NONLINEAR FILTERING OF DIFFUSION PROCESSES IN CORRELATED NOISE: ANALYSIS BY SEPARATION OF VARIABLES

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Abstract. An approximation to the solution of a stochastic parabolic equation is constructed using the Galerkin approximation followed by the Wiener Chaos decomposition. The result is applied to the nonlinear filtering problem for the time homogeneous diffusion model with correlated noise. An algorithm is proposed for computing recursive approximations of the unnormalized filtering density and filter, and the errors of the approximations are estimated. Unlike most existing algorithms for nonlinear filtering, the real-time part of the algorithm does not require solving partial differential equations or evaluating integrals. The algorithm can be used for both continuous and discrete time observations.

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1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a Wiener process $W = W(t),\ 0 \le t \le T$. Consider a random field $u = u(x)$, $x \in \mathbb{R}^d$, so that $u \in L_2(\Omega; L_2(\mathbb{R}^d))$ and u is measurable with respect to the sigma-algebra $\mathcal{F}_T^W$ generated by the Wiener process up to time T. If $\{e_k, k \ge 1\}$ is an orthonormal basis in $L_2(\mathbb{R}^d)$ and $\{\xi_m, m \ge 1\}$ is an orthonormal basis in $L_2(\Omega, \mathcal{F}_T^W)$, then we can write

$$u(x) = \sum_{m,k \ge 1} \varphi_{m,k}^K \xi_m e_k(x), \tag{1.1}$$

where $\varphi_{m,k}$ are some deterministic coefficients. The objective of the current work is to study an approximation of u using representation (1.1) when the random field u is a solution of a stochastic parabolic equation. For such random fields it is possible to derive explicit representation for the coefficients $\varphi_{m,k}$ and to get an upper bound on the approximation

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error in (1.1). The results are then used to derive an approximate algorithm for solving the nonlinear filtering problem of diffusion processes with correlated noise.

The problem of nonlinear filtering can be briefly described as follows. Assume that $(X = X(t), Y = Y(t)),\ t \geq 0$, are two diffusion processes with values in $\mathbb{R}^d$ and $\mathbb{R}^r$ respectively, so that X is the unobservable component and the observable component Y is given by

$$Y(t) = \int_0^t h(X(s))ds + W(t).$$

The problem is called noise-uncorrelated, if the Wiener process $W = W(t)$, representing the observation noise, is independent of X. The problem is called noise-correlated, if there is correlation between W and X. If $f = f(x)$ is a measurable function satisfying $\mathbb{E}|f(X(t))|^2 < \infty,\ t \geq 0$, then the problem of nonlinear filtering is to find the best mean square estimate $\hat{f}_t$ of $f(X(t))$ given the trajectory $Y(s),\ s \leq t$. It is known [12, 17, 26] that, under certain regularity assumptions, we have

$$\hat{f}_t = \frac{\int_{\mathbb{R}^d} f(x)p(t,x)dx}{\int_{\mathbb{R}^d} p(t,x)dx}, \tag{1.2}$$

where $p = p(t,x)$ is a random field called the unnormalized filtering density (UFD). The problem of estimating $f(X(t))$ is thus reduced to the problem of computing the UFD p. It is also known [26] that $p = p(t,x)$ is the solution of the Zakai filtering equation, a stochastic parabolic equation, driven by the observation process. The exact solution of this equation can be found only in some special cases, and the development of numerical schemes for solving the Zakai equation has become an area of active research.

Many of the existing numerical schemes for the Zakai equation use various generalizations of the corresponding algorithms for the deterministic partial differential equations. Examples of the corresponding algorithms can be found in Bennaton [1], Florchinger and LeGland [7], Ito [11], etc. Because of the large amount of calculations, these algorithms cannot be implemented in real time when the dimension of the state process is more than three. An alternative approach is based on the Monte-Carlo method; see, for example, Del Moral et. al [6].

In some applications, like target tracking, the filter estimate must be computed in real time. Such applications require filtering algorithms with fast on line computations. When the parameters of the model are known in advance, the real time computations can be simplified by separating the deterministic and stochastic components of the Zakai equation and performing the computations related to the deterministic component in advance. The separation is based on the Wiener chaos decomposition of solutions of stochastic parabolic equations. Starting with the works of Kunita [15], Ocone [25], and Lo and Ng [18], this approach was further developed by Budhiraja and Kallianpur [2, 3, 4] and Mikulevicius and Rozovskii [20, 21, 22, 23]. An algorithm to solve the Zakai equation using this approach for the noise uncorrelated problem was suggested in Lototsky et al. [19]. The algorithm in [19] was based on the following representation of the unnormalized filtering density. First, the

UFD $p(T,x)$ was expanded in the Wiener Chaos:

$$p(T,x) = \sum_{m=1}^{\infty} \varphi_m(T,x)\xi_m. \tag{1.3}$$

After that, the coefficients φ_m were expanded in the basis $\{e_k\}$ in $L_2(\mathbb{R}^d)$, resulting in the representation

$$p(T,x) = \sum_{m=1}^{\infty} \left(\sum_{k=1}^{\infty} \varphi_{m,k}(T)e_k(x) \right) \xi_m. \tag{1.4}$$

In other words, first, the stochastic variable was separated, and then, the spacial variable.

Alternatively, one can start with the Galerkin approximation of p:

$$p^K(t,x) = \sum_{k=1}^{K} p_k^K(t)e_k(x). \tag{1.5}$$

The coefficients $p_k^K(t)$ satisfy a system of stochastic ordinary differential equations driven by the observation process. The solution of this system at time T is then expanded using the Wiener Chaos decomposition. When combined with (1.5), the result is

$$p^K(T,x) = \sum_{k=1}^{K} \left(\sum_{m=1}^{\infty} \varphi_{m,k}^K(T)\xi_m \right) e_k(x). \tag{1.6}$$

In other words, first, the spacial variable is separated, then, the stochastic variable. First suggested in [8] as a computational alternative to (1.4), this approach was further analyzed in [9].

The order in which the variables are separated does make a difference. The algorithms based on (1.4) and on (1.6) have different approximation errors and, unlike (1.4), analysis of (1.6) is possible for noise correlated problem.

Recall that the Zakai filtering equation for the unnormalized filtering density $p = p(t,x)$ is

$$dp = \mathcal{L}^* p \; dt + \mathcal{M}^* p \; dY(t). \tag{1.7}$$

The elliptic differential operator $\mathcal{L}$ is the generator of the unobserved process X, while the operator $\mathcal{M}$ is bounded in the noise uncorrelated problem and in unbounded in the noise correlated problem. The presence of the unbounded operator in the stochastic part of equation (1.7) for the noise correlated problem makes the analysis and implementation of the numerical methods for the Zakai equation much more difficult (see, for example, Florchinger and LeGland [7]).

The objective of the current work is to analyze the algorithm for solving the Zakai equation using approximation (1.6). First, (1.6) is studied for an abstract stochastic evolution system. In Section 2, the Galerkin approximation is investigated, and in Section 3, the Wiener chaos decomposition for a system of stochastic ordinary differential equations. In each situation, the rate of convergence is established in terms of the numbers of the basis function used. The filtering problem is introduced in Section 4, the filtering algorithm is presented in Section

5, and the convergence of the algorithm is studied in Section 6. The real time part of the proposed algorithm does not require solving differential equations or using quadrature methods to evaluate integrals in (1.2). The algorithm can also be used if the observations are available in discrete time.

Unlike the previous works on the subject, this paper presents a unified treatment of both noise-correlated and noise-uncorrelated problems with possibly degenerate diffusion in the un-observed component. Another difference from the previous works on the subject is that the error bound is derived not only for the filtering density but also for the optimal filter $\hat{f}_t$ with a large class of functions f.

2. Galerkin approximation of stochastic evolution equations

Consider the stochastic evolution system

$$u(t) = u_0 + \int_{T_0}^{t} \mathcal{A}u(s)ds + \int_{T_0}^{t} \sum_{l=1}^{r} \mathcal{B}_l u(s)dW_l(s),\ T_0 \geq 0, \tag{2.1}$$

where $W = W(t)$ is an r - dimensional standard Wiener process on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, u_0 is independent of W, and $\mathcal{A}$ and $\mathcal{B}_l$, $l = 1, \ldots, r$, are linear operators acting in the scale of infinite dimensional Hilbert spaces $\{\mathbf{H}^a, a \in \mathbb{R}\}$. To simplify the notation, both the inner product in $\mathbf{H}^0$ and the duality between $\mathbf{H}^1$ and $\mathbf{H}^{-1}$ will be denoted by $(\cdot,\cdot)_0$; $\|\cdot\|_a$ is the norm in the space $\mathbf{H}^a$. It will be assumed that equation (2.1) is either coercive or dissipative [26, Chapter 3]. In particular, there exists a constant $C^* > 0$ so that, for every $v \in \mathbf{H}^1$,

$$\|\mathcal{A}v\|_{-1} \leq C^*\|v\|_1,\ \|\mathcal{B}_l v\|_0 \leq C^*\|v\|_1,\ \text{and } 2(\mathcal{A}v, v)_0 + \sum_{l=1}^{r} \|\mathcal{B}_l v\|_0^2 \leq C^*\|v\|_0^2. \tag{2.2}$$

If $u_0 \in \mathbf{H}^1$, then there is a unique solution $u = u(t)$ in the space $L_2(\Omega \times [T_0, T]; \mathbf{H}^1) \cap L_2(\Omega; \mathbf{C}([T_0, T]; \mathbf{H}^0))$ (see Theorems 3.1.4 and 3.2.2 in [26]).

Suppose there exists an orthonormal basis $\{e_k, k \geq\}$ in $\mathbf{H}^0$ so that $e_k \in \mathbf{H}^1$ for all k. Consider the following system of stochastic ordinary differential equations:

$$\begin{aligned} du_k^K(t) &= \sum_{n=1}^{K} (\mathcal{A}e_n, e_k)_0 u_n^K(t)dt \\ &+ \sum_{l=1}^{r}\sum_{n=1}^{K} (\mathcal{B}_l e_n, e_k)_0 u_n^K(t)dW_l(t),\ T_0 < t \leq T, \\ u_k^K(T_0) &= (u_0, e_k)_0,\ k = 0, \ldots, K. \end{aligned} \tag{2.3}$$

The function

$$u^K(t) = \sum_{k=1}^{K} u_k^K(t)e_k$$

is called the **Galerkin approximation** of $u(t)$. It is proved in the following theorem that, under some natural assumptions,

$$\lim_{K\to\infty} \sup_{T_0\le t\le T} \mathbb{E}\|u(t;T_0;u_0) - u^K(t)\|_0^2 = 0,$$

and the rate of convergence is determined.

2.1. Theorem. *Let the following conditions be fulfilled:*

1. *The basis $\{e_k\}$ consists of the eigenfunctions of a linear operator Λ with the corresponding eigenvalues λ_k. The operator Λ is a symmetric operator in $\mathcal{H}^0$ and there exist numbers $0 < c_1 < c_2$ and $\theta > 0$ so that, for all k,*

$$c_1 \le \lambda_k k^{-\theta} \le c_2; \tag{2.4}$$

2. *$e_k \in \mathbf{H}^1$ and $\|e_k\|_1 \le C_e k^q$, $q \ge 0$;*
3. *$\sup_{T_0\le t\le T} \mathbb{E}\|\Lambda^\nu u(t)\|_0^2 < \infty$ for some positive integer ν so that $\theta_1 := \nu\theta - 2q > 1$.*

Then

$$\sup_{T_0\le t\le T} \mathbb{E}\|u(t) - u^K(t)\|_0^2 \le \sup_{T_0\le t\le T} \mathbb{E}\|\Lambda^\nu u(t)\|_0^2 \, \frac{Ce^{C(T-T_0)}}{K^{2(\theta_1-1)}}, \tag{2.5}$$

where C is a constant depending only on the constant C^* in (2.2) and the numbers $c_1, c_2, C_e, \nu,\ \theta,\ q$.

Proof. If $\psi_k(t) := (u(t), e_k)_0$, then

$$\mathbb{E}\|u(t) - u^K(t)\|_0^2 = \sum_{k=0}^{K} \mathbb{E}|\psi_k(t) - u_k^K(t)|^2 + \sum_{k>K} \mathbb{E}|\psi_k(t)|^2. \tag{2.6}$$

By assumptions 1 and 3 of the theorem,

$$|\psi_k(t)| \le \frac{\|\Lambda^\nu u(t)\|_0}{\lambda_k^\nu} \tag{2.7}$$

so that

$$\sup_{T_0\le t\le T} \sum_{k>K} \mathbb{E}|\psi_k(t)|^2 \le \sup_{T_0\le t\le T} \mathbb{E}\|\Lambda^\nu u(t)\|_0^2 \frac{C}{K^{2\nu\theta-1}} \le \sup_{T_0\le t\le T} \mathbb{E}\|\Lambda^\nu u(t)\|_0^2 \frac{C}{K^{2(\theta_1-1)}}. \tag{2.8}$$

For $1 \le k \le K$ define $\delta_k(t) := \psi_k(t) - u_k^K(t)$, so that $\sum_{k=1}^K \mathbb{E}|\psi_k(t) - u_k^K(t)|^2 = \sum_{k=1}^K \mathbb{E}|\delta_k|^2$, and also define

$$\delta_{1,n}(t) := \sum_{k>K} (\mathcal{A}e_k, e_n)_0 \psi_k(t), \quad \delta_{2,n}^l(t) := \sum_{k>K} (\mathcal{B}_l e_k, e_n)_0 \psi_k(t).$$

Both $\delta_{1,n}(t)$ and $\delta^l_{2,n}(t)$ are well defined due to (2.7) and assumptions 2 and 3 of the theorem. Then

$$
\begin{aligned}
d\delta_n(t) &= \sum_{k=1}^K (\mathcal{A}e_k, e_n)_0 \delta_k(t)dt + \sum_{l=1}^r \sum_{k=1}^K (\mathcal{B}_l e_k, e_n)_0 \delta_k(t) dW_l(t) \\
&+ \delta_{1,n}(t)dt + \sum_{l=1}^r \delta^l_{2,n} dW_l(t),\ T_0 < t \leq T; \\
\delta_n(T_0) &= 0,\ 1 \leq n \leq K,
\end{aligned} \tag{2.9}
$$

and by the Ito formula,

$$
\begin{aligned}
&\sum_{n=1}^K \mathbb{E}|\delta_n(t)|^2 = 2\int_{T_0}^t \sum_{n,k=1}^K (\mathcal{A}e_k, e_n)_0 \mathbb{E}\delta_n(s)\delta_k(s)ds \\
&+\sum_{l=1}^r \sum_{n=1}^K \int_{T_0}^t \mathbb{E}\left(\sum_{k=1}^K (\mathcal{B}_l e_k, e_n)_0 \delta_k(s)\right)^2 ds + 2\sum_{n=1}^K \int_{T_0}^t \mathbb{E}\delta_{1,n}(s)\delta_n(s)ds \\
&+2\sum_{l=1}^r \sum_{n,k=1}^K \int_{T_0}^t (\mathcal{B}_l e_k, e_n)_0 \mathbb{E}\delta^l_{2,n}(s)\delta_k(s)ds + \sum_{l=1}^r \sum_{n=1}^K \int_{T_0}^t \mathbb{E}(\delta^l_{2,n}(s))^2 ds.
\end{aligned} \tag{2.10}
$$

It follows from the third inequality in (2.2) that

$$
\begin{aligned}
&2\int_{T_0}^t \sum_{n,k=1}^K (\mathcal{A}e_k, e_n)_0 \mathbb{E}\delta_n(s)\delta_k(s)ds \\
&+\sum_{l=1}^r \sum_{n=1}^K \int_{T_0}^t \mathbb{E}\left(\sum_{k=1}^K (\mathcal{B}_l e_k, e_n)_0 \delta_k(s)\right)^2 ds \leq C\sum_{k=1}^K \int_{T_0}^t \mathbb{E}(\delta_k(s))^2 ds.
\end{aligned} \tag{2.11}
$$

The first two inequalities in (2.2) and assumption 2 imply

$$
|(\mathcal{A}e_k, e_n)_0| \leq C\|e_k\|_1 \|e_n\|_1 \leq Ck^q n^q, \quad |(\mathcal{B}_l e_k, e_n)_0| \leq C\|e_k\|_1 \|e_n\|_0 \leq Ck^q,
$$

so that by (2.7),

$$
|\delta_{1,n}(t)| \leq Cn^q \frac{\|\Lambda^\nu u(t)\|_0}{K^{\nu\theta-q-1}}, \quad |\delta^l_{2,n}| \leq C\frac{\|\Lambda^\nu u(t)\|_0}{K^{\nu\theta-q-1}},
$$

and

$$
\begin{aligned}
&\sum_{n=1}^K \int_{T_0}^T \mathbb{E}(\delta_{1,n}(s))^2 ds + \sum_{l=1}^r \sum_{n=1}^K \int_{T_0}^T \mathbb{E}(\delta^l_{2,n}(s))^2 ds \\
&\leq (T-T_0) \sup_{T_0 \leq t \leq T} \mathbb{E}\|\Lambda^\nu u(t)\|_0^2 \, \frac{(r+1)C}{K^{2(\theta_1-1)}}.
\end{aligned} \tag{2.12}
$$

After that (2.10)–(2.12) and the obvious inequality $2|ab| \leq a^2 + b^2$ imply

$$
\sum_{n=1}^K \mathbb{E}|\delta_n(t)|^2 \leq C\sum_{n=1}^K \int_{T_0}^t \mathbb{E}|\delta_n(s)|^2 ds + (T-T_0) \sup_{T_0 \leq t \leq T} \mathbb{E}\|\Lambda^\nu u(t)\|_0^2 \, \frac{(r+1)C}{K^{2(\theta_1-1)}},
$$

so that by the Gronwall inequality

$$\sup_{T_0\le t\le T}\sum_{n=1}^{K}\mathbb{E}|\delta_n(t)|^2 \le (T-T_0)\sup_{T_0\le t\le T}\mathbb{E}\|\Lambda^\nu u(t)\|_0^2\, e^{C(T-T_0)}\,\frac{(r+1)C}{K^{2(\theta_1-1)}}.$$

Together with (2.6) and (2.8), the last inequality implies (2.5). Theorem 2.1 is proved. □

3. Wiener Chaos Expansion

On a complete probability space $(\Omega,\mathcal{F},\mathbb{P})$ consider a system of stochastic ordinary differential equations:

$$U(t) = U_0 + \int_{T_0}^{t} AU(s)ds + \int_{T_0}^{t}\sum_{l=1}^{r} B_l U(s)dW_l(s)\; T_0\ge 0, \tag{3.1}$$

where $U(t), U_0 \in \mathbb{R}^K$, $A, B_l \in \mathbb{R}^{K\times K}$, the matrices A, B_l are deterministic, and U_0 is independent of the r-dimensional Wiener process W. The solution of (3.1) is denoted by $U(t;T_0;U_0)$.

In what follows, the Wiener chaos decomposition of $U(t;T_0;U_0)$ will be derived and the properties of the decomposition studied.

As the first step, recall the construction of an orthonormal basis in the space $L_2(\Omega,\mathcal{F}^W_{T_0,t},\mathbb{P})$ of square integrable random variables that are measurable with respect to the σ-algebra, generated by the Wiener process up to time t. Let α be an **r-dimensional multi-index**, that is, a collection $\alpha=(\alpha^l_k)_{1\le l\le r,\ k\ge 1}$ of nonnegative integers such that only finitely many of α^l_k are different from zero. The set of all such multi-indices will be denoted by J. For $\alpha\in J$ define $\alpha! := \prod_{k,l}(\alpha^l_k!)$.

For a fixed $t^*>T_0$ choose a complete orthonormal system $\{m_k\}=\{m_k(s)\}_{k\ge1}$ in $L_2([T_0,t^*])$ and define

$$\xi_{k,l} = \int_{T_0}^{t^*} m_k(s)dW_l(s)$$

so that $\xi_{k,l}$ are independent Gaussian random variables with zero mean and unit variance.

If

$$H(x) := (-1)^n e^{x^2/2}\frac{d^n}{dx^n}e^{-x^2/2} \tag{3.2}$$

is the n-th Hermite polynomial, then the collection

$$\left\{\xi_\alpha(W_{T_0,t^*}) := \prod_{k,l}\left(\frac{H_{\alpha^l_k}(\xi_{k,l})}{\sqrt{\alpha^l_k!}}\right),\quad \alpha\in J\right\}$$

is an orthonormal system in $L_2(\Omega,\mathcal{F}^W_{T_0,t^*},\mathbb{P})$. A theorem of Cameron and Martin [5] shows that $\{\xi_\alpha(W_{T_0,t^*})\}_{\alpha\in J}$ is actually a basis in that space.

3.1. Theorem. *If $\eta \in L_2(\Omega, \mathcal{F}^W_{T_0,t^*}, \mathbb{P})$, then*

$$\eta = \sum_{\alpha \in J} \mathbb{E}[\eta \xi_\alpha(W_{T_0,t^*})] \xi_\alpha(W_{T_0,t^*}) \tag{3.3}$$

and

$$\mathbb{E}|\eta|^2 = \sum_{\alpha \in J} |\mathbb{E}\eta\xi_\alpha(W_{T_0,t^*})|^2.$$

Proof. This theorem is proved in [5] and [10]. □

3.2. Theorem. *If $t^* > T_0$ is fixed, then, for every $s \in [T_0, t^*]$, the solution $U(s; T_0; U_0)$ can be written as*

$$U(s; T_0; U_0) = \sum_{\alpha \in J} \frac{1}{\sqrt{\alpha!}} \varphi_\alpha(s; T_0; U_0) \xi_\alpha(W_{T_0,t^*}), \tag{3.4}$$

and the following Parseval's equality holds:

$$\mathbb{E}|U(s; T_0; U_0)|^2 = \sum_{\alpha \in J} \frac{1}{\alpha!} E|\varphi_\alpha(s; T_0; U_0)|^2. \tag{3.5}$$

The coefficients of the expansion are $\mathbb{R}^K$-vector functions and satisfy the recursive system of deterministic equations

$$\begin{aligned} \frac{\partial \varphi_\alpha(s; T_0; U_0)}{\partial s} &= A\varphi_\alpha(s; T_0; U_0) + \sum_{k,l} \alpha^l_k m_k(s) B_l \varphi_{\alpha(k,l)}(s; T_0; U_0), \quad T_0 < s \le t^*; \\ \varphi_\alpha(T_0; T_0; U_0) &= U_0 1_{\{|\alpha|=0\}}, \end{aligned} \tag{3.6}$$

where $\alpha = (\alpha^l_k)_{1 \le l \le r,\ k \ge 1} \in J$ and $\alpha(i,j)$ stands for the multi-index $\tilde{\alpha} = (\tilde{\alpha}^l_k)_{1 \le l \le r,\ k \ge 1}$ with

$$\tilde{\alpha}^l_k = \begin{cases} \alpha^l_k & \text{if } k \ne i \text{ or } l \ne j \text{ or both} \\ \max(0, \alpha^j_i - 1) & \text{if } k = i \text{ and } l = j. \end{cases} \tag{3.7}$$

Proof. Assume first that $U_0 = g$ is deterministic; the Markov property of the solution of (3.1) implies that, once the derivation is complete, we can replace g with U_0.

If g is deterministic, then $U(s; T_0; g) \in L_2(\Omega, \mathcal{F}^W_{T_0,t^*}, \mathbb{P})$ for $s \le t^*$, and Theorem 3.1 implies (3.4) and (3.5).

To prove that the coefficients satisfy (3.6), define

$$P_t(z) = \exp\Big\{ \int_{T_0}^t \sum_{l=1}^r m^l_z(s) dW_l(s) - \frac{1}{2} \int_{T_0}^t \sum_{l=1}^r |m^l_z(s)|^2 ds \Big\},\ T_0 \le t \le t^*,$$

where $m^l_z = \sum_{k \ge 1} m_k(s) z^l_k$ and $\{z^l_k\},\ l = 1, \dots, r,\ k = 1, 2, \dots,$ is a sequence of real numbers such that $\sum_{k,l} |z^l_k|^2 < \infty$. Then direct computations show that

$$\xi_\alpha(W_{T_0,t^*}) = \frac{1}{\sqrt{\alpha!}} \frac{\partial^\alpha}{\partial z^\alpha} P_{t^*}(z)\Big|_{z=0},$$

where

$$\frac{\partial^\alpha}{\partial z^\alpha} = \prod_{k,l} \frac{\partial^{\alpha_k^l}}{(\partial z_k^l)^{\alpha_k^l}} ,$$

and also, that

$$\mathbb{E}[\eta \xi_\alpha(W_{T_0,t^*})] = \frac{\partial^\alpha}{\partial z^\alpha} \mathbb{E}[\eta P_{t^*}(z)]\Big|_{z=0}$$

for every $\eta \in L_2(\Omega, \mathcal{F}^W_{T_0,t^*}, \mathbb{P})$. Consequently,

$$\begin{aligned} \varphi_\alpha(s; T_0; g) &= \frac{\partial^\alpha}{\partial z^\alpha} \mathbb{E}[U(s; T_0; g) P_{t^*}(z)]\Big|_{z=0} \\ &= \frac{\partial^\alpha}{\partial z^\alpha} \mathbb{E}[U(s; T_0; g) P_s(z)]\Big|_{z=0} , \end{aligned}$$

where the second equality follows from the martingale property of $P_s(z)$ on $(\Omega, \{\mathcal{F}^W_{T_0,t}\}_{T_0 \le t \le t^*}, \mathbb{P})$. It follows from the definition of $P_s(z)$ that

$$dP_s(z) = \sum_{l=1}^r m_z^l(s) P_s(z) dW_l(s),\ T_0 \le s \le t;\ P_{T_0}(z) = 1.$$

Then (3.1) and the Ito formula imply that

$$\begin{aligned} U(s; T_0; g) P_s(z) &= g \\ &+ \int_{T_0}^s \Big(AU(\tau; T_0; g) + \sum_{l=1}^r B_l U(\tau; T_0; g) \Big) m_z^l(\tau) P_\tau(z) d\tau \\ &+ \int_{T_0}^s \sum_{l=1}^r \Big(B_l U(\tau; T_0; g) + U(\tau; T_0; g) m_z^l(s) \Big) P_s(z) dW_l(\tau). \end{aligned}$$

Taking the expectation on both sides of the last equality and setting $\varphi(s, z; T_0; g) := \mathbb{E}U(s; T_0; g) P_s(z)$ results in

$$\varphi(s, z; T_0; g) = g + \int_{T_0}^s \Big(A\varphi(\tau, z; T_0; g) + \sum_{l=1}^r m_z^l(\tau) B_l \varphi(\tau, z; T_0; g) \Big) d\tau.$$

Applying the operator $\frac{1}{\sqrt{\alpha!}} \frac{\partial^\alpha}{\partial z^\alpha}$ and setting $z=0$ yields that the functions $\varphi_\alpha(s; T_0; g)$ satisfy (3.6). Theorem 3.2 is proved. □

For a multi-index $\alpha \in J$ define

- $|\alpha| := \sum_{l,k} \alpha_k^l$ (length of α);
- $d(\alpha) := \max\{k \ge 1 : \alpha_k^l > 0 \text{ for some } 1 \le l \le r\}$ (order of α).

To study the rate of convergence of the series in (3.4), it is necessary to note that the summation $\sum_{\alpha \in J}$ is double infinite:

(3.8)
$$\sum_{\alpha \in J} = \sum_{k=0}^{\infty} \sum_{|\alpha|=k}$$

and there are infinitely many multi-indices α with $|\alpha| = k > 0$.

Define $J_N^n = \{\alpha \in J : |\alpha| \le N,\ d(\alpha) \le n\}$ and then

$$U_N^n(s;T_0;U_0) = \sum_{\alpha \in J_N^n} \frac{1}{\sqrt{\alpha!}} \varphi_\alpha(s;T_0;U_0)\xi_\alpha(W_{T_0,t^*}). \tag{3.9}$$

Now the summation in (3.9) is over a finite set: if $d(\alpha) \le n$, then there are at most $(nr)^k$ multi-indices α with $|\alpha| = k$.

3.3. Theorem. *Let the constants C_0, C_1, C_2 be such that $|Av|^2 \le C_0|v|^2$, $|e^{At}v|^2 \le e^{C_1 t}|v|^2$, $|B_l v|^2 \le C_2|v|^2$ for every vector $v \in \mathbb{R}^K$. If the basis $\{m_k\}$ is the Fourier cosine basis*

$$m_1(s) = \frac{1}{\sqrt{t^* - T_0}};\ m_k(s) = \sqrt{\frac{2}{t^* - T_0}} \cos\left(\frac{\pi(k-1)(s - T_0)}{t^* - T_0}\right),\ k > 1;\ T_0 \le s \le t^*, \tag{3.10}$$

then

$$\begin{aligned} &\mathbb{E}|U(t^*;T_0;U_0) - U_N^n(t^*;T_0;U_0)|^2 \le 2e^{\bar{C}(t^* - T_0)} \Bigg(\frac{[C_2 r(t^* - T_0)]^{N+1}}{(N+1)!} \\ &+ 2C_2 r \frac{(t^* - T_0)^2}{n} [\epsilon(B) + C_0(1 + C_2 r(t^* - T_0))(t^* - T_0)] \Bigg) \mathbb{E}|U_0|^2, \end{aligned} \tag{3.11}$$

where $\bar{C} = C_1 + C_2 r$ and $0 \le \epsilon(B) \le 4$; $\epsilon(B) = 0$ if the matrices B_l commute (in particular, if $r = 1$).

This Theorem is proved below in Section 7.

If $t^* - T_0 = \Delta$, then (3.11) becomes

$$\mathbb{E}|U(t^*;T_0;U_0) - U_N^n(t^*;T_0;U_0)|^2 \le e^{C\Delta} \left(\frac{(C\Delta)^{N+1}}{(N+1)!} + \frac{\Delta^2}{n}(\epsilon(B) + C\Delta) \right) \mathbb{E}|U_0|^2, \tag{3.12}$$

and the constant C depends only on the matrices A and B_l in (3.1).

4. Diffusion Filtering Model

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with independent standard Wiener processes $W = W(t)$ and $V = V(t)$ of dimensions d_1 and r respectively. Let X_0 be a random variable independent of W and V. In the *diffusion filtering model*, the unobserved d - dimensional state (or signal) process $X = X(t)$ and the r-dimensional observation process $Y = Y(t)$ are defined by the stochastic ordinary differential equations

$$\begin{aligned} &dX(t) = b(X(t))dt + \sigma(X(t))dW(t) + \rho(X(t))dV(t), \\ &dY(t) = h(X(t))dt + dV(t),\ 0 < t \le T; \\ &X(0) = X_0, \quad Y(0) = 0, \end{aligned} \tag{4.1}$$

where $b(x) \in \mathbb{R}^d$, $\sigma(x) \in \mathbb{R}^{d \times d_1}$, $\rho(x) \in \mathbb{R}^{d \times r}$, $h(x) \in \mathbb{R}^r$.

Assumption R1. The functions σ and ρ are $\mathbf{C}_b^3(\mathbb{R}^d)$, that is, bounded and three times continuously differentiable on $\mathbb{R}^d$ so that all the derivatives are also bounded; the functions b and h are $\mathbf{C}_b^2(\mathbb{R})$, and the random variable X_0 has a density p_0.

Under Assumption R1 system (4.1) has a unique strong solution [13, Theorems 5.2.5 and 5.2.9].

If $f = f(x)$ is a scalar measurable function on $\mathbb{R}^d$ so that $\sup_{0\le t\le T}\mathbb{E}|f(X(t))|^2 < \infty$, then the *filtering problem* for (4.1) is to find the best mean square estimate $\hat{f}_t$ of $f(X(t)),\ t \le T$, given the observations $Y(s),\ 0 < s \le t$. Denote by $\mathcal{F}_t^Y$ the σ-algebra generated by $Y(s),\ 0 \le s \le t$. Then the properties of the conditional expectation imply that the solution of the filtering problem is

$$\hat{f}_t = \mathbb{E}\left(f(X(t))|\mathcal{F}_t^Y\right).$$

To derive an alternative representation of $\hat{f}_t$, some additional constructions will be necessary.

Define a new probability measure $\tilde{\mathbb{P}}$ on $(\Omega, \mathcal{F})$ as follows: for $A \in \mathcal{F}$,

$$\tilde{\mathbb{P}}(A) = \int_A Z_T^{-1} d\mathbb{P},$$

where

$$Z_t = \exp\left\{\int_0^t h^*(X(s))dY(s) - \frac{1}{2}\int_0^t |h(X(s))|^2 ds\right\}$$

(here and below, if $\zeta \in \mathbb{R}^k$, then ζ is a *column* vector, $\zeta^* = (\zeta_1, \dots, \zeta_k)$, and $|\zeta|^2 = \zeta^*\zeta$). If the function h is bounded, then the measures $\mathbb{P}$ and $\tilde{\mathbb{P}}$ are equivalent. The expectation with respect to the measure $\tilde{\mathbb{P}}$ will be denoted by $\tilde{\mathbb{E}}$.

The following properties of the measure $\tilde{\mathbb{P}}$ are well known [12, 26]:

P1. Under the measure $\tilde{\mathbb{P}}$, the distributions of the Wiener process W and the random variable X_0 are unchanged, the observation process Y is a standard Wiener process, and the state process X satisfies

$$dX(t) = b(X(t))dt + \sigma(X(t))dW(t) + \rho(X(t))\left(dY(t) - h(X(t))dt\right),\ 0 < t \le T;$$
$$X(0) = X_0;$$

P2. Under the measure $\tilde{\mathbb{P}}$, the Wiener processes W and Y and the random variable X_0 are independent of one another;

P3. The optimal filter $\hat{f}_t$ satisfies

$$\hat{f}_t = \frac{\tilde{\mathbb{E}}\left[f(X(t))Z_t|\mathcal{F}_t^Y\right]}{\tilde{\mathbb{E}}[Z_t|\mathcal{F}_t^Y]}. \tag{4.2}$$

Because of property **P2** of the measure $\tilde{\mathbb{P}}$ the filtering problem will be studied on the probability space $(\Omega, \mathcal{F}, \tilde{\mathbb{P}})$. If the function h is bounded, then there is a continuous embedding

$$L_2(\Omega, \tilde{\mathbb{P}}) \subset L_1(\Omega, \mathbb{P}). \tag{4.3}$$

Indeed, if $\xi \in L_2(\Omega, \tilde{\mathbb{P}})$, then

$$\mathbb{E}\xi = \tilde{\mathbb{E}}(Z_T \xi) \le \sqrt{\tilde{\mathbb{E}} Z_T^2} \sqrt{\tilde{\mathbb{E}}\xi^2} \le C \sqrt{\tilde{\mathbb{E}}\xi^2},$$

because

$$\begin{aligned} \tilde{\mathbb{E}} Z_T^2 &= \tilde{\mathbb{E}} \left(\exp\left\{ \int_0^T |h(X(t))|^2 dt \right\} \exp\left\{ 2\int_0^T h^*(X(t)) dY(t) - 2\int_0^T |h(X(t))|^2 dt \right\} \right) \\ &\le C\tilde{\mathbb{E}} \exp\left\{ 2\int_0^T h^*(X(t)) dY(t) - 2\int_0^T |h(X(t))|^2 dt \right\} \le C \end{aligned}$$

where the last inequality follows from the property **P2** of $\tilde{\mathbb{P}}$ and Proposition 3.5.12 in [13].
Next, consider the partial differential operators

$$\mathcal{L}g(x) = \frac{1}{2} \sum_{i,j=1}^{d} \left((\sigma(x)\sigma^*(x))_{ij} + (\rho(x)\rho^*(x))_{ij} \right) \frac{\partial^2 g(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x) \frac{\partial g(x)}{\partial x_i};$$

$$\mathcal{M}_l g(x) = h_l(x) g(x) + \sum_{i=1}^{d} \rho_{il}(x) \frac{\partial g(x)}{\partial x_i}, \ l = 1, \dots, r;$$

and their adjoints

$$\mathcal{L}^* g(x) = \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial^2}{\partial x_i \partial x_j} \left((\sigma(x)\sigma^*(x))_{ij} g(x) + (\rho(x)\rho^*(x))_{ij} g(x) \right) - \sum_{i=1}^{d} \frac{\partial}{\partial x_i} \left(b_i(x) g(x) \right);$$

$$\mathcal{M}_l^* g(x) = h_l(x) g(x) - \sum_{i=1}^{d} \frac{\partial}{\partial x_i} \left(\rho_{il}(x) g(x) \right), \ l = 1, \dots, r.$$

Let $\mathbf{H}^a$ be the Sobolev space $\{f : (1+|w|^2)^{a/2} \hat{f} \in L_2(\mathbb{R}^d)\}$, where $\hat{f} = \hat{f}(w)$ is the Fourier transform of f; $\mathbf{H}^0 = L_2(\mathbb{R}^d)$ with the norm $\|\cdot\|_0$. The inner product in $L_2(\mathbb{R}^d)$ and the duality between $\mathbf{H}^1$ and $\mathbf{H}^{-1}$ will be denoted by $(\cdot,\cdot)_0$. Note that the operators $\mathcal{L}, \mathcal{L}^*$ are bounded from $\mathbf{H}^1$ to $\mathbf{H}^{-1}$, operators $\mathcal{M}, \mathcal{M}^*$ are bounded from $\mathbf{H}^1$ to $L_2(\mathbb{R}^d)$, and, for every $g \in \mathbf{H}^1$,

$$2(\mathcal{L}^* g, g)_0 + \sum_{l=1}^{r} \|\mathcal{M}_l^* g\|_0^2 \le C\|g\|_0^2. \tag{4.4}$$

The following result is well known [26, Theorem 6.2.1].

4.1. Proposition. *In addition to Assumption R1 suppose that the initial density p_0 belongs to the space $\mathbf{H}^1$. Then there is a random field $p = p(t,x),\ t \in [0,T],\ x \in \mathbb{R}^d$, with the following properties:*

1. $p \in L_2(\Omega \times (0,T), d\tilde{\mathbb{P}} \times dt; \mathbf{H}^1) \cap L_2(\Omega, \tilde{\mathbb{P}}; \mathbf{C}([0,T], L_2(\mathbb{R}^d)))$.

2. The function $p(t,x)$ is a generalized solution of the stochastic partial differential equation

$$
\begin{aligned}
dp(t,x) &= \mathcal{L}^* p(t,x)dt + \sum_{l=1}^{r} \mathcal{M}_l^* p(t,x)dY_l(t),\ 0 < t \le T,\ x \in \mathbb{R}^d; \\
p(0,x) &= p_0(x).
\end{aligned} \tag{4.5}
$$

3. The equality

$$
\tilde{\mathbb{E}}\left[f(X(t))Z_t|\mathcal{F}_t^Y\right] = \int_{\mathbb{R}^d} f(x)p(t,x)dx \tag{4.6}
$$

holds for all bounded measurable functions f.

The random field $p = p(t,x)$ is called the *unnormalized filtering density* (UFD) and the random variable $\phi_t[f] = \tilde{\mathbb{E}}\left[f(X(t))Z_t|\mathcal{F}_t^Y\right]$, the *unnormalized optimal filter.* Under Assumption R1, equation (4.5) is at least dissipative. If the matrix $\sigma\sigma^*$ is uniformly positive definite, then equation (4.5) is coercive rather than dissipative, and it is enough to assume that $p_0 \in L_2(\mathbb{R}^d)$.

5. Approximation of the optimal filter

Let $\{e_i,\ i \ge 1\}$ be an orthonormal basis in $L_2(\mathbb{R}^d)$ so that every e_i belongs to $\mathbf{H}^1$. Fix a positive integer number K. Define the matrices $A^K = (A_{ij}^K,\ i,j = 1,\dots,K)$ and $B_l^K = (B_{l,ij}^K,\ i,j = 1,\dots,K;\ l = 1,\dots,r)$, by

$$
A_{ij}^K = (\mathcal{L}^* e_j, e_i)_0, \quad B_{l,ij}^K = (\mathcal{M}_l^* e_j, e_i)_0.
$$

Since $e_i \in \mathbf{H}^1$ for all i, the matrices are well defined. The Galerkin approximation $p^K(t,x)$ of $p(t,x)$ is given by

$$
p^K(t,x) = \sum_{i=1}^{K} p_i^K(t) e_i(x), \tag{5.1}
$$

where the vector $p^K(t) = \{p_i^K(t),\ i = 1,\dots,K\}$ is the solution of the system of stochastic ordinary differential equations

$$
dp^K(t) = A^K p^K(t)dt + \sum_{l=1}^{r} B_l^K p^K(t)dY_l(t) \tag{5.2}
$$

with the initial condition $p_i^K(0) = (p_0, e_i)_0$. Note that the matrices $B_l^K,\ l = 1,\dots,r$, do not, in general, commute with each other even if $\rho(x) \equiv 0$.

We next use Theorem 3.2 to derive the Cameron-Martin version of the Wiener chaos expansion of the solution of (5.2).

Let $0 = t_0 < t_1 \ldots < t_M = T$ be a uniform (for simplicity) partition of the interval $[0,T]$ with step Δ and let $\{m_k(t),\ k \ge 1\}$ be an orthonormal basis in $L_2([0,\Delta])$. Denote by J the set of all multi-indices $\alpha = \{\alpha_k^l,\ l = 1,\dots,r,\ k \ge 1, \alpha_k^l = 0,1,2,\ldots\}$ so that $|\alpha| = \sum_{l,k} \alpha_k^l < \infty$.

Define random variables

$$\xi^i_{k,l} = \int_{t_{i-1}}^{t_i} m_k(s - t_{i-1}) dY_l(s), \tag{5.3}$$

and then, for $\alpha \in J$,

$$\xi^i_\alpha = \frac{1}{\sqrt{\alpha!}} \prod_{k,l} H_{\alpha^l_k}(\xi^i_{k,l}), \tag{5.4}$$

where $H_n(t) = (-1)^n e^{t^2/2} \dfrac{d^n}{dt^n} e^{-t^2/2}$.

The following result is a direct consequence of Theorem 3.2.

5.1. Theorem. *For every $i = 1, \ldots, M$, the solution of (5.2) can be written in $L_2(\Omega; \mathbb{R}^K)$ as*

$$p^K(t_i) = \sum_{\alpha \in J} \frac{1}{\sqrt{\alpha!}} \varphi^K_\alpha(\Delta; p^K(t_{i-1})) \xi^i_\alpha, \quad i = 1, \ldots, M, \tag{5.5}$$

where, for $s \in (0, \Delta]$ and $\zeta \in \mathbb{R}^K$, the functions $\varphi^K_\alpha(s; \zeta)$ are the solutions of

$$\begin{aligned} \frac{\partial \varphi^K_\alpha(s;\zeta)}{\partial s} &= A^K \varphi^K_\alpha(s;\zeta) + \sum_{k,l} \alpha^l_k m_k(s) B^K_l \varphi^K_{\alpha(k,l)}(s;\zeta), \ 0 < s \le \Delta, \\ \varphi^K_\alpha(0;\zeta) &= \zeta 1_{\{|\alpha|=0\}}, \end{aligned} \tag{5.6}$$

and $\alpha(i,j)$ stands for the multi-index $\tilde{\alpha} = (\tilde{\alpha}^l_k)_{1 \le l \le r,\ k \ge 1}$ with

$$\tilde{\alpha}^l_k = \begin{cases} \alpha^l_k & \text{if } k \ne i \text{ or } l \ne j \text{ or both} \\ \max(0, \alpha^j_i - 1) & \text{if } k = i \text{ and } l = j. \end{cases} \tag{5.7}$$

For fixed positive integers N and n define the set J^n_N as the collection of multi-indices α from J such that $|\alpha| \le N$ and $\alpha^l_k = 0$ if $k > n$. The approximation $p^{K,n}_N(t_i)$ of $p^K(t_i)$ is defined by

$$p^{K,n}_N(t_0) = p^K(0), \quad p^{K,n}_N(t_i) = \sum_{\alpha \in J^n_N} \frac{1}{\sqrt{\alpha!}} \varphi^K_\alpha(\Delta; p^{K,n}_N(t_{i-1})) \xi^i_\alpha, \quad i = 1, \ldots, M. \tag{5.8}$$

Note the $p^{K,n}_N(t_i)$ is a vector in $\mathbb{R}^K$. Let $\mathbf{U} = \{\mathbf{u}^j,\ j = 1, \ldots, K\}$ be a basis in $\mathbb{R}^K$. The vector $p^{K,n}_N(t_i)$ can then be written as

$$p^{K,n}_N(t_i) = \sum_{j=1}^K p^{K,n}_{N,j}(t_i; \mathbf{U}) \mathbf{u}^j,$$

and by the recursive definition of $p_N^{K,n}(t_i)$,

$$\begin{aligned} p_N^{K,n}(t_{i+1}) &= \sum_{\alpha\in J_N^n} \varphi_\alpha^K(\Delta; p_N^{K,n}(t_i))\xi_\alpha^i \\ &= \sum_{\alpha\in J_N^n}\sum_{j=1}^K \varphi_\alpha^K(\Delta; \mathbf{u}^j) p_{N,j}^{K,n}(t_i;\mathbf{U})\xi_\alpha^i. \end{aligned}$$

Once again, $\varphi_\alpha^k(\Delta, \mathbf{u}^i)$ is a vector in $\mathbb{R}^K$, so we write

$$\varphi_\alpha^K(\Delta, \mathbf{u}^j) = \sum_{k=1}^K q_{jk}^{K,\alpha}(\mathbf{U})\mathbf{u}^k,$$

and conclude that

$$p_{N,j}^{K,n}(t_{i+1};\mathbf{U}) = \sum_{\alpha\in J_N^n}\sum_{k=1}^K q_{jk}^{K,\alpha}(\mathbf{U}) p_{N,k}^{K,n}(t_i;\mathbf{U})\xi_\alpha^i. \tag{5.9}$$

Then

$$p_N^{K,n}(t_i, x) = \sum_{j,k=1}^K p_{N,j}^{K,n}(t_{i+1};\mathbf{U})\mathbf{u}_k^j e_k(x) \tag{5.10}$$

is an approximation of the unnormalized filtering density.

Suppose that the basis functions e_k and the function f are such that

$$f_k = \int_{\mathbb{R}^d} f(x)e_k(x)dx \tag{5.11}$$

is defined for every $k = 1, \dots, K$. It follows from (5.10) that

$$\tilde{\phi}_i[f] = \sum_{j,k=1}^K p_{N,j}^{K,n}(t_{i+1};\mathbf{U})\mathbf{u}_k^j f_k \tag{5.12}$$

is an approximation of the unnormlized optimal filter.

The following is a possible algorithm for computing approximations of the unnormlized filtering density and optimal filter using (5.10) and (5.12).

1. <u>Preliminary computations</u> (before the observations are available):

(1) *Choose suitable basis functions* $\{e_k, k = 1, \dots, K\}$ *in* $L_2(\mathbb{R}^d)$, $\{m_i, i = 1, \dots, n\}$ *in* $L_2([0, \Delta])$, *and a* **standard unit basis** $\{\mathbf{u}^j, j = 1, \dots K\}$ *in* $\mathbb{R}^K$, *that is,* $\mathbf{u}_i^i = 1$, $\mathbf{u}_i^j = 0$ *otherwise.*

(2) *for* $\alpha \in J_N^n$ *and* $j, k = 1, \dots, K$ *compute*

$$q_{jk}^{K,\alpha} = \varphi_{\alpha,j}^K(\Delta; \mathbf{u}^k) \text{ (using (5.6))},\ f_k = \int_{\mathbb{R}^d} f(x)e_k(x)dx,\ p_{N,k}^{K,n}(t_0) = \int_{\mathbb{R}^d} p_0(x)e_k(x)dx;$$

2. $\underline{Real-time\ computations,\ i-th\ step}$ (as the observations become available): compute ξ^i_α, $\alpha \in J^n_N$ (according to (5.3) and (5.4));

$$Q^K_{jk}(\xi^i) = \sum_{\alpha \in J^n_N} q^{K,\alpha}_{jk} \xi^i_\alpha;$$

$$p^{K,n}_{N,j}(t_i) = \sum_{k=1}^{K} Q^K_{jk}(\xi^i) p^{K,n}_{N,k}(t_{i-1}), \quad j = 1, \ldots, K; \tag{5.13}$$

then, if necessary, compute

$$p^{K,n}_N(t_i, x) = \sum_{j=1}^{K} p^{K,n}_{N,j}(t_i) e_j(x), \tag{5.14}$$

$$\tilde{\phi}_{t_i}[f] = \sum_{j=1}^{K} f_j p^{K,n}_{N,j}(t_i), \tag{5.15}$$

and

$$\tilde{f}_{t_i} = \frac{\tilde{\phi}_{t_i}[f]}{\tilde{\phi}_{t_i}[1]}. \tag{5.16}$$

5.2. Remark. The main advantage of the above algorithm as compared to most other schemes for solving the Zakai equation is that the time consuming computations, including solving partial differential equations and computing integrals, are performed in advance, while the real-time part is relatively simple even when the dimension d of the state process is large. Here are some other features of the algorithm:

(1) The overall amount of preliminary computations does not depend on the number of the on-line time steps;
(2) Formulas (5.15) and (5.16) can be used to compute an approximation to $\hat{f}_{t_i}$, for example, conditional moments, without the time consuming computations of $p^{\kappa,n}_N(t_i, x)$ and the related integrals;
(3) Only the coefficients $p^{K,n}_{N,j}(t_i)$ must be computed at every time step while the approximate filter $\tilde{f}_{t_i}$ and/or UFD $p^{K,n}_N(t_i, x)$ can be computed as needed, for example, at the final time moment.
(4) The real-time part of the algorithm can be easily parallelized.
(5) Even though the coefficients $q^{K,\alpha}_{jk}$ are computed according to (5.6), their values can be further adjusted by simulating the state and observation processes and computing the corresponding filter estimates.
(6) If $n = 1$, then each ξ^i_α depends only on the increments $Y_l(t_i) - Y_l(t_{i-1})$ of the observation process. For $n > 1$ and $k > 1$, the integral $\int_{t_{i-1}}^{t_i} m_k(s - t_{i-1}) dY_l(s)$ can be reduced to a usual Riemann integral and then approximated by the trapezoidal

rule. In general, successful implementation and testing of the algorithm will require effective numerical methods for stochastic ODEs (see, for example, [14, 16, 24]).

6. Rate of convergence

To study the convergence of the algorithm, it is necessary to specify the bases $\{e_k, k \geq 1\}$ on $\mathbb{R}^d$ and $\{m_i, i \geq 1\}$ on $[0, \Delta]$.

Let $\{e_k, k \geq 1\}$ be the Hermite basis in $L_2(\mathbb{R}^d)$. The basis can be described as follows. Denote by Γ the set of ordered d-tuples $\gamma = (\gamma_1, \ldots, \gamma_d)$ with $\gamma_j = 0, 1, 2, \ldots$. For $\gamma \in \Gamma$ define

$$\mathcal{H}_\gamma(x) = \prod_{j=1}^{d} \mathcal{H}_{\gamma_j}(x_j),$$

where

$$\mathcal{H}_k(t) = \frac{(-1)^n}{\sqrt{2^n \pi^{1/2} n!}} e^{t^2} \frac{d^n}{dt^n} e^{-t^2}$$

With this definition, $\mathcal{H}_\gamma$ is the eigenfunction of the self-adjoint operator $\Lambda = -\nabla^2 + (1+|x|^2)$:

$$\Lambda \mathcal{H}_\gamma = \lambda_\gamma e_\gamma,$$

where ∇^2 is the Laplace operator and $\lambda_\gamma = (2|\gamma| + d + 1)$.

To define an ordering of the set Γ, we define $|\gamma| = \sum_{j=1}^{d} \gamma_j$ and then say that $\gamma < \tau$ if $|\gamma| < |\tau|$ or if $|\gamma| = |\tau|$ and $\gamma < \tau$ under the lexicographic ordering, that is, $\gamma_{i_0} < \tau_{i_0}$, where i_0 is the first index for which $\gamma_i \neq \tau_i$. The basis $\{e_k\}_{k \geq 1}$ is then the set $\{\mathcal{H}_\gamma(x),\ \gamma \in \Gamma\}$ together with the above ordering of the set Γ so that $\Lambda e_k = \lambda_k e_k$ and $\lambda_k \asymp k^{1/d}$.

Next, we define an orthonormal basis $\{m_k\}$ in $L_2([0, \Delta])$ by

$$m_1(s) = \frac{1}{\sqrt{\Delta}}; \quad m_k(s) = \sqrt{\frac{2}{\Delta}} \cos\left(\frac{\pi(k-1)s}{\Delta}\right),\ k > 1;\ 0 \leq s \leq \Delta.$$

6.1. Definition. The filtering model (4.1) is called ν-regular for some positive integer ν if the functions σ and ρ belong to $\mathbf{C}_b^{2\nu+3}$, the functions b and h belong to $\mathbf{C}_b^{2\nu+2}$, and $\Lambda^\nu p_0 \in \mathbf{H}^1$.

6.2. Theorem. *If the filtering model (4.1) is ν-regular, in the sense of Definition 6.1, for some $\nu > d + 1$ and*

$$C_\rho = \max_{i,l} \sup_{x \in \mathbb{R}^d} |\rho_{il}(x)|^2,$$

then

(6.1)

$$\max_{0 \leq i \leq M} \tilde{\mathbb{E}} \|p(t_i, \cdot) - p_N^{K,n}(t_i, \cdot)\|_0^2 \leq \frac{C(\nu, T)}{K^{2(\nu-d-1)/d}} + \left(C \frac{(1 + C_\rho K^{1/d})\Delta + (K^{2/d} + C_\rho K^{3/d})\Delta^2}{n} + \frac{(C(1 + C_\rho K^{1/d}))^{N+1} \Delta^N}{(N+1)!} \right) e^{C(1+C_\rho K^{1/d})T}.$$

The number $C(\nu,T)$ depends on ν, T, and the parameters of the model (coefficients of the equations (4.1)). The number C depends only on the parameters of the model.

If, in addition, $(1+|x|^2)^{-w}f \in L_2(\mathbb{R}^d)$ for some $w \geq 0$ so that $\nu > d+1+w$ and $\Lambda^\nu((1+|x|^2)^w p_0) \in \mathbf{H}^1$, then

(6.2)

$$\max_{0\leq i\leq M} \tilde{\mathbb{E}}|\phi_{t_i}[f]-\tilde{\phi}_{t_i}[f]|^2 \leq \frac{C(\nu,T,w)C_f}{K^{2(\nu-w-d-1)/d}}$$
$$+C_f\left(C\frac{(1+C_\rho K^{1/d})\Delta+(K^{2/d}+C_\rho K^{3/d})\Delta^2}{n}+\frac{(C(1+C_\rho K^{1/d}))^{N+1}\Delta^N}{(N+1)!}\right)e^{C(1+C_\rho K^{1/d})T}.$$

The number $C(\nu,T,w)$ depends on ν, T, w, and the parameters of the model; the number C depends only on w and the parameters of the model; $C_f = \int_{\mathbb{R}^d}(1+|x|^2)^{-2w}|f(x)|^2dx$.

Proof. By Theorem 2.1,

$$\tilde{\mathbb{E}}\|p(t_i,\cdot)-p^K(t_i,\cdot)\|_0^2 \leq \frac{C(\nu,T)}{K^{2(\nu-d-1)/d}}. \tag{6.3}$$

Indeed, by Theorem 4.3.2 in [26], $\sup_{0<t<T}\tilde{\mathbb{E}}\|\Lambda^\nu p(t,\cdot)\|_0^2 \leq e^{CT}\|\Lambda^\nu p_0\|_0^2$, where C depends only on ν and the parameters of the model. Also, in the notations of Theorem 2.1, $\theta = 1/d$, $q = 1/(2d)$, and $\theta_1 = (\nu-1)/d$.

To simplify the further presentation, set $\kappa = K^{1/d}$ and define $C_\kappa = 1 + C_\rho\kappa$. Then, to prove (6.1), it remains to show that

$$\tilde{\mathbb{E}}|p^K(t_i)-p_N^{K,n}(t_i)|^2 \leq \left(C\frac{C_\kappa\Delta+\kappa^2C_\kappa\Delta^2}{n}+\frac{(CC_\kappa)^{N+1}\Delta^N}{(N+1)!}\right)e^{CC_\kappa T},$$

and by Theorem 3.3 this inequality holds if, for every vector $\zeta\in\mathbb{R}^K$,

$$|A^K\zeta|^2 \leq C\kappa^2|\zeta|^2,\ |B_l^K\zeta|^2 \leq CC_\kappa|\zeta|^2,\ |e^{tA^K}\zeta|^2 \leq e^{Ct}|\zeta|^2. \tag{6.4}$$

Because of the multi-step approximation, we, as usual, loose one power of Δ in (3.12). Inequalities (6.4) are verified by direct calculations using that the operators $\Lambda^{-1}\mathcal{L}$ and $\Lambda^{-1/2}\mathcal{M}_l$ are bounded in $L_2(\mathbb{R}^d)$.

To prove (6.2), let $\beta(x) = \sqrt{1+|x|^2}$ and, for $w\in\mathbb{R}$, define the space $L_{2,w}(\mathbb{R}^d) = \{f : f\beta^w \in L_2(\mathbb{R}^d)\}$. Clearly, $L_{2,w}(\mathbb{R}^d)$ is a Hilbert space with inner product $(f,g)_{0,w} = (f\beta^w, g\beta^w)_0$ and norm $\|f\|_{0,w}^2 = (f,f)_{0,w}$. Then

$$|\phi_{t_i}[f]-\tilde{\phi}_{t_i}[f]|^2 \leq C_f\|p(t_i,\cdot)-p_N^{K,n}(t_i,\cdot)\|_{0,2w}^2. \tag{6.5}$$

Using the calculus of pseudo-differential operators [27, Chapter 4], we conclude that, for every $g\in L_{2,2w}(\mathbb{R}^d)$,

$$\|g\|_{0,2w} = \|\beta^{2w}g\|_0 \leq C\|\Lambda^{-w}\beta^{2w}\Lambda^w g\|_0 \leq C\|\Lambda^w g\|_0. \tag{6.6}$$

Therefore, by Theorem 2.1 and Theorem 4.3.2 in [26],

$$\tilde{\mathbb{E}}\|p(t_i,\cdot)-p^K(t_i,\cdot)\|_{0,2w}^2 \leq \frac{C(\nu,T)}{K^{2(\nu-w-d-1)/d}}. \tag{6.7}$$

Next, (6.6) implies

$$\tilde{\mathbb{E}}\|p^K(t_i,\cdot)-p_N^{K,n}(t_i,\cdot)\|_{0,2w}^2 \le C\sum_{k=1}^{K}\lambda_k^{2w}\tilde{\mathbb{E}}|p_k^K(t_i)-p_{N,k}^{K,n}(t_i)|^2.$$

Define the diagonal matrix $\hat{\Lambda}=(\hat{\Lambda}_{ij})_{i,j=1,\ldots,K}$ by $\hat{\Lambda}_{ii}=\lambda_i^w$. Then define the matrices

$$\hat{A}^K=\hat{\Lambda}A^K\hat{\Lambda}^{-1},\ \hat{B}_l^K=\hat{\Lambda}B_l^K\hat{\Lambda}^{-1},$$

and the vectors $\hat{p}(t)=\hat{\Lambda}p^K(t),\ \hat{p}_N^n(t_i)=\hat{\Lambda}p_N^{K,n}(t_i)$. With these definitions, the vector $\hat{p}^K(t)$ is the solution of

$$d\hat{p}^K(t)=\hat{A}^K\hat{p}^K(t)dt+\sum_{l=1}^{r}\hat{B}_l^K\hat{p}(t)dY_l(t)$$

with the initial condition $\hat{p}_k^K(0)=\lambda_k^w(p_0,e_k)_0$, the vector $\hat{p}_N^{K,n}(t)$ satisfies

$$\hat{p}_N^{K,n}(t_0)=\hat{p}^K(0),\ \hat{p}_N^{K,n}(t_i)=\sum_{\alpha\in J_N^n}\frac{1}{\sqrt{\alpha!}}\hat{\varphi}_\alpha^K(\Delta;\hat{p}_N^{K,n}(t_{i-1}))\xi_\alpha^i,\quad i=1,\ldots,M,$$

and

$$\tilde{\mathbb{E}}\|p^K(t_i,\cdot)-p_N^{K,n}(t_i,\cdot)\|_{0,w}^2\le C\tilde{\mathbb{E}}|\hat{p}^K(t_i)-\hat{p}_N^{K,n}(t_i)|^2. \tag{6.8}$$

The functions $\hat{\varphi}_\alpha^K$ satisfy the equations (5.6) with $\hat{A}^K$ and $\hat{B}_l^K$ instead of A^K and B^K. Direct computations show that, for all $\zeta\in\mathbb{R}^K$,

$$|\hat{A}^K\zeta|^2\le C\kappa^2|\zeta|^2,\quad |\hat{\mathcal{B}}_l^K\zeta|^2\le CC_\kappa|\zeta|^2,\quad |e^{t\hat{A}^K}\zeta|^2\le e^{Ct}|\zeta|^2, \tag{6.9}$$

with C depending on w and the parameters of the filtering model. By Theorem 3.3 we then conclude that

$$\tilde{\mathbb{E}}|\hat{p}^K(t_i)-\hat{p}_N^{K,n}(t_i)|^2\le\left(C\frac{C_\kappa\Delta+\kappa^2C_\kappa\Delta^2}{n}+\frac{(CC_\kappa)^{N+1}\Delta^N}{(N+1)!}\right)e^{CC_\kappa T}.$$

Together with (6.5), (6.6), and (6.8), the last inequality implies (6.2).
Theorem 6.2 is proved. □

7. Proof of Theorem 3.3

The proof requires an explicit formula for the solution of (3.6). We begin with some auxiliary constructions.

Every multi-index α with $|\alpha|=k$ can be identified with the set $K_\alpha=\{(i_1^\alpha,q_1^\alpha),\ldots,(i_k^\alpha,q_k^\alpha)\}$ so that $i_1^\alpha\le i_2^\alpha\le\ldots\le i_k^\alpha$ and if $i_j^\alpha=i_{j+1}^\alpha$, then $q_j^\alpha\le q_{j+1}^\alpha$. The first pair (i_1^α,q_1^α) in K_α is the position numbers of the first nonzero element of α. The second pair is the same as the first if the first nonzero element of α is greater than one; otherwise, the second pair is the position numbers of the second nonzero element of α and so on. As a result, if $\alpha_j^q>0$, then exactly α_j^q pairs in K_α are (j,q). The set K_α will be referred to as **the characteristic set** of the multi-index α. For example, if $r=2$ and

$$\alpha=\begin{pmatrix} 0 & 1 & 0 & 2 & 3 & 0 & 0 & \dots \\ 1 & 2 & 0 & 0 & 0 & 1 & 0 & \dots \end{pmatrix},$$

then the nonzero elements are $\alpha_1^2=\alpha_2^1=\alpha_1^6=1,\ \alpha_2^2=\alpha_4^1=2,\ \alpha_5^1=3$, and the characteristic set is $K_\alpha=\{(1,2),\,(2,1),\,(2,2),\,(2,2),\,(4,1),\,(4,1),\,(5,1),\,(5,1),\,(5,1),\,(6,2)\}$. In the future, when there is no danger of confusion, the superscript α in i and q will be omitted so that (i_j,q_j) will be written instead of (i_j^α,q_j^α).

Let $\mathcal{P}^k$ be the permutation group of the set $\{1,\dots,k\}$. For a given $\alpha\in J$ with $|\alpha|=k$ and the characteristic set $\{(i_1,q_1),\dots,(i_k,q_k)\}$ define

$$E_\alpha(s^k;l^k):=\sum_{\sigma\in\mathcal{P}^k} m_{i_1}(s_{\sigma(1)})1_{\{l_{\sigma(1)}=q_1\}}\cdots m_{i_k}(s_{\sigma(k)})1_{\{l_{\sigma(k)}=q_k\}}.$$

The following notations are introduced to simplify the further presentation:

- s^k, the ordered set $(s_1,\dots,s_k)$; $ds^k:=ds_1\dots ds_k$;
- l^k, the ordered set $(l_1,\dots,l_k)$;
- $\Phi_t=e^{At}$;
- $F(t;s^k;l^k;g):=\Phi_{t-s_k}B_{l_k}\Phi_{s_k-s_{k-1}}\dots B_{l_1}\Phi_{s_1-T_0}g,\ k\ge 1$;
- $\displaystyle\int_{T_0}^{(k,t)}(\cdots)ds^k:=\int_{T_0}^{t}\int_{T_0}^{s_k}\dots\int_{T_0}^{s_2}(\cdots)ds_1\dots ds_k$;
- $\displaystyle\sum_{l^k}:=\sum_{l_1,\dots,l_k=1}^{r}$.

Note that

$$|F(t;s^k;l^k;g)|^2\le C_2^k e^{C_1(t-T_0)}|g|^2,\quad \int_{T_0}^{(k,t)}ds^k=\frac{(t-T_0)^k}{k!},\quad \sum_{l^k}1=r^k. \tag{7.1}$$

7.1. Proposition. *If $\alpha\in J$ is a multi-index with $|\alpha|=k$ and the characteristic set $\{(i_1,q_1),\dots,(i_k,q_k)\}$, then, for $t\in[T_0,t^*]$, the corresponding solution $\varphi_\alpha(t;T_0;U_0)$ of (3.6) is given by*

$$\begin{aligned}
&\varphi_\alpha(t;T_0;U_0)=\\
&\sum_{\sigma\in\mathcal{P}^k}\sum_{l^k}\int_{T_0}^{(k,t)}F^k(t;s^k;l^k;U_0)m_{i_{\sigma(k)}}(s_k)1_{\{l_k=q_{\sigma(k)}\}}\cdots m_{i_{\sigma(1)}}(s_1)1_{\{l_1=q_{\sigma(1)}\}}ds^k,\ k>1;\\
&\varphi_\alpha(t;T_0;U_0)=\int_{T_0}^{t}\Phi_{t-s_1}B_{q_1}\Phi_{s_1-T_0}U_0m_{i_1}(s_1)ds_1,\ k=1;\\
&\varphi_\alpha(t;T_0;U_0)=\Phi_{t-T_0}U_0,\ k=0,
\end{aligned}\tag{7.2}$$

and

$$\sum_{|\alpha|=k} \frac{|\varphi_\alpha(t;T_0;U_0)|^2}{\alpha!} = \sum_{l^k} \int_{T_0}^{(k,t)} |F(t;s^k;l^k;U_0)|^2 ds^k. \tag{7.3}$$

Proof. To simplify the notations, the arguments T_0 and U_0 will be omitted wherever possible. Representation (7.2) is obviously true for $|\alpha| = 0$. Then the general case $|\alpha| \geq 1$ follows by induction from the variation of parameters formula.

To prove (7.3), first of all note that

$$\sum_{\sigma\in\mathcal{P}^k} m_{i_{\sigma(k)}}(s_k) 1_{\{l_k=q_{\sigma(k)}\}} \cdots m_{i_{\sigma(1)}}(s_1) 1_{\{l_1=q_{\sigma(1)}\}}$$
$$= \sum_{\sigma\in\mathcal{P}^k} m_{i_k}(s_{\sigma(k)}) 1_{\{l_{\sigma(k)}=q_k\}} \cdots m_{i_1}(s_{\sigma(1)}) 1_{\{l_{\sigma(1)}=q_1\}}.$$

Indeed, every term on the left corresponding to a given $\sigma_0 \in \mathcal{P}^k$ coincides with the term on the right corresponding to $\sigma_0^{-1} \in \mathcal{P}^k$.

Then (7.2) can be written as $\varphi_\alpha(t) = \sum_{l^k} \int_{T_0}^{(k,t)} F(t;s^k;l^k) E_\alpha(s^k;l^k) ds^k$. Using the notation

$$G(t;s^k;l^k) := \sum_{\sigma\in\mathcal{P}^k} \Phi_{t-s_{\sigma(k)}} B_{l_{\sigma(k)}} \ldots \Phi_{s_{\sigma(2)}-s_{\sigma(1)}} B_{l_{\sigma(1)}} \Phi_{s_{\sigma(1)}-T_0} g 1_{s_{\sigma(1)}<\ldots<s_{\sigma(k)}<t},$$

it can be rewritten as

$$\varphi_\alpha(t) = \frac{1}{k!} \sum_{l^k} \int\limits_{[T_0,t^*]^k} G(t;s^k;l^k) E_\alpha(s^k;l^k) ds^k. \tag{7.4}$$

Since for every $t \in [T_0, t^*]$ the function $G(t;s^k;l^k)$ is symmetric,

$$G(t;s^k;l^k) = \sum_{|\beta|=k} \frac{c_\beta(t) E_\beta(s^k;l^k)}{\sqrt{\beta! k!}}$$

with some vector coefficients $c_\beta(t)$. This and (7.4) imply $|\varphi_\alpha(t)|^2/\alpha! = |c_\alpha|^2/k!$ and so

$$\sum_{|\alpha|=k} \frac{|\varphi_\alpha(t)|^2}{\alpha!} = \frac{1}{k!} \sum_{|\alpha|=k} |c_\alpha(t)|^2 = \frac{1}{k!} \int\limits_{[T_0,t^*]^k} |G(t;s^k;l^k)|^2 ds^k$$
$$= \frac{1}{k!} \sum_{l^k} \int\limits_{[T_0,t^*]^k} \Big| \sum_{\sigma\in\mathcal{P}^k} \Phi_{t-s_{\sigma(k)}} B_{l_{\sigma(k)}} \ldots \Phi_{s_{\sigma(2)}-s_{\sigma(1)}} B_{l_{\sigma(1)}} \Phi_{s_{\sigma(1)}-T_0} g 1_{s_{\sigma(1)}<\ldots<s_{\sigma(k)}<t} \Big|^2 ds^k$$
$$= \sum_{l^k} \int_{T_0}^{(k,t)} |F(t;s^k;l^k)|^2 ds^k,$$

which proves (7.3). Proposition 7.1 is proved. $\square$

We continue by considering the truncation only of the length of α. Define $J_N = \{\alpha \in J : |\alpha| \le N\}$ and

$$U^N(s;T_0;U_0) = \sum_{\alpha \in J_N} \frac{1}{\sqrt{\alpha!}} \varphi_\alpha(s;T_0;U_0)\xi_\alpha(W_{T_0,t^*}). \tag{7.5}$$

Note that the summation in (7.5) is still infinite.

7.2. Proposition. *In the notations of Theorem 3.3,*

$$\sup_{s\in[T_0,t^*]} \mathbb{E}|U(s;T_0;U_0) - U^N(s;T_0;U_0)|^2 \le \frac{[C_2 r(t^*-T_0)]^{N+1}}{(N+1)!} e^{\bar{C}(t^*-T_0)}\, \mathbb{E}|U_0|^2. \tag{7.6}$$

Proof. To simplify the presentation, the arguments T_0 and U_0 will be omitted wherever possible.

By Theorem 7.1,

$$\sum_{|\alpha|=k} \frac{|\varphi_\alpha(s)|^2}{\alpha!} = \sum_{l^k} \int_{T_0}^{(k,s)} |F(s;s^k;l^k)|^2 ds^k. \tag{7.7}$$

Since the random variables $\xi_\alpha(W_{T_0,t})$ are uncorrelated and are independent of U_0, formulas (5.5) and (7.5) imply $\mathbb{E}|U(s) - U^N(s)|^2 = \sum_{k>N}\sum_{|\alpha|=k} \frac{\mathbb{E}|\varphi_\alpha(s)|^2}{\alpha!}$. By (7.1),

$$\begin{aligned}\sum_{k>N}\sum_{|\alpha|=k} \frac{\mathbb{E}|\varphi_\alpha(s)|^2}{\alpha!} &\le e^{C_1(s-T_0)}\, \mathbb{E}|U_0|^2 \sum_{k>N} \frac{(C_2 r(s-T_0))^k}{k!}\\ &\le \frac{(C_2 r(t^*-T_0))^{N+1}}{(N+1)!} e^{\bar{C}(t^*-T_0)}\, \mathbb{E}|U_0|^2,\end{aligned}$$

which completes the proof of Proposition 7.2. □

Now we truncate the sum in (7.5) even more by restricting α to the set J_N^n.

7.3. Proposition. *In the notations of Theorem 3.3 and Proposition 7.2,*

$$\begin{aligned}\mathbb{E}|U^N(t^*;T_0;U_0) - U_N^n(t^*;T_0;U_0)|^2 \;\le\;& 2C_2 r\, e^{\bar{C}(t^*-T_0)}\Big(\epsilon(B)\frac{(t^*-T_0)^2}{n}\\ &+ C_0\,(1+(t^*-T_0)C_2 r)\,\frac{(t^*-T_0)^3}{n}\Big)\mathbb{E}|U_0|^2.\end{aligned} \tag{7.8}$$

Proof. To simplify the presentation, the arguments T_0 and U_0 will be omitted wherever possible.

If α is a multi-index with $|\alpha| = k$ and the characteristic set $\{(i_1^\alpha, q_1^\alpha)\ \ldots, (i_k^\alpha, q_k^\alpha)\}$, then $i_k^\alpha = d(\alpha)$, the order of α, and so the set J_N^n can be described as $\{\alpha \in J : |\alpha| \le N;\ i_{|\alpha|}^\alpha \le n\}$. Since the random variables ξ_α are uncorrelated and are independent of U_0,

$$\mathbb{E}|U_N^n(t^*) - U^N(t^*)|^2 = \sum_{b=n+1}^{\infty}\sum_{k=1}^{N}\sum_{|\alpha|=k;\, i_k^\alpha=b} \frac{\mathbb{E}|\varphi_\alpha(t^*)|^2}{\alpha!}.$$

The problem is thus to estimate $\displaystyle\sum_{b=n+1}^{\infty}\sum_{k=1}^{N}\sum_{|\alpha|=k;i_k^\alpha=b}\frac{|\varphi_\alpha(t^*)|^2}{\alpha!}$.

By Theorem 7.1 the corresponding solution φ_α of (3.6) can be written as

$$\varphi_\alpha(t^*)=\sum_{l^k}\int_{T_0}^{(k,t^*)}F(t^*;s^k;l^k)E_\alpha(s^k,l^k)ds^k. \tag{7.9}$$

According to (3.7), the characteristic set of $\alpha(i_k,q_k)$ is $\{(i_1,q_1),\ldots,(i_{k-1},q_{k-1})\}$; therefore, it is possible to write

$$E_\alpha(s^k)=\sum_{j=1}^{k}m_{i_k}(s_j)1_{\{l_j=q_k\}}E_{\alpha(i_k,q_k)}(s_j^k;l_j^k),$$

where s_j^k (resp. l_j^k) denotes the same set $(s_1,\ldots,s_k)$ (resp. $(l_1,\ldots,l_k)$) with omitted s_j (resp. l_j); for example, $s_1^k=(s_2,\ldots,s_k)$.

As a result, after changing the order of integration in the multiple integral, equality (7.9) can be rewritten as

$$\varphi_\alpha(t^*)=\sum_{j=1}^{k}\sum_{l_j^k}\int_{T_0}^{(k-1,t^*)}\Big(\int_{s_{j-1}}^{s_{j+1}}F(t^*;s^k;l^k)m_{i_k}(s_j)1_{\{l_j=q_k\}}ds_j\Big)E_{\alpha(i_k,q_k)}(s_j^k;l_j^k)ds_j^k, \tag{7.10}$$

where $s_0:=T_0;\ s_{k+1}:=t^*$.

Denote

$$M_k(s):=\frac{\sqrt{2(t^*-T_0)}}{\pi(k-1)}\sin\Big(\frac{\pi(k-1)(s-T_0)}{(t^*-T_0)}\Big);\ k>1,\ T_0\le s\le t^*,$$

and $F_j:=\dfrac{\partial F(t^*;s^k;l^k)}{\partial s_j}$. Then, as long as $i_k=b>1$, integration by parts in the inner integral on the right hand side of (7.10) yields:

$$\begin{aligned}&\int_{s_{j-1}}^{s_{j+1}}F(t^*;s^k;l^k)m_b(s_j)ds_j\\&=F(t^*;s^k;l^k)M_b(s_j)\Big|_{s_j=s_{j-1}}^{s_j=s_{j+1}}-\int_{s_{j-1}}^{s_{j+1}}F_j(t^*;s^k;l^k)M_b(s_j)ds_j.\end{aligned}$$

For each j, let us rename the remaining variables s_j^k in (7.10) as follows: $t_i:=s_i, i\le j-1$; $t_i:=s_{i+1}, i>j-1$, or, symbolically, $t^{k-1}:=s_j^k$. We will set $t_0:=T_0$, $t_k:=t^*$ and denote by $t^{k-1,j}, j=1,\ldots,k-1$, the set t^{k-1} in which t_j is repeated twice (e.g. $t^{k-1,1}=(t_1,t_1,\ldots,t_{k-1})$, etc.); also $t^{k-1,0}:=(t_0,t_1,t_2,\ldots,t_{k-1})$, $t^{k-1,k}:=(t_1,\ldots,t_{k-1},t_k)$.

The similar changes will also be made with the set l^k: for fixed j, there are $k-1$ free indices $l_1,\ldots,l_{j-1},l_{j+1},\ldots,l_k$ and they are renamed just like s^k to form the set l^{k-1} (in this case, the same symbols are used). Similarly, $l^{k-1,j}$ denotes the set $(l_1,\ldots,l_{j-1},q_k,l_j,\ldots,l_{k-1})$. After these transformations, $E_{\alpha(i_k,q_k)}(s_j^k;l_j^k)$ becomes $E_{\alpha(i_k,q_k)}(t^{k-1};l^{k-1})$ - independent of j, and

$$F(t^*;s^k;l^k)1_{\{l_j=q_k\}}M_b(s_j)\Big|_{s_j=s_{j-1}}^{s_j=s_{j+1}}$$
$$= F(t^*;t^{k-1,j};l^{k-1,j})M_b(t_j) - F(t^*;t^{k-1,j-1};l^{k-1,j})M_b(t_{j-1}),\ j=1,\ldots,k.$$

Therefore, if $d(\alpha)=b>1$ and $|\alpha|=k>0$, then

$$\begin{aligned}\varphi_\alpha(t^*) &= \sum_{l^{k-1}} \int\limits_{T_0}^{(k-1,t^*)} \Big(f_b^{(1)}(t^*;t^{k-1};l^{k-1}) \\ &+ f_b^{(2)}(t^*;t^{k-1};l^{k-1})\Big)E_{\alpha(i_k,q_k)}(t^{k-1};l^{k-1})dt^{k-1},\end{aligned}$$

where

$$\begin{aligned}f_b^{(1)}(t^*;t^{k-1};l^{k-1}) &= \sum_{j=1}^{k}\Big(F(t^*;t^{k-1,j};l^{k-1,j})M_b(t_j) \\ &- F(t^*;t^{k-1,j-1};l^{k-1,j})M_b(t_{j-1})\Big) \quad \text{if } k>1,\end{aligned}$$

$f_b^{(1)}=0$ if $k=1$ – because $M_b(t_0)=M_b(t_k)=0$ (this is the only place where the choice of $\{m_k\}$ really makes the difference), and

$$\begin{aligned}f_b^{(2)}(t^*;t^{k-1};l^{k-1}) &= -\int_{T_0}^{t_1} F_1(t^*;s,t^{k-1};q_k,l^{k-1})M_b(s)ds \\ &- \sum_{j=2}^{k-1}\int_{t_{j-1}}^{t_j} F_j(t^*;\ldots,t_{j-1},s,t_j,\ldots;l^{k-1,j})M_b(s)ds \\ &- \int_{t_{k-1}}^{t_k} F_k(t^*;t^{k-1},s;l^{k-1},q_k)M_b(s)ds.\end{aligned}$$

Note that if the operators B_l commute with each other, then $f_b^{(1)}(t^*;t^{k-1};l^{k-1})$ is identically equal to zero for all k.

Since $|\alpha(i_{|\alpha|},q_{|\alpha|})|=|\alpha|-1$ and $\alpha!\ge\alpha(i_{|\alpha|},q_{|\alpha|})!$, it now follows from (7.10) that

$$\begin{aligned}&\sum_{|\alpha|=k;i_k^\alpha=b}\frac{|\varphi_\alpha(t^*)|^2}{\alpha!} \\ &= \sum_{|\alpha|=k;i_k^\alpha=b}\sum_{q_k=1}^{r}\Big|\frac{1}{\sqrt{\alpha!}}\sum_{l^{k-1}}\int_{T_0}^{(k-1,t^*)}(f_b^{(1)}+f_b^{(2)})E_{\alpha(b,q_k)}dt^{k-1}\Big|^2 \\ &\le \sum_{q_k=1}^{r}\sum_{|\beta|=k-1}\Big|\frac{1}{\sqrt{\beta!}}\sum_{l^{k-1}}\int_{T_0}^{(k-1,t^*)}(f_b^{(1)}+f_b^{(2)})E_\beta dt^{k-1}\Big|^2,\end{aligned}$$

and the proof of Proposition 7.1 shows that the last expression is equal to

$$\sum_{q_k=1}^{r}\sum_{l^{k-1}}\int_{T_0}^{(k-1,t^*)}\Big|f_b^{(1)}(t^*;t^{k-1};l^{k-1})+f_b^{(2)}(t^*;t^{k-1};l^{k-1})\Big|^2dt^{k-1}. \tag{7.11}$$

Definition of $f_b^{(1)}$ implies

$$|f_b^{(1)}|^2 = 0,\ k=1;\ \ |f_b^{(1)}|^2 \le \frac{k(C_2)^k\epsilon(B)(t^*-T_0)}{(b-1)^2}e^{C_1(t^*-T_0)}\ |U_0|^2,\ k\ge 2. \tag{7.12}$$

Next, direct computations yield

$$\begin{array}{rcl} F_j(t^*;s^k;l^k) & = & \Phi_{t^*-s_k}B_{l_k}\dots\Phi_{s_{j+1}-s_j}B_{l_j}A\Phi_{s_j-s_{j-1}}\dots\Phi_{s_1-T_0}U_0 \\ & - & \Phi_{t^*-s_k}B_{l_k}\dots A\Phi_{s_{j+1}-s_j}B_{l_j}\Phi_{s_j-s_{j-1}}\dots\Phi_{s_1-T_0}U_0, \end{array}$$

so that by assumption (3) of the theorem, $|F_j(t^*;s^k;l^k)|^2 \le C_0(C_2)^k e^{C_1(t^*-T_0)}\ |U_0|^2$. After that the definition of $f_b^{(2)}$ implies:

$$\begin{array}{rcl} |f_b^{(2)}|^2 & \le & 4C_0k(C_2)^k e^{C_1(t^*-T_0)}|U_0|^2(t^*-T_0)\displaystyle\int_{T_0}^{t^*}(M_b(s))^2ds \\ & \le & \dfrac{C_0k(C_2)^k(t^*-T_0)^3}{(b-1)^2}e^{C_1(t^*-T_0)}\ |U_0|^2\ ; \end{array}$$

so, since $\displaystyle\int\limits_{T_0}^{(k-1,t^*)} dt^{k-1} = (t^*-T_0)^{k-1}/(k-1)!$, (7.11), (7.12) and the last inequality yield

$$\mathbb{E}|U^N(t^*) - U_N^n(t^*)|^2 = \sum_{b\ge n+1}\sum_{k=1}^{N}\sum_{|\alpha|=k;i_k^\alpha=b}\frac{\mathbb{E}|\varphi_\alpha(t^*)|^2}{\alpha!}$$
$$\le C_2re^{C_1(t^*-T_0)}\Big[\epsilon(B)(t^*-T_0^2)\sum_{k\ge 0}\frac{k+2}{k+1}\frac{(C_2r(t^*-T_0))^k}{k!}$$
$$+C_0(t^*-T_0)^3\sum_{k\ge 0}\frac{(k+1)(C_2r(t^*-T_0))^k}{k!}\Big]\mathbb{E}|U_0|^2\sum_{b\ge n}\frac{1}{b^2}$$
$$\le \frac{2C_2r\ e^{\bar{C}(t^*-T_0)}}{n}\left[\epsilon(B)(t^*-T_0)^2 + (1+(t^*-T_0)C_2r)C_0(t^*-T_0)^3\right]\mathbb{E}|U_0|^2.$$

This completes the proof of Proposition 7.3. The statement of Theorem 3.3 now follows from Propositions 7.2 and 7.3.

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