analytic function. On the other hand, if φ is infinitely differentiable, then, by Theorem 8.4, the Wiener Chaos solution exists and can be recovered from v.

Theorem 14.1. *Assume that the initial condition φ belongs to the Schwarz space $\mathcal{S}=\mathcal{S}(\mathbb{R})$ of tempered distributions. Then there exists a generalized random process $u=u(t,x)$, $t\ge 0$, $x\in\mathbb{R}$, so that, for every $\gamma\in\mathbb{R}$ and $T>0$, the process u is the unique $w(H_2^\gamma(\mathbb{R}),H_2^{\gamma-1}(\mathbb{R}))$ Wiener Chaos solution of equation (14.1).*

Proof. The propagator for (14.1) is

$$u_\alpha(t,x)=\varphi(x)I(|\alpha|=0)+\int_0^t\sum_i\sqrt{\alpha_i}(u_{\alpha^-(i)}(s,x))_x m_i(s)ds. \tag{14.3}$$

Even though Theorem 6.4 is not applicable, the system can be solved by induction if φ is sufficiently smooth. Denote by $C_\varphi(k)$, $k \geq 0$, the square of the $L_2(\mathbb{R})$ norm of the k^{th} derivative of φ:

$$C_\varphi(k) = \int_{-\infty}^{+\infty} |\varphi^{(k)}(x)|^2 dx. \tag{14.4}$$

By Corollary 6.6, for every $k \geq 0$ and $n \geq 0$,

$$\sum_{|\alpha|=k} \|(u_\alpha^{(n)})_x\|^2_{L_2(\mathbb{R})}(t) = \frac{t^k C_\varphi(n+k)}{k!}. \tag{14.5}$$

The statement of the theorem now follows. □

Remark 14.2. *Once interpreted in a suitable sense, the Wiener Chaos solution of (14.1) is $\mathcal{F}_t^W$-adapted and does not depend on the choice of the Cameron-Martin basis in $L_2(\mathbb{W})$. Indeed, choose the wight sequence so that*

$$r_\alpha^2 = \frac{1}{1 + C_\varphi(|\alpha|)}.$$

By (14.5), we have $u \in \mathcal{R}L_2(\mathbb{W}; L_2(\mathbb{R}))$.

Next, define

$$\psi_N(x) = \frac{1}{\pi}\frac{\sin(Nx)}{x}.$$

Direct computations show that the Fourier transform of ψ_N is supported in $[-N, N]$ and $\int_{\mathbb{R}} \psi_N(x)dx = 1$. Consider equation (14.1) with initial condition

$$\varphi_N(x) = \int_{\mathbb{R}} \varphi(x-y)\psi_N(y)dy.$$

By (14.2), this equation has a unique solution u_N so that $u_N(t,\cdot) \in L_2(\mathbb{W}; H_2^\gamma(\mathbb{R}))$, $t \geq 0$, $\gamma \in \mathbb{R}$. Relation (14.5) and the definition of u_N imply

$$\lim_{N\to\infty} \sum_{|\alpha|=k} \|u_\alpha - u_{N,\alpha}\|^2_{L_2(\mathbb{R})}(t) = 0, \ t \geq 0, \ k \geq 0,$$

so that, by the Lebesgue dominated convergence theorem,

$$\lim_{N\to\infty} \|u - u_N\|^2_{\mathcal{R}L_2(\mathbb{W};L_2(\mathbb{R}))}(t) = 0, \ t \geq 0.$$

In other words, the solution of the propagator (14.3) corresponding to any basis $\mathfrak{m}$ in $L_2((0,T))$ is a limit in $\mathcal{R}L_2(\mathbb{W}; L_2(\mathbb{R}))$ of the sequence $\{u_N, \ N \geq 1\}$ of $\mathcal{F}_t^W$-adapted processes.

The properties of the Wiener Chaos solution of (14.1) depend on the growth rate of the numbers $C_\varphi(n)$. In particular,

- If $C_\varphi(n) \leq C^n (n!)^\gamma, \ C > 0, \ 0 \leq \gamma < 1$, then $u \in L_2\left(\mathbb{W}; L_2((0,T); H_2^n(\mathbb{R}))\right)$ for all $T > 0$ and every $n \geq 0$.
- If $C_\varphi(n) \leq C^n n!, \ C > 0$, then
  - for every $n \geq 0$, there is a $T > 0$ so that $u \in L_2\left(\mathbb{W}; L_2((0,T); H_2^n(\mathbb{R}))\right)$. In other words, the square-integrable solution exists only for sufficiently small T.
  - for every $n \geq 0$ and every $T > 0$, there exists a number $\delta \in (0,1)$ so that $u \in L_{2,Q}\left(\mathbb{W}; L_2((0,T); H_2^n(\mathbb{R}))\right)$ with $Q = (\delta, \delta, \delta, \ldots)$.

- If the numbers $C_\varphi(n)$ grow as $C^n(n!)^{1+\rho}$, $\rho \geq 0$, then, for every $T > 0$, there exists a number $\gamma > 0$ so that $u \in (\mathcal{S})_{-\rho,-\gamma}\left(L_2(\mathbb{W}); L_2((0,T); H_2^n(\mathbb{R}))\right)$. If $\rho > 0$, then this solution does not belong to any $L_{2,Q}\left(\mathbb{W}; L_2((0,T); H_2^n(\mathbb{R}))\right)$. If $\rho > 1$, then this solution does not have an S-transform.
- If the numbers $C_\varphi(n)$ grow faster than $C^n(n!)^b$ for any $b, C > 0$, then the Wiener Chaos solution of (14.1) does not belong to any $(\mathcal{S})_{-\rho,-\gamma}\left(L_2((0,T); H_2^n(\mathbb{R}))\right)$, $\rho,\ \gamma > 0$, or $L_{2,Q}\left(\mathbb{W}; L_2((0,T); H_2^n(\mathbb{R}))\right)$.

To construct a function φ with the required rate of growth of $C_\varphi(n)$, consider

$$\varphi(x) = \int_0^\infty \cos(xy) e^{-g(y)} dy,$$

where g is a suitable positive, unbounded, even function. Note that, up to a multiplicative constant, the Fourier transform of φ is $e^{-g(y)}$, and so $C_\varphi(n)$ grows with n as $\int_0^{+\infty} |y|^{2n} e^{-2g(y)} dy$.

A more general first-order equation can be considered:

$$du(t,x) = \sigma_{ik}(t,x) D_i u(t,x) dw_k(t),\ t > 0,\ x \in \mathbb{R}^d. \tag{14.6}$$

Theorem 14.3. *Assume that in equation (14.6) the initial condition $u(0,x)$ belongs to $\mathcal{S}(\mathbb{R}^d)$ and each σ_{ik} is infinitely differentiable with respect to x so that $\sup_{(t,x)} |D^n \sigma_{ik}(t,x)| \leq C_{ik}(n)$, $n \geq 0$. Then there exists a generalized random process $u = u(t,x)$, $t \geq 0$, $x \in \mathbb{R}^d$, so that, for every $\gamma \in \mathbb{R}$ and $T > 0$, the process u is the unique $w(H_2^\gamma(\mathbb{R}^d), H_2^{\gamma-1}(\mathbb{R}^d))$ Wiener Chaos solution of equation (14.1).*

Proof. The arguments are identical to the proof of Theorem 14.1. □

Note that the S-transformed equation (14.6) is $v_t = h_k \sigma_{ik} D_i v$ and has a unique solution if each σ_{ik} is a Lipschitz continuous function of x. Still, without additional smoothness, it is impossible to relate this solution to any generalized random process.

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