PASSIVE SCALAR EQUATION IN A TURBULENT INCOMPRESSIBLE GAUSSIAN VELOCITY FIELD

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Abstract. Time evolution of a passive scalar is considered in a turbulent homogeneous incompressible Gaussian flow. The turbulent nature of the flow results in non-smooth coefficients in the corresponding evolution equation. A strong, in the probabilistic sense, solution of the equation is constructed using Wiener Chaos expansion, and the properties of the solution are studied. Among the results obtained are a certain L_p-regularity of the solution and Feynman-Kac-type, or Lagrangian, representation formula. The results apply to both viscous and conservative flows.

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1. Introduction

If $\mathbf{v} = \mathbf{v}(t,x)$ is a smooth vector field in $\mathbb{R}^d$, then there exists a unique classical solution $\theta = \theta(t,x)$ of a non-viscous transport equation

$$\frac{\partial \theta}{\partial t} + \mathbf{v} \cdot \nabla \theta = 0,\ t > 0,\ \theta(0,x) = \theta_0(x). \tag{1.1}$$

This solution can be written as

$$\theta(t,x) = \theta_0(X^x_{t,0}), \tag{1.2}$$

where $X = X^x_{s,t}$ is the flow generated by the vector field $\mathbf{v}$:

$$\frac{dX^x_{s,t}}{dt} = \mathbf{v}(t, X^x_{s,t}),\ t > s,\ X^x_{s,s} = x. \tag{1.3}$$

If the velocity field $\mathbf{v}$ is not smooth as a function of x, then, in general, there are no existence results for equation (1.1), while equation (1.3) can have more than one solution. Consequently, the connection between (1.1) and (1.3) becomes unclear and representation (1.2) dubious.

For non-smooth vector fields $\mathbf{v}$, analysis of either (1.1) or (1.3) is impossible without further specifying the function $\mathbf{v}$. It is shown in [11] that in many physical models,

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such as turbulent flows, etc., the function $\mathbf{v} = (v^1(t,x), \dots, v^d(t,x))$ can be represented as

$$v^i(t,x) = \sum_{k\geq 1} \sigma_k^i(x)\dot{w}_k(t), \tag{1.4}$$

where each $\sigma_k^i(x)$ is Hölder continuous in x of order less than one, $\sum_{k\geq 1}\sigma_k^i(x)\sigma_k^j(y) = C^{ij}(x-y)$ for certain functions C^{ij}, and $(w_k, k \geq 1)$ are independent standard Brownian motions on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Equation (1.3) then becomes a stochastic differential equation

$$d(X_{s,t}^x)^i = \sigma_k^i(X_{s,t}^x)dw_k(t),\ t > s,\ (X_{s,s}^x)^i = x^i. \tag{1.5}$$

This equation does not, in general, have a strong solution, that is, $X_{s,t}^x$ is not, in general, measurable with respect to the sigma-algebra $\mathcal{F}_{s,t}^W$ generated by the increments $w_k(v) - w_k(u)$, $s \leq u < v \leq t$, $k \geq 1$. On the other hand, it is a standard fact that a unique weak solution of (1.5) always exists under the above assumptions on σ_k^i (see e.g. [1]).

In [11], the authors provide a systematic study of the family of operators

$$S_{s,t} : f(x) \mapsto \mathbb{E}\left(f(X_{s,t}^x)|\mathcal{F}_{s,t}^W\right) \tag{1.6}$$

for suitable functions f, where X is a weak solution of (1.5). If the functions σ_k^i are Lipschits continuous, then $X_{s,t}^x$ is $\mathcal{F}_{s,t}^W$-measurable. Moreover, it was shown in [13] that in this case $\theta(t,x) = \theta_0(X_{t,0}^x) = S_{t,0}\theta_0(x)$ is a unique generalized solution of the Stratonovich stochastic partial differential equation

$$d\theta + \sum_{i,k}\sigma_k^i\frac{\partial\theta}{\partial x^i}\circ dw_k,\ t > 0,\ \theta(0,x) = \theta_0(x). \tag{1.7}$$

When the functions σ_k^i are not Lipschits continuous, the connection between the operators $S_{s,t}$ and equation (1.7) is not clear. In particular, (1.7) does not, in general, have a solution in the traditional sense, classical or generalized, and, even if one *defines* the solution to be $S_{t,0}\theta_0(x)$ (cf. [5]), it is not clear in what sense the equation will be satisfied. Finally, no existing results provide a way of computing the conditional expectation in (1.6) for a particular vector field $\mathbf{v}$.

The objective of the current paper is to show that if the vector field $\mathbf{v}$ is not smooth but still divergence-free in the generalized sense and $\theta_0 \in L_p(\mathbb{R}^d)$, $2 \leq p \leq \infty$, then equation (1.7) has a unique strong generalized solution. More precisely, it is shown that there exists a random field $\theta = \theta(t,x)$ so that, for each $t > 0$,

- $\theta(t,\cdot) \in L_p(\Omega \times \mathbb{R}^d),\ 2 \leq p \leq \infty.$
- $\theta(t,\cdot)$ is $\mathcal{F}_{0,t}^W$-measurable.
- For every smooth compactly supported function φ,

$$(\theta,\varphi)(t) = (\theta_0,\varphi) + \sum_{i,k}\int_0^t\left(\theta, \sigma_k^i\frac{\partial\varphi}{\partial x^i}\right)(s)\circ dw_k(s) \tag{1.8}$$

  with probability one, where $(\cdot,\cdot)$ denotes the inner product in $\mathbb{R}^d$.

This random field admits a Lagrangian representation

$$\theta\left(t,x\right)=\mathbb{E}\left(\theta_{0}\left(X_{t,x}\left(0\right)\right)|\mathcal{F}_{t}^{W}\right)\quad(\mathbb{P}-\text{a.s.}) \tag{1.9}$$

where

$$X_{t,x}\left(s\right)=x+\int_{s}^{t}\sigma\left(X_{t,x}\left(r\right)\right)\overleftarrow{dW}\left(r\right)+\sqrt{2\nu}\left(\tilde{W}\left(t\right)-\tilde{W}\left(s\right)\right)$$

where $\tilde{W}$ is a standard Brownian motion independent of W. The solution of equation (1.8) as well as its moments can be computed using the Wiener Chaos expansion (see also Remark 5.6).

2. Passive Scalar in a Gaussian Field

We consider the following transport equation to describe the evolution of a passive scalar θ in a random velocity field $\mathbf{v}$:

$$\dot{\theta}(t,x)=\frac{1}{2}\nu\Delta\theta(t,x)-\mathbf{v}(t,x)\cdot\nabla\theta(t,x)+f(t,x);\ x\in\mathbb{R}^{d},\ d>1. \tag{2.1}$$

In (2.1), Δ and ∇ denote the Laplace operator and the gradient, respectively. Our interest in this equation is motivated by the on-going progress in the study of the turbulent transport problem (E and Vanden Eijnden [5], Gawędzki and Kupiainen [7], Gawędzki and Vergasola [6], Kraichnan [10], etc.)

We assume in (2.1) that $\mathbf{v}=\mathbf{v}(t,x)\in\mathbb{R}^{d}$, $d\geq 2$, is an isotropic Gaussian vector field with zero mean and covariance $\mathbb{E}(v^{i}(t,x)v^{j}(s,y))=\delta(t-s)C^{ij}(x-y)$, where $C=(C^{ij}(x),i,j=1,\dots,d)$ is a matrix-valued function. It is well-known (see, for example, LeJan [11]) that in the physically interesting models, such as Kraichnan velocity [10], the matrix-valued function $C=C(x)$ has the Fourier transform $\hat{C}=\hat{C}(z)$ given by

$$\hat{C}(z)=\frac{A_{0}}{(1+|z|^{2})^{(d+\alpha)/2}}\left(a\frac{zz^{*}}{|z|^{2}}+\frac{b}{d-1}\left(I-\frac{zz^{T}}{|z|^{2}}\right)\right), \tag{2.2}$$

where z^{*} is the row vector $(z_{1},\dots,z_{d})$, z is the corresponding column vector, $|z|^{2}=z^{*}z$, I is the identity matrix; $\alpha>0,a\geq 0,b\geq 0,A_{0}>0$ are real numbers. Similar to [11], we will assume that $0<\alpha<2$.

By direct computation (cf. [2]), the vector field $\mathbf{v}=(v^{1},\dots,v^{d})$ can be written as

$$v^{i}(t,x)=\sum_{k\geq 0}\sigma_{k}^{i}(x)\dot{w}_{k}(t), \tag{2.3}$$

where $\dot{w}_{k}(t)$, $k\geq 1$, are independent standard Gaussian white noises and $\{\sigma_{k},\ k\geq 1\}$ is a CONS in the space H_{C}, the reproducing kernel Hilbert space corresponding to the kernel function C. The space H_{C} is all or part of the Sobolev space $H_{2}^{(d+\alpha)/2}(\mathbb{R}^{d};\mathbb{R}^{d})$. It follows from (2.3) that $\sum_{k}\sigma_{k}^{i}(x)\sigma_{k}^{j}(y)=C^{ij}(x-y)$ for all x,y; in particular, $\sigma_{k}^{i}(x)\sigma_{k}^{j}(x)=C^{ij}(0)$ for all x.

If $a>0$ and $b>0$, then the matrix $\hat{C}$ is invertible and

$$H_{C}=\left\{f\in\mathbb{R}^{d}:\int_{\mathbb{R}^{d}}\hat{f}^{*}(z)\hat{C}^{-1}(z)\hat{f}(z)dz<\infty\right\}=H_{2}^{(d+\alpha)/2}(\mathbb{R}^{d};\mathbb{R}^{d}),$$

because $\|\hat{C}(z)\| \sim (1+|z|^2)^{-(d+\alpha)/2}$.

If $a > 0$ and $b = 0$, then

$$H_C = \left\{ f \in \mathbb{R}^d : \int_{\mathbb{R}^d} |\hat{f}(z)|^2 (1+|z|^2)^{(d+\alpha)/2} dz < \infty;\ zz^* \hat{f}(z) = |z|^2 \hat{f}(z) \right\},$$

the subset of gradient fields in $H_2^{(d+\alpha)/2}(\mathbb{R}^d;\mathbb{R}^d)$, that is, the collection of vector fields f with $\hat{f}(z) = z\hat{F}(z)$ for some scalar $F \in H_2^{(d+\alpha+1)/2}(\mathbb{R}^d)$.

If $a = 0$ and $b > 0$, then

$$H_C = \left\{ f \in \mathbb{R}^d : \int_{\mathbb{R}^d} |\hat{f}(z)|^2 (1+|z|^2)^{(d+\alpha)/2} dz < \infty;\ z^* \hat{f}(z) = 0 \right\},$$

the subset of divergence free fields in $H_2^{(d+\alpha)/2}(\mathbb{R}^d;\mathbb{R}^d)$.

By the embedding theorems, each σ_k^i is a bounded continuous function on $\mathbb{R}^d$; in fact, every σ_k^i is Hölder continuous of order $\alpha/2$. In addition, being an element of the corresponding space H_C, each σ_k is a gradient field if $b = 0$ and is divergence free if $a = 0$.

To simplify the further presentation and to make the model (2.1) more physically relevant, we consider the divergence-free velocity field and assume that the original stochastic differential in (2.3) is in the sense of Stratonovich. Under these assumptions, equation (2.1) becomes

$$d\theta(t,x) = \frac{1}{2}\nu\Delta\theta(t,x)dt - \sum_k \sigma_k(x) \cdot \nabla\theta(t,x) \circ dw_k(t). \tag{2.4}$$

With divergence-free functions σ_k, the equivalent Itô formulation is

$$d\theta(t,x) = \frac{1}{2}(\nu\Delta\theta(t,x) + C^{ij}(0)D_iD_j\theta(t,x))dt - \sigma_k^i(x)D_i\theta(t,x)dw_k(t), \tag{2.5}$$

where $D_i = \partial/\partial x^i$ and summation is carried out over the repeated indices.

We will study equation (2.5) under the following assumptions:

- **A1** There is a fixed stochastic basis $\mathbb{F} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ with the usual assumptions and $(w_k(t), k \geq 1, t \geq 0)$ is a collection of independent standard Wiener processes on $\mathbb{F}$.
- **A2** For each k, the vector field σ_k is a divergence-free element of the Sobolev space $H_2^{(d+\alpha)/2}(\mathbb{R}^d;\mathbb{R}^d)$, $0 < \alpha < 2$, $d \geq 2$.
- **A3** For all x, y in $\mathbb{R}^d$, $\sum_k \sigma_k^i(x)\sigma_k^j(y) = C^{ij}(x-y)$ and the matrix-valued function $C = C(x)$ satisfies (2.2).
- **A4** The initial condition θ_0 is non-random and belongs to $L_2(\mathbb{R}^d)$; $\nu \geq 0$ is a real number.

The objective is to establish existence, uniqueness, and regularity properties of the solution of (2.5) using Wiener Chaos.

3. A Review of the Wiener Chaos

Let $\mathbb{F} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a stochastic basis with the usual assumptions. On $\mathbb{F}$ consider a collection $(w_k(t), k \geq 1, t \geq 0)$ of independent standard Wiener processes. For a fixed $0 < T < \infty$, let $\mathcal{F}_T^W$ be the sigma-algebra generated by $w_k(t),\ k \geq 1,\ 0 < t < T$, and denote by $L_2(\mathcal{F}_T^W)$ the collection of $\mathcal{F}_T^W$-measurable square integrable random variables.

Fix the Fourier cosine basis $\{m_k,\ k \geq 1,\}$ in $L_2((0,T))$ with

$$m_1(t) = \frac{1}{\sqrt{T}},\ m_k(t) = \sqrt{\frac{2}{T}}\cos\left(\frac{\pi(k-1)t}{T}\right),\ k \geq 2. \tag{3.1}$$

Consider the collection of multi-indices

$$\mathcal{J} = \Big\{\alpha = (\alpha_i^k,\ i,k \geq 1),\ \alpha_i^k \in \{0,1,2,\ldots\},\ \sum_{i,k}\alpha_i^k < \infty\Big\}.$$

The set $\mathcal{J}$ is countable, and, for every $\alpha \in \mathcal{J}$, only finitely many of α_i^k are not equal to zero. For $\alpha \in \mathcal{J}$, define $|\alpha| = \sum_{i,k}\alpha_i^k,\ \alpha! = \prod_{i,k}\alpha_i^k!$, and

$$\xi_\alpha = \frac{1}{\sqrt{\alpha!}}\prod_{i,k} H_{\alpha_i^k}(\xi_{ik}), \quad \text{where} \quad \xi_{ik} = \int_0^T m_i(s)dw_k(s) \tag{3.2}$$

and

$$H_n(t) = e^{t^2/2}\frac{d^n}{dt^n}e^{-t^2/2}$$

is n-th Hermite polynomial. In particular, if $\alpha \in \mathcal{J}$ is such that $\alpha_i^k = 1$ if $i = j$ and $k = l$, and $\alpha_i^k = 0$ otherwise, then $\xi_\alpha = \xi_{jl}$.

Definition 3.1. *The space $L_2(\mathcal{F}_T^W)$ is called the Wiener Chaos space. The N-th Wiener Chaos is the linear subspace of $L_2(\mathcal{F}_T^W)$, generated by $\xi_\alpha,\ |\alpha| = N$.*

The following is a classical result of Cameron and Martin [4].

Theorem 3.1. *The collection $\{\xi_\alpha,\ \alpha \in \mathcal{J}\}$ is an orthonormal basis in the space $L_2(\mathcal{F}_T^W)$.*

In addition to the original source [4], the proof of this theorem can be found in many other places, for example, in [8]. By Theorem 3.1 every element v of $L_2(\mathcal{F}_T^W)$ can be written as $v = \sum_{\alpha\in\mathcal{J}} v_\alpha\xi_\alpha$, where $v_\alpha = \mathbb{E}(v\xi_\alpha)$.

4. The Wiener Chaos Solution of the Passive Scalar Equation

With summation convention in force, define the operators $\mathcal{A} = \frac{1}{2}(\nu\Delta + C^{ij}(0)D_iD_j)$ and $\mathcal{M}_k = \sigma_k^i(x)D_i$. Equation (2.5) then becomes

$$\theta(t,x) = \theta_0(x) + \int_0^t \mathcal{A}\theta(s,x)ds + \int_0^t \mathcal{M}_k\theta(s,x)dw_k(s)ds.$$

From now on in this section, dependence of various functions on x will not be shown explicitly.

Notice that, for every function $f \in H_2^1(\mathbb{R}^d)$,

$$\sum_{k\geq 1} \|\mathcal{M}_k f\|^2_{L_2(\mathbb{R}^d)} = (\sigma^j_k \sigma^i_k D_i f, D_j f) = (C^{ij}(0) D_i f, D_j f), \tag{4.1}$$

where $(\cdot,\cdot)$ is the inner product in $L_2(\mathbb{R}^d)$. Since the matrix $C(0)$ is positive definite, we conclude that there exist positive numbers c_1, c_2 so that, for every function $f \in H_2^1(\mathbb{R}^d)$,

$$c_1 \|\nabla f\|^2_{L_2(\mathbb{R}^d)} \leq \sum_{k\geq 1} \|\mathcal{M}_k f\|^2_{L_2(\mathbb{R}^d)} \leq c_2 \|\nabla f\|^2_{L_2(\mathbb{R}^d)}. \tag{4.2}$$

Equality (4.1) also shows that equation (2.5) is a stochastic parabolic equation [14]; the equation is super-parabolic if $\nu > 0$ and is fully degenerate if $\nu = 0$.

For $\alpha \in \mathcal{J}$, define functions θ_α by

$$\theta_\alpha(t) = \theta_0 I(|\alpha| = 0) + \int_0^t \mathcal{A}\theta_\alpha(s)ds + \int_0^t \sum_{i,k} \sqrt{\alpha_i^k} \mathcal{M}_k \theta_{\alpha^-(i,k)}(s) m_i(s) ds, \tag{4.3}$$

where $\alpha^-(i,k)$ is the multi-index with components

$$\Big(\alpha^-(i,k)\Big)_j^l = \begin{cases} \max(\alpha_i^k - 1, 0), & \text{if } i = j \text{ and } k = l, \\ \alpha_j^l, & \text{otherwise.} \end{cases}$$

Lemma 4.1. *Under assumptions* ***A2***–***A4****, the system of equations (4.3) has a unique solution so that every θ_α is a smooth bounded function of x for $t > 0$ and, if T_t, $t \geq 0$, is the heat semigroup generated by the operator $\mathcal{A}$, then, for every $N \geq 0$,*

$$\begin{aligned} &\sum_{|\alpha|=N} |\theta_\alpha(t,x)|^2 \\ &= \sum_{k_1,\ldots,k_N=1}^{\infty} \int_0^t \int_0^{s_N} \ldots \int_0^{s_2} |T_{t-s_N}\mathcal{M}_{k_N} \ldots T_{s_2-s_1}\mathcal{M}_{k_1} T_{s_1}\theta_0(x)|^2 ds^N, \end{aligned} \tag{4.4}$$

where $ds^N = ds_1 \ldots ds_N$, and

$$\begin{aligned} &\sum_{|\alpha|=N} |\nabla\theta_\alpha(t,x)|^2 \\ &= \sum_{k_1,\ldots,k_N=1}^{\infty} \int_0^t \int_0^{s_N} \ldots \int_0^{s_2} |\nabla T_{t-s_N}\mathcal{M}_{k_N} \ldots T_{s_2-s_1}\mathcal{M}_{k_1} T_{s_1}\theta_0(x)|^2 ds^N \end{aligned} \tag{4.5}$$

where $ds^N = ds_1 \ldots ds_N$

Proof. The results follow directly from (4.3) by induction on $|\alpha|$; details can be found in [12, Proposition A.1]. □

Theorem 4.2. *Under assumptions* ***A1***–***A4****, fix $T > 0$ and, for $\alpha \in \mathcal{J}$, define $\theta_\alpha(t)$ and ξ_α by (4.3) and (3.2), respectively. Then the following holds.*

(1) *For every $\nu \geq 0$ and every $t \in [0,T]$, the series*

$$\sum_{\alpha\in\mathcal{J}} \theta_\alpha(t)\xi_\alpha \tag{4.6}$$

converges in $L_2(\Omega; L_2(\mathbb{R}^d))$ to a process $\theta = \theta(t)$.

(2) *If $\nu > 0$, then, for every $\varphi \in C_0^\infty(\mathbb{R}^d)$, the process θ satisfies*

$$\begin{aligned}(\theta,\varphi)(t) = (\theta_0,\varphi) - \frac{1}{2}\nu\int_0^t(\nabla\theta,\nabla\varphi)(s)ds - \frac{1}{2}\int_0^t C^{ij}(0)(D_i\theta, D_j\varphi)(s)ds\\ - \int_0^t(\sigma_k^i D_i\theta,\varphi)dw_k(s)\end{aligned} \tag{4.7}$$

with probability one for all $t \in [0,T]$ at once, where $(\cdot,\cdot)$ is the inner product in $L_2(\mathbb{R}^d)$. Also,

$$\mathbb{E}\|\theta\|^2_{L_2(\mathbb{R}^d)}(t) + \nu\int_0^t \mathbb{E}\|\nabla\theta\|^2_{L_2(\mathbb{R}^d)}(s)ds = \|\theta_0\|^2_{L_2(\mathbb{R}^d)}. \tag{4.8}$$

(3) *If $\nu = 0$, then, for every $\varphi \in C_0^\infty(\mathbb{R}^d)$, the process θ satisfies*

$$(\theta,\varphi)(t) = (\theta_0,\varphi) + \frac{1}{2}\int_0^t C^{ij}(0)(\theta, D_iD_j\varphi)(s)ds + \int_0^t(\theta,\sigma_k^iD_i\varphi)dw_k(s) \tag{4.9}$$

with probability one for all $t \in [0,T]$ at once. Also,

$$\mathbb{E}\|\theta\|^2_{L_2(\mathbb{R}^d)}(t) \leq \|\theta_0\|^2_{L_2(\mathbb{R}^d)}. \tag{4.10}$$

Remark 4.3. *Equalities (4.7) and (4.9) mean that $\theta = \theta(t,x)$ is the solution of the transport equation (2.5) in the traditional sense of the theory of stochastic partial differential equations, that is, it is a strong solution in the stochastic sense, satisfying the corresponding equation in the generalized function sense. The solution is also unique in the class of $L_2((0,T)\times\Omega; L_2(\mathbb{R}^d))$ random functions, because any other solution will automatically have the same Wiener Chaos expansion. Without going into the details, we mention that the uniqueness can, in fact, be established in a much wider class of generalized random functions.*

The proof of Theorem 4.2 is carried out in three steps: (1) establishing convergence of (4.6) and the corresponding energy estimates; (2) establishing predictability of θ; (3) establishing equalities (4.7) and (4.9).

Convergence of (4.6) and the corresponding energy estimates (**Step 1**) will follow from the following two lemmas.

Lemma 4.4. *Assume that $\nu \geq 0$. Define $\theta_N(t,x) = \sum_{n=0}^N\sum_{|\alpha|=n}\theta_\alpha(t,x)\xi_\alpha$. Then, for all $t \in [0,T]$,*

$$\begin{aligned}\mathbb{E}\|\theta_N\|^2_{L_2(\mathbb{R}^d)}(t) = \|\theta_0\|^2_{L_2(\mathbb{R}^d)} - \nu\sum_{n=0}^N\sum_{|\alpha|=n}\int_0^t\|\nabla\theta_\alpha\|^2_{L_2(\mathbb{R}^d)}(s)ds\\ - \sum_{k_1,\ldots,k_{N+1}=1}^\infty\int_0^t\cdots\int_0^{s_2}\|\mathcal{M}_{k_{N+1}}T_{s-s_N}\mathcal{M}_{k_N}\ldots T_{s_2-s_1}\mathcal{M}_{k_1}T_{s_1}\theta_0\|^2_{L_2(\mathbb{R}^d)}ds^N ds.\end{aligned} \tag{4.11}$$

Proof. By Lemma 4.1, after integration with respect to x,

$$\sum_{|\alpha|=N} \|\theta_\alpha\|^2_{L_2(\mathbb{R}^d)}(t) = \sum_{k_1,\dots,k_N=1}^{\infty} \int_0^t \int_0^{s_N} \dots \int_0^{s_2} \|T_{t-s_N}\mathcal{M}_{k_N}\dots T_{s_2-s_1}\mathcal{M}_{k_1}T_{s_1}\theta_0\|^2_{L_2(\mathbb{R}^d)} ds^N. \tag{4.12}$$

If $F_N(t) = \sum_{|\alpha|=N} \|\theta_\alpha(t)\|^2_{L_2(\mathbb{R})}$, then

$$\begin{aligned} \frac{d}{dt}F_N(t) &= \sum_{k_1,\dots,k_N=1}^{\infty} \int_0^t \int_0^{s_{N-1}} \dots \int_0^{s_2} \|\mathcal{M}_{k_N}T_{t-s_{N-1}}\mathcal{M}_{k_{N-1}}\dots T_{s_2-s_1}\mathcal{M}_{k_1}T_{s_1}\theta_0\|^2_{L_2(\mathbb{R})} ds^{N-1} \\ &+ 2\sum_{k_1,\dots,k_N=1}^{\infty} \int_0^t \dots \int_0^{s_2} \left(\mathcal{A}T_{t-s_N}\mathcal{M}\dots T_{s_1}u_0, T_{t-s_N}\mathcal{M}\dots T_{s_2-s_1}\mathcal{M}_{k_N}T_{s_1}\theta_0\right) ds^N. \end{aligned} \tag{4.13}$$

It remains to notice that, for every smooth function $f = f(x)$,

$$2(\mathcal{A}f, f) = -\nu\|\nabla f\|^2_{L_2(\mathbb{R})} - \sum_{k\geq 1} \|\mathcal{M}_k f\|^2_{L_2(\mathbb{R})}.$$

Equality (4.11) now follows. □

Notice that (4.11) implies both the $L_2(\Omega; L_2(\mathbb{R}^d))$ convergence of the series $\sum_\alpha \theta_\alpha(t)\xi_\alpha$ for every $t \in [0, T]$ and inequality (4.10).

Lemma 4.5. *If $\nu > 0$, then, for every $t \in [0, T]$,*

$$\lim_{N\to\infty} \sum_{k_1,\dots,k_{N+1}=1}^{\infty} \int_0^t \dots \int_0^{s_2} \|\mathcal{M}_{k_{N+1}}T_{s-s_N}\mathcal{M}_{k_N}\dots T_{s_2-s_1}\mathcal{M}_{k_1}T_{s_1}\theta_0\|^2_{L_2(\mathbb{R}^d)} ds^N ds = 0. \tag{4.14}$$

Proof. Define

$$F_N(t) = \sum_{k_1,\dots,k_{N+1}=1}^{\infty} \int_0^t \dots \int_0^{s_2} \|\mathcal{M}_{k_{N+1}}T_{s-s_N}\mathcal{M}_{k_N}\dots T_{s_2-s_1}\mathcal{M}_{k_1}T_{s_1}\theta_0\|^2_{L_2(\mathbb{R}^d)} ds^N ds. \tag{4.15}$$

By (4.2) and Lemma 4.1, $F_N(t) \leq c_2 \sum_{|\alpha|=N} \int_0^t \|\nabla\theta_\alpha\|^2_{L_2(\mathbb{R}^d)}(s)ds$. Lemma 4.4 then implies that the series $\sum_{N\geq 0} F_N(t)$ converges for all $t \in [0, T]$. Therefore, $\lim_{N\to\infty} F_N(t) = 0$ and the statement of the lemma follows.

□

Note that (4.11) and (4.14) imply (4.8).

Remark 4.6. *Analysis of the above proofs shows that the conclusions of Lemmas 4.1, 4.4 and 4.5 do not depend on the choice of the basis* $\{m_i, i \geq 1\}$ *in* $L_2((0,T))$*. In other words, if* $\{m_i, i \geq 1\}$ *is any orthonormal basis in* $L_2((0,T))$ *and* θ *is defined by (4.6), then, for each* $t \in [0,T]$, $\theta(t)$ *belongs to* $L_2(\Omega; L_2(\mathbb{R}^d))$*, satisfies the corresponding energy estimate, and is* $\mathcal{F}_T^W$*-measurable.*

The next lemma shows that the construction of the process θ does not depend on the choice of the basis $\{m_i, i \geq 1\}$ in $L_2((0,T))$. The lemma is also the key to establishing predictability of θ.

Lemma 4.7. *Let* θ *be the process defined by (4.6) and let* $\bar{m}_i(t), i \geq 1$*, be another orthonormal basis in* $L_2((0,T))$*. Then* $\theta(t) = \sum_{\alpha \in \mathcal{J}} \bar{\theta}_\alpha(t) \bar{\xi}_\alpha$*, where*

$$\bar{\xi}_\alpha = \frac{1}{\sqrt{\alpha!}} \prod_{i,k} H_{\alpha_i^k}(\bar{\xi}_{ik}), \qquad \bar{\xi}_{ik} = \int_0^T \bar{m}_i(s) dw_k(s),$$

and the coefficients $\bar{\theta}_\alpha(t)$ *satisfy the system of equations (4.3) with* $\bar{m}_i$ *instead of* m_i*.*

Proof. Let $h = (h_1(t), \ldots, h_N(t))$ be a finite collection of bounded measurable function on $(0,T)$. Define

$$\mathcal{E}(h) = \exp\left(\sum_{k=1}^N \int_0^T h_k(t) dw_k(t) - \frac{1}{2} \sum_{k=1}^N \int_0^T |h_k(t)|^2 dt \right). \tag{4.16}$$

Setting $h_{k,i} = \int_0^T h_k(t) m_i(t) dt$, we rewrite (4.16) as

$$\mathcal{E}(h) = \exp\left(\sum_{i,k} \left(h_{k,i} \xi_{ki} - \frac{1}{2} |h_{k,i}|^2 \right) \right)$$

and conclude that

$$\mathcal{E}(h) = \sum_{\alpha \in \mathcal{J}} \frac{h^\alpha}{\sqrt{\alpha!}} \xi_\alpha, \quad \text{where } \; h^\alpha = \prod_{i,k} h_{k,i}^{\alpha_i^k}. \tag{4.17}$$

If $\theta_h(t) = \mathbb{E}\left(\theta(t)\mathcal{E}(h)\right)$, then, by combining equations (4.3), (4.6), and (4.17), we get

$$\theta_h(t) = \theta_0 + \int_0^t \mathcal{A}\theta_h(s) ds + \int_0^t h_k(s) \mathcal{M}_k \theta_h(s) ds. \tag{4.18}$$

Next, let $\bar{\theta}_\alpha(t)$ be the solution of the system of equations (4.3) with $\bar{m}_i$ instead of m_i. It follows from Remark 4.6 that, for each $t \in [0,T]$, the process $\bar{\theta}(t) = \sum_\alpha \bar{\theta}_\alpha(t) \bar{\xi}_\alpha(t)$ is an element of $L_2(\Omega; L_2(\mathbb{R}^d))$. Also, defining $\bar{\theta}_h(t) = \mathbb{E}\left(\bar{\theta(t)}\mathcal{E}(h)\right)$ and observing that $\mathcal{E}(h) = \exp\left(\sum_{i,k} \left(\bar{h}_{k,i} \bar{\xi}_{ki} - \frac{1}{2} |\bar{h}_{k,i}|^2 \right) \right)$, where $\bar{h}_{k,i} = \int_0^T h_k(t) \bar{m}_i(t) dt$, we get

$$\bar{\theta}_h(t) = \theta_0 + \int_0^t \mathcal{A}\bar{\theta}_h(s) ds + \int_0^t h_k(s) \mathcal{M}_k \bar{\theta}_h(s) ds.$$

Uniqueness of solution of this parabolic equation implies the equality $\theta_h(t) = \bar{\theta}_h(t)$ in $L_2(\mathbb{R}^d)$ for all $t \in [0,T]$ and all finite collections $h_1, \ldots, h_N$ of bounded measurable functions on $(0,T)$. Since the corresponding collection of $\mathcal{E}(h)$ is everywhere dense in $L_2(\mathcal{F}_T^W)$, it follows that, for each $t \in [0,T]$, $\theta(t) = \bar{\theta}(t)$ as elements of $L_2(\Omega; L_2(\mathbb{R}^d))$. Lemma 4.7 is proved. □

We can now establish predictability of θ (**Step 2**).

Lemma 4.8. *The process θ defined by (4.6) is predictable. If, in addition, $\nu > 0$, then $\nabla\theta$ is also predictable.*

Proof. Fix $t^* \in (0,T)$ and consider a special basis $\bar{m}_i(t)$ in $L_2((0,T))$ so that each $\bar{m}_i$ is supported either in $[0,t^*]$ or in $[t^*,T]$. Denote by $\bar{\xi}_\alpha$, $\alpha \in \mathcal{J}$, the corresponding orthonormal basis in $L_2(\mathcal{F}_T^W)$. Then the definition of $\bar{\xi}_\alpha$ implies that $\bar{\xi}_\alpha = \bar{\xi}_\beta(0,t^*)\bar{\xi}_\gamma(t^*,T)$, where $\bar{\xi}_\beta(0,t^*)$, the up-to-t^* component, is $\mathcal{F}_{t^*}^W$-measurable and $\bar{\xi}_\gamma(t^*,T)$, the after-t^* component, is independent of $\mathcal{F}_{t^*}^W$. Accordingly, each multi-index $\alpha \in \mathcal{J}$ will be represented as $\alpha = (\beta,\gamma)$ to account for the up-to-t^* and after-t^* components. By Lemma 4.7,

$$\theta(t) = \sum_{\alpha\in\mathcal{J}} \bar{\theta}_\alpha(t)\bar{\xi}_\alpha = \sum_{\alpha\in\mathcal{J}} \bar{\theta}_{\beta,\gamma}(t)\bar{\xi}_\beta(0,t^*)\bar{\xi}_\gamma(t^*,T),\ t \in [0,T],$$

and the coefficients $\bar{\theta}_\alpha$ satisfy the system of equations (4.3) with $\bar{m}_i$ instead of m_i. Then, for $t \le t^*$, the function $\bar{m}_i$ appears in the system if and only if $\bar{m}_i$ is supported in $[0,t^*]$. It follows by induction on β that, for $t \in [0,t^*]$, $\bar{\theta}_{\beta,\gamma}(t) = 0$ if $|\gamma| > 0$. On the other hand, $\mathbb{E}(\bar{\xi}_\gamma(t^*,T)|\mathcal{F}_{t^*}^W) = \mathbb{E}\bar{\xi}_\gamma(t^*,T) = 0$ for all γ with $|\gamma| > 0$. As a result,

$$\mathbb{E}(\theta(t^*)|\mathcal{F}_{t^*}^W) = \sum_{\alpha\in\mathcal{J}} \bar{\theta}_{\beta,0}\bar{\xi}_\beta(0,t^*) = \theta(t^*),$$

that is, $\theta(t^*)$ is $\mathcal{F}_{t^*}^W$-measurable for every $t^* \in (0,T)$. The same arguments prove predictability of $\nabla\theta$ if $\nu > 0$. Lemma 4.8 is proved. □

To complete the proof of Theorem 4.2, it remains to establish equalities (4.7) and (4.9) (**Step 3**). As in the proof of Lemma 4.7, set $\theta_h(t) = \mathbb{E}(\theta(t)\mathcal{E}(h))$ with $\mathcal{E}(h)$ defined by (4.16).

If $\varphi \in C_0^\infty(\mathbb{R}^d)$, then equation (4.3) implies

$$(\theta_h,\varphi)(t) = (\theta(0),\varphi) + \int_0^t (\theta_h, \mathcal{A}^*\varphi)(s)ds + \sum_{\alpha\in\mathcal{J}} \frac{h^\alpha}{\alpha!} \sum_{i,k} \int_0^t \sqrt{\alpha_i^k} m_i(s)(\theta_{\alpha^-(i,k)}, \mathcal{M}_k^*\varphi)(s)ds,$$

where * means the adjoint of the operator, $(\cdot,\cdot)$ is the inner product in $\mathbb{R}^n$, and h^α is defined in (4.17). Predictability and integrability properties of θ imply that the stochastic integral $I(t) = \int_0^t (\theta(s), \mathcal{M}_k^*\varphi)dw_k(s)$ is well defined. If $\xi_\alpha(t) = \mathbb{E}(\xi_\alpha|\mathcal{F}_t^W)$, then, according to [13],

$$d\xi_\alpha(t) = \sum_{i,k} \sqrt{\alpha_i^k}\,\xi_{\alpha^-(i,k)}(t) m_i(t) dw_k(t). \tag{4.19}$$

By $\mathcal{F}_t^W$-measurability of $I(t)$,

$$I_\alpha(t) = \mathbb{E}\Big(I(t)\mathbb{E}(\xi_\alpha|\mathcal{F}_t^W)\Big) = \mathbb{E}(I(t)\xi_\alpha(t)), \tag{4.20}$$

and then, by the Itô formula,

$$\mathbb{E}(I(t)\xi_\alpha(t)) = \int_0^t \sum_{i,k} \sqrt{\alpha_i^k}\, m_i(s)(\theta_{\alpha^-(i,k)}, \mathcal{M}_k^*\varphi)(s)ds. \tag{4.21}$$

Together with (4.17), the last equality implies

$$\sum_{\alpha\in\mathcal{J}} \frac{z^\alpha}{\alpha!} \sum_{i,k} \int_0^t \sqrt{\alpha_i^k}\, m_i(s)(\theta_{\alpha^-(i,k)}, \mathcal{M}_k^*\varphi)(s)ds = \mathbb{E}\left(\mathcal{E}(h)\int_0^t (\theta(s), \mathcal{M}_k^*\varphi)dw_k(s)\right).$$

As a result,

$$\begin{aligned}\mathbb{E}\left(\mathcal{E}(h)(\theta,\varphi)(t)\right) = (\theta(0),\varphi) + \mathbb{E}\left(\mathcal{E}(h)\int_0^t (\theta, \mathcal{A}^*\varphi)(s)ds\right)\\ + \mathbb{E}\left(\mathcal{E}(h)\int_0^t (\theta(s), \mathcal{M}_k\varphi)dw_k(s)\right).\end{aligned} \tag{4.22}$$

Equality (4.22) and density of the collection $\{\mathcal{E}(h)\}$ in $L_2(\mathcal{F}_T^W)$, together with assumption **A4** and equality $\mathbb{E}\mathcal{E}(h) = 1$, imply

$$(\theta,\varphi)(t) = (\theta(0),\varphi) + \int_0^t (\theta, \mathcal{A}^*\varphi)(s)ds + \int_0^t (\theta, \mathcal{M}_k^*\varphi)(s)dw_k(s). \tag{4.23}$$

If $\nu = 0$, then the last equality coincides with (4.9). If $\nu > 0$, then (4.8) implies that (4.23) can be rewritten as (4.7).

Theorem 4.2 is proved. □

The Cameron-Martin Theorem 3.1 and Theorem 4.2 provide the following simple formulae for computing the first and the second moments of a solution of the passive scalar equations (4.7) and (4.9):

Corollary 1. *Under the assumptions of Theorem 4.2, for all s, t and almost all x, y,*

$$\mathbb{E}\theta\,(t,x) = \theta_\alpha\,(t,x)\, I_{|\alpha|=0}$$

and

$$\mathbb{E}\theta\,(t,x)\,\theta\,(s,y) = \sum_{\alpha\in\mathcal{J}} \theta_\alpha\,(t,x)\,\theta_\alpha\,(s,y)\,.$$

5. Lagrangian Representation of a Solution

Let $\tilde{W} = (\tilde{w}_k(s), k = 1, ..., d, s \geq 0)$ be a collection of independent standard Wiener Processes. Assume that $\tilde{W}$ is independent of $W = (w_k(s), k \geq 1, s \geq 0)$ and $\nu \geq 0$. For a fixed $t < \infty$, consider the following backward Itô equation (see e.g. [14]):

$$
\begin{aligned}
-d_s X^i_{t,x}(s) &= \sqrt{2\nu}\overleftarrow{d\tilde{w}_i}(s) + \sigma^i_k(X_{t,x}(s))\overleftarrow{dw_k}(s), s \in [0,t), \\
X^i_{t,x}(t) &= x.
\end{aligned}
\tag{5.1}
$$

Due to our assumption that $div(\sigma_k) = 0$, the Itô and the Stratonovitch forms of this equation coincide.

Since each $\sigma^i_k(x)$ is a continuous function and, by assumption, $\sigma^i_k(x)\sigma^j_k(x) = C^{ij}(0)$, it follows that, for some $\delta > 0$, $(\sigma(x)\sigma^*(x)y, y) \geq \delta|y|^2$ for all $y \in \mathbb{R}^d$. Therefore, equation (5.1) has a martingale solution, that is, there exist a stochastic basis $\mathbb{F} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ with the usual assumptions, a collection $B = \left(\tilde{W}, W\right) = (\tilde{w}_k(s), k = 1, ..., d,\ w_k(s), k \geq 1, s \geq 0)$ of independent standard Wiener processes adapted to $\{\mathcal{F}_s\}_{s\geq 0}$, and a process $X_{t,x}(s)$ on $\mathbb{F}$ with values in $\mathbb{R}^d$ so that the following equality holds $\mathbb{P} - a.s.$ for all $s < t$:

$$
X_{t,x}(s) = x + \int_s^t \sigma(X_{t,x}(r))\overleftarrow{dW}(r) + \sqrt{2\nu}\left(\tilde{W}(t) - \tilde{W}(s)\right). \tag{5.2}
$$

Remark 5.1. *With $Y_{t,x}(s) = X_{t,x}(t-s)$ and $B_t(s) = B(t) - B(t-s)$, equation (5.2) can be rewritten as follows*

$$
Y_{t,x}(s) = x + \int_0^s \sigma(Y_{t,x}(r))\, dW_t(r) + \sqrt{2\nu}\tilde{W}_t(t-s),
$$

where $W_t(r) := W(t) - W(t-r)$.

Note that the martingale, or weak, solution of (5.1) is not necessarily $\mathcal{F}^B_s-$adapted. Moreover, a priori there is no guarantee that a solution of equation (5.1) can be constructed on a preselected stochastic basis and for a given collection B of Wiener process. Roughly speaking, these limitations constitute the difference between a martingale solution and a strong, in probabilistic sense, solution, also known as pathwise solution; see, for example, Ikeda, Watanabe [9] or Anulova et. al [1].

On the contrary, we have proved in Theorem 4.2 that equation (2.5) has a unique $\mathcal{F}^W_s-$adapted solution for any $\nu \geq 0$ on any stochastic basis and for any collection W of independent standard Wiener processes on this basis. In particular, we can and will assume that there is a pathwise unique solution $\theta = \theta(t,x)$ of (2.5) on $\mathbb{F}$ driven by W. Therefore, unless $X_{t,x}(0)$ is $\mathcal{F}^W_t-$measurable, it is unrealistic to expect the classic Lagrangian representation of the solution to the passive scalar equation $\theta(t,x) = \theta_0(X_{t,x}(0))$.

However, the following generalization of this formula holds true.

Theorem 5.2. *For every $T < \infty$ and almost all x,*

$$
\theta(T,x) = \mathbb{E}\left(\theta_0(X_{T,x}(0))\,|\mathcal{F}^W_T\right) \quad (\mathbb{P} - \text{a.s.})
$$

Proof. As in the proof of Lemma 4.7, define $\mathcal{E}(h)$ according to (4.16) and the function θ_h so that (4.18) holds.

On the other hand, by Girsanov's theorem

$$\mathbb{E}\left(\mathcal{E}(h)\mathbb{E}\left(\theta_0\left(X_{T,x}\left(0\right)\right)|\mathcal{F}_t^W\right)\right) = \mathbb{E}\left(\mathcal{E}(h)\theta_0\left(X_{T,x}\left(0\right)\right)\right) = \mathbb{E}'\left(\theta_0\left(X_{T,x}\left(0\right)\right)\right),$$

where $\mathbb{E}'$ is expectation with respect to the measure $d\mathbb{P}'_T = \mathcal{E}(h)d\mathbb{P}_T$ and $\mathbb{P}_T$ is a restriction of the measure $\mathbb{P}$ to $\mathcal{F}_T$. Moreover, $\mathbb{P}_T\ -a.s.$ the process $X_{T,x}\left(s\right)$ is a martingale solution of the equation

$$\begin{aligned} X_{T,x}\left(s\right) &= x + \int_s^T \sigma_k\left(X_{T,x}\left(r\right)\right)h_k\left(r\right)dr + \int_s^T \sigma\left(X_{T,x}\left(r\right)\right)\overleftarrow{dW'}\left(r\right) \\ &+ \sqrt{2\nu}\left(\tilde{W}'\left(T\right) - \tilde{W}'\left(s\right)\right), \quad s \le T. \end{aligned}$$

By the Feynman-Kac formula, the function $\psi_h\left(s\right) := \mathbb{E}'\left(\theta_0\left(X_{t,x}\left(s\right)\right)\right)$ is also a solution of the equation (4.18). From the uniqueness of solution of (4.18), it follows that $\mathbb{E}\left(\mathcal{E}(h)\mathbb{E}\left(\theta_0\left(X_{T,x}\left(0\right)\right)|\mathcal{F}_T^W\right)\right) = \mathbb{E}\left(\mathcal{E}\left(h\right)\theta\left(T,x\right)\right)$ for every finite collection h of bounded measurable functions on $[0,T]$. Since the collection of all such $\mathcal{E}(h)$ is everywhere dense in $L_2\left(\Omega,\mathcal{F}_t^W\right)$, we have that, $\mathbb{P}-a.s.$, $\theta\left(T,\cdot\right) = \mathbb{E}\left(\theta_0\left(X_{t,\cdot}\left(0\right)\right)|\mathcal{F}_T^W\right)$ as elements of $L_2\left(\Omega;\mathbb{R}^d\right)$. This completes the proof of Theorem 5.2. □

Theorem 5.3. *The energy equality* $\mathbb{E}\|\theta\|^2_{L_2(\mathbb{R}^d)}(t) = \|\theta_0\|^2_{L_2(\mathbb{R}^d)}$ *holds if and only if, for almost all* x*,*

$$\mathbb{E}\left(\theta_0\left(X_{t,x}\left(0\right)\right)|\mathcal{F}_t^W\right) = \theta_0\left(X_{t,x}\left(0\right)\right), \quad (\mathbb{P}-a.s.) \tag{5.3}$$

Proof. Suppose that condition (5.3) holds. The transition probability density of the process X is homogeneous, i.e. it is of the form $p\left(t,x-y\right)$. Then, we have

$$\begin{aligned} \mathbb{E}\|\theta\|^2_{L_2(\mathbb{R}^d)}(t) &= \int_{\mathbb{R}^d}\mathbb{E}\left|\theta_0\left(X_{t,x}\left(0\right)\right)\right|^2dx = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left|\theta_0\left(y\right)\right|^2 p\left(t,x-y\right)dxdy \\ &= \|\theta_0\|^2_{L_2(\mathbb{R}^d)}. \end{aligned}$$

Assume now that for every t, there exists a set $\Gamma \subseteq \mathbb{R}^d$ of positive Lebesgue measure such that $\mathbb{E}\left(\theta_0\left(X_{t,x}\left(0\right)\right)|\mathcal{F}_t^W\right) \ne \theta_0\left(X_{t,x}\left(0\right)\right)$. By Minkovski's inequality,
$\mathbb{E}\left|\mathbb{E}\left(\theta_0\left(X_{t,x}\left(0\right)\right)|\mathcal{F}_t^W\right)\right|^2 \le \mathbb{E}\left|\theta_0\left(X_{t,x}\left(0\right)\right)\right|^2$ and the equality holds only if
$\mathbb{E}\left(\theta_0\left(X_{t,x}\left(0\right)\right)|\mathcal{F}_t^W\right) = \theta_0\left(X_{t,x}\left(0\right)\right)$. Therefore,

$$\mathbb{E}\|\theta\|^2_{L_2(\mathbb{R}^d)}(t) = \mathbb{E}\int_{\mathbb{R}^d}\left|\mathbb{E}\left(\theta_0\left(X_{t,x}\left(0\right)\right)|\mathcal{F}_t^W\right)\right|^2dx < \int_{\mathbb{R}^d}\mathbb{E}\left|\theta_0\left(X_{t,x}\left(0\right)\right)\right|^2dx$$

$$= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\mathbb{E}\left|\theta_0\left(y\right)\right|^2 p\left(t,x-y\right)dydx = \|\theta_0\|^2_{L_2(\mathbb{R}^d)}.$$

Theorem 5.3 is proved. □

Remark 5.4. *Obviously, in all interesting scenarios,* (5.3) *does not hold if* $\nu > 0$. *If* $\nu = 0$, *condition* (5.3) *is equivalent to the assumption that* $X_{t,x}\left(s\right)$ *is a strong solution of equation* (5.1).

Theorems 4.2 and 5.2 imply the following estimate on the norm of the solution of equation (2.5).

Theorem 5.5. *If* $\theta_0 \in L_p(\mathbb{R}^d)$, $2 \le p < \infty$, *then, for every* $t \ge 0$, *the solution* $\theta = \theta(t,x)$ *of (2.5) satisfies*

$$\mathbb{E}\|\theta\|^p_{L_p(\mathbb{R}^d)}(t) \le \|\theta_0\|^p_{L_p(\mathbb{R}^d)}. \tag{5.4}$$

Proof. Denote by $S_t : \theta_0(\cdot) \mapsto \theta(t,\cdot)$, $t > 0$, the solution operator for equation (2.5). By Theorem 4.2, S_t is a bounded linear operator from $L_2(\mathbb{R}^d)$ to $L_2(\Omega \times \mathbb{R}^d)$ and

$$\|S_t\theta_0\|_{L_2(\Omega\times\mathbb{R}^d)} \le \|\theta_0\|_{L_2(\mathbb{R}^d)}.$$

By Theorem 5.2, S_t is a bounded linear operator from $L_\infty(\mathbb{R}^d)$ to $L_\infty(\Omega \times \mathbb{R}^d)$ and

$$\|S_t\theta_0\|_{L_\infty(\Omega\times\mathbb{R}^d)} \le \|\theta_0\|_{L_\infty(\mathbb{R}^d)}.$$

Inequality (5.4) now follows from the Riesz Convexity Theorem (see, for example, [3, Theorem 4.1.7].) □

Remark 5.6. *It could be shown that the adjoint to the solution operator in Theorem 3.2* [11] *is a generalized solution of equation (4.7) with* $\nu \ge 0$. *However, since we are considering a much more specialized situation, the proof of the existence presented in this paper is simpler. The main novel elements of our paper include:*

(1) The uniqueness of a strong solution of the unforced incompressible passive scalar equation for a suitable class of initial conditions.

(2) The L_p *regularity of the solution (in terms of the* L_p *regularity of the initial condition).*

(3) A procedure for computing the solution and the moments of the passive scalar equation via the stochastic Fourier coefficients θ_α.

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