Sobolev Spaces with Weights in Domains and Boundary Value Problems for Degenerate Elliptic Equations

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Abstract. A family of Banach spaces is introduced to control the interior smoothness and boundary behavior of functions in a general domain. Interpolation, embedding, and other properties of the spaces are studied. As an application, a certain degenerate second-order elliptic partial differential equation is considered.

1. Introduction

Let G be a domain in $\mathbb{R}^d$ with a non-empty boundary ∂G and $\rho_G(x) = dist(x, \partial G)$. For $1 \leq p < \infty$ and $\theta \in \mathbb{R}$ define the space $L_{p,\theta}(G)$ as follows:

$$L_{p,\theta}(G) = \{u : \int_G |u(x)|^p \rho_G^{\theta-d}(x)dx < \infty\}.$$

Then we can define the spaces $H^m_{p,\theta}(G)$, $m = 1, 2, \ldots$, so that

$$H^m_{p,\theta}(G) = \{u : u, \rho_G Du, \ldots, \rho_G^m D^m u \in L_{p,\theta}\},$$

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where D^k denotes generalized derivative of order k. The objective of the current paper is to define spaces $H^{\gamma}_{p,\theta}(G)$, $\gamma \in \mathbb{R}$, so that, for positive integer γ, the spaces $H^{\gamma}_{p,\theta}(G)$ coincide with the ones introduced above. It will be shown that these spaces can be easily defined using the spaces $H^{\gamma}_{p}(\mathbb{R}^d)$ of Bessel potentials. Note that $u \in H^1_{p,d-p}(G)$ if and only if $u/\rho_G, Du \in L_p(G)$, which means that, for bounded G, the space $H^1_{p,d-p}(G)$ coincides with the space $\overset{\circ}{H}{}^1_p(G)$. As a result, the spaces $H^{\gamma}_{p,\theta}(G)$ can be considered as a certain generalization of the usual Sobolev spaces on G with zero boundary conditions. A major application of the spaces $H^{\gamma}_{p,\theta}(G)$ is in the analysis of the Dirichlet problem for stochastic parabolic equations [5, 7].

Some of the spaces $H^{\gamma}_{p,\theta}(G)$ have been studied before. Lions and Magenes [6] introduced what corresponds to $H^{\gamma}_{2,d}(G)$. They constructed the scale by interpolating between the positive integer γ for $\gamma > 0$ and used duality for $\gamma < 0$. Krylov [3] defined the spaces $H^{\gamma}_{p,\theta}(\mathbb{R}^d_+)$, where $\mathbb{R}^d_+$ is the half-space. After that, if G is sufficiently regular and bounded, then $H^{\gamma}_{p,\theta}(G)$ can be defined using the partition of unity, and this was done in [7]. Other related examples and references can be found in Chapter 3 of [10].

In this paper, an intrinsic definition (not involving $\mathbb{R}^d_+$) of the spaces $H^{\gamma}_{p,\theta}(G)$ is given for a general domain G, and the basic properties of the spaces are studied. Once a suitable definition of the spaces is found, most of the properties follow easily from the known results. Definition and properties of the spaces $H^{\gamma}_{p,\theta}(G)$ are presented in Sections 2, 3, and 4. Roughly speaking, the index γ controls the smoothness inside the domain, and the index θ controls the boundary behavior. In particular, the space $H^{\gamma}_{p,\theta}(G)$ with sufficiently large γ and $\theta < 0$ contains functions that are continuous in the closure of G and vanish on the boundary. In Section 5 some results are presented about solvability of certain degenerate elliptic equations in a general domain G.

Throughout the paper, D^m denotes a partial derivative of order m, that is, $D^m = \partial^m/\partial x_1^{m_1} \cdots \partial x_d^{m_d}$ for some $m_1 + \cdots + m_d = m$. For two Banach spaces, X, Y, notation $X \subset Y$ means that X is continuously embedded into Y.

2. Definition and main properties of the weighted spaces in domains

Let $G \subset \mathbb{R}^d$ be a domain (open connected set) with non-empty boundary ∂G, and $c > 1$, a real number. Denote by $\rho_G(x)$, $x \in G$, the distance from x to ∂G. For $n \in \mathbb{Z}$ and a fixed integer $k_0 > 0$ define the subsets G_n of G by

$$G_n = \{x \in G : c^{-n-k_0} < \rho_G(x) < c^{-n+k_0}\}.$$

Let $\{\zeta_n,\ n \in \mathbb{Z}\}$ be a collection of non-negative functions with the following properties:

$$\zeta_n \in C_0^\infty(G_n),\ |D^m \zeta_n(x)| \le N(m)c^{mn},\ \sum_{n\in\mathbb{Z}} \zeta_n(x) = 1.$$

The function $\zeta_n(x)$ can be constructed by mollifying the characteristic (indicator) function of G_n. If G_n is an empty set, then the corresponding ζ_n is identical zero.

If $u \in \mathcal{D}'(G)$, that is, u is a distribution on $C_0^\infty(G)$, then $\zeta_n u$ is extended by zero to $\mathbb{R}^d$ so that $\zeta_n u \in \mathcal{D}'(\mathbb{R}^d)$. The space $H^\gamma_{p,\theta}(G)$ is defined as a collection of those $u \in \mathcal{D}'(G)$, for which $\zeta_n u$ is in H^γ_p and the norms $\|\zeta_n u\|_{H^\gamma_p}$, $n \in \mathbb{Z}$, behave in a certain way. Recall [10, Section 2.3.3] that the space of Bessel potentials H^γ_p is the closure of $C_0^\infty(\mathbb{R}^d)$ in the norm $\|\mathcal{F}^{-1}(1+|\xi|^2)^{\gamma/2}\mathcal{F}\cdot\|_{L_p(\mathbb{R}^d)}$, where $\mathcal{F}$ is the Fourier transform with inverse $\mathcal{F}^{-1}$.

Definition 2.1. Let G be a domain in $\mathbb{R}^d$, θ and γ, real numbers, and $p \in (1, +\infty)$. Take a collection $\{\zeta_k,\ n \in \mathbb{Z}\}$ as above. Then

$$H^\gamma_{p,\theta}(G) := \left\{ u \in \mathcal{D}'(G) : \|u\|^p_{H^\gamma_{p,\theta}(G)} := \sum_{n\in\mathbb{Z}} c^{n\theta} \|\zeta_{-n}(c^n\cdot)u(c^n\cdot)\|^p_{H^\gamma_p} < \infty \right\}. \tag{2.1}$$

Since $H^{\gamma_1}_p \subset H^{\gamma_2}_p$ for $\gamma_1 > \gamma_2$, the definition implies that $H^{\gamma_1}_{p,\theta}(G) \subset H^{\gamma_2}_{p,\theta}(G)$ for $\gamma_1 > \gamma_2$ and all $\theta \in \mathbb{R}$, $1 \le p < \infty$. Still, it is necessary to establish correctness of Definition 2.1 by showing that the norms defined according to

(2.1) are equivalent for every admissible choice of the numbers $c,\ k_0$ and the functions ζ_n. Proving this equivalence is the main goal of this section.

Proposition 2.2. *1. If u is compactly supported in G, then $u \in H^\gamma_{p,\theta}(G)$ if and only if $u \in H^\gamma_p$.*

2. The set $C^\infty_0(G)$ is dense in every $H^\gamma_{p,\theta}(G)$.

3. If $\gamma = m$ is a non-negative integer, then

$$H^\gamma_{p,\theta}(G) = \left\{ u : \rho^k_G D^k u \in L_{p,\theta}(G),\ 0 \le k \le m \right\}, \tag{2.2}$$

where $L_{p,\theta}(G) = L_p(G, \rho^{\theta-d}_G(x)dx)$.

4. If $\{\xi_n,\ n \in \mathbb{Z}\}$ is a system of function so that $\xi_n \in C^\infty_0(G_n),\ |D^m \xi_n(x)| \le N(m)c^{mn}$, then

$$\sum_{n\in\mathbb{Z}} c^{n\theta} \|\xi_{-n}(c^n \cdot) u(c^n \cdot)\|^p_{H^\gamma_p} \le N \|u\|^p_{H^\gamma_{p,\theta}}$$

with N independent of u, and if in addition $\sum_n \xi(x) \ge \delta > 0$ for all $x \in G$, then the reverse inequality also holds.

Proof. 1. The result is obvious because, for compactly supported u, the sum in (2.1) contains only finitely many non-zero terms.

2. Given $u \in H^\gamma_{p,\theta}(G)$, first approximate u by $u_K = u \cdot \sum_{|k|\le K} \zeta_k$, and then mollify u_K.

3. The result follows because, for all $\nu \in \mathbb{R}$ and all x in the support of ζ_{-n}, $N_1 \le c^{-\nu n} \rho^\nu_G(x) \le N_2$ with N_1 and N_2 independent of n, ν, x.

4. Use that, by Theorem 4.2.2 in [9], $C^\infty_0(\mathbb{R}^d)$ functions are pointwise multipliers in every H^γ_p.

Remark 2.3. In the future we will also use a system of non-negative $C^\infty_0(\mathbb{R}^d)$ functions $\{\eta_n,\ n \in \mathbb{Z}\}$ with the following properties: η_n is supported in $\{x : c^{-n-k_0-1} < \rho_G(x) < c^{-n+k_0+1}\}$, $\eta(x) = 1$ on the support of ζ_n, $|D^m \eta_n(x)| \le N(m)c^{mn}$. By Proposition 2.2(4) the functions η_n can replace ζ_n in (2.1).

Proposition 2.4. *1. For every $p \in (1,\infty)$ and $\theta,\gamma \in \mathbb{R}$, the space $H^\gamma_{p,\theta}(G)$ is a reflexive Banach space with the dual $H^{-\gamma}_{p',\theta'}(G)$, where $1/p + 1/p' = 1$ and $\theta/p + \theta'/p' = d$.*

2. If $0 < \nu < 1$, $\gamma = (1-\nu)\gamma_0 + \nu\gamma_1$, $1/p = (1-\nu)/p_0 + \nu/p_1$, and $\theta = (1-\nu)\theta_0 + \nu\theta_1$, then

$$H^\gamma_{p,\theta}(G) = [H^{\gamma_0}_{p_0,\theta_0}(G), H^{\gamma_1}_{p_1,\theta_1}(G)]_\nu, \tag{2.3}$$

where $[X,Y]_\nu$ is the complex interpolation space of X and Y (see [10, Section 1.9] for the definition and properties of the complex interpolation spaces).

Proof. Let $l^\theta_p(H^\gamma_p)$ be the set of sequences with elements from H^γ_p and the norm

$$\|\{f_n\}\|^p_{l^\theta_p(H^\gamma_p)} = \sum_{n\in\mathbb{Z}} c^{n\theta}\|f_n\|^p_{H^\gamma_p}.$$

Define bounded linear operators $S_{p,\theta} : H^\gamma_{p,\theta}(G) \to l^\theta_p(H^\gamma_p)$ and $R_{p,\theta} : l^\theta_p(H^\gamma_p) \to H^\gamma_{p,\theta}(G)$ as follows:

$$(S_{p,\theta}u)_n(x) = \zeta_{-n}(c^n x)u(c^n x), \quad R_{p,\theta}(\{f_n\})(x) = \sum_{n\in\mathbb{Z}} \eta_{-n}(x)f_n(c^{-n}x).$$

Note that $R_{p,\theta}S_{p,\theta} = \mathrm{Id}_{H^\gamma_{p,\theta}(G)}$. Then, by Theorem 1.2.4 in [10], the space $H^\gamma_{p,\theta}(G)$ is isomorphic to $S_{p,\theta}(H^\gamma_{p,\theta}(G))$, which is a closed subspace of a reflexive Banach space $l^\theta_p(H^\gamma_p)$. This means that $H^\gamma_{p,\theta}(G)$ is also a reflexive Banach space. The interpolation result (2.3) follows from Theorems 1.2.4 and 1.18.1 in [10].

Denote by $(\cdot,\cdot)$ the duality between H^γ_p and $H^{-\gamma}_{p'}$. If $v \in H^{-\gamma}_{p',\theta'}(G)$, then, by the Hölder inequality, v defines a bounded linear functional on $H^\gamma_{p,\theta}(G)$ as follows:

$$u \mapsto \langle v,u\rangle = \sum_n c^{nd}(v_n,u_n),$$

where $u_n(x) = \zeta_{-n}(c^n x)u(c^n x)$ and $v_n(x) = \eta_{-n}(c^n x)v(c^n x)$. Note that if $u,v \in C^\infty_0(G)$, then $\langle v,u\rangle = \int_G u(x)v(x)dx$.

Conversely, if V is a bounded linear functional on $H^\gamma_{p,\theta}(G)$, then we use the Hahn-Banach theorem and the equality $(l^\theta_p(H^\gamma_p))' = l^{-\theta p'/p}_{p'}(H^{-\gamma}_{p'})$ to construct $v \in H^{-\gamma}_{p',\theta'}(G)$ so that $V(u) = \langle v,u\rangle$.

One consequence of (2.3) is the interpolation inequality

$$\|u\|_{H^\gamma_{p,\theta}(G)} = \epsilon\|u\|_{H^{\gamma_0}_{p,\theta_0}(G)} + N(\nu,p,\epsilon)\|u\|_{H^{\gamma_1}_{p,\theta_1}(G)},\ \epsilon > 0. \tag{2.4}$$

Corollary 2.5. *The space $H^\gamma_{p,\theta}$ does not depend, up to equivalent norms, on the specific choice of the numbers c and k_0 and the functions ζ_n. Moreover, the distance function ρ_G can be replaced with any measurable function ρ satisfying $N_1\rho_G(x) \le \rho(x) \le N_2\rho_G(x)$ for all $x \in G$, with N_1, N_2 independent of x.*

Proof. By Proposition 2.2(3), we have the result for non-negative integer γ. For general $\gamma > 0$ the result then follows from (2.3), where we take $p_0 = p_1 = p$, $\theta_0 = \theta_1 = \theta$, and integer γ_0, γ_1. After that, the result for $\gamma < 0$ follows by duality.

In view of Corollary 2.5, it will be assumed from now on that $c = 2$ and $k_0 = 1$.
Remark 2.6. If X is a Banach space of generalized functions on $\mathbb{R}^d$, then we can define the space $X_\theta(G)$ according to (2.1) by replacing the norm $\|\cdot\|_{H^\gamma_p}$ with $\|\cdot\|_X$. In particular, we can define the spaces $B^\gamma_{p,q;\theta}(G)$ and $F^\gamma_{p,q;\theta}(G)$ using the spaces $B^\gamma_{p,q}$ and $F^\gamma_{p,q}$ described in Section 2.3.1 of [10]. Results similar to Propositions 2.2 and 2.4 can then be proved in the same way.

Example. (cf. [5, Definition 1.1].) Let $G = \mathbb{R}^d_+ = \{x = (x_1,\ldots,x_d) \in \mathbb{R}^d : x_1 > 0\}$ and $\zeta \in C^\infty_0((b_1,b_2))$, $0 < b_1,\ b_2 > 3b_1$. Define $\zeta(x) = \zeta(x_1)$ and

$$H^\gamma_{p,\theta} = \left\{ u \in \mathcal{D}'(G) : \|u\|^p_{H^\gamma_{p,\theta}} := \sum_{n\in\mathbb{Z}} e^{n\theta}\|\zeta u(e^n\cdot)\|^p_{H^\gamma_p} < \infty \right\}.$$

It follows that $H^\gamma_{p,\theta} = H^\gamma_{p,\theta}(\mathbb{R}^d_+)$ with $H^\gamma_{p,\theta}(\mathbb{R}^d_+)$ defined according to (2.1), where $c = e$, $\rho_G(x) = x_1$, $\zeta_n(x) = \zeta(e^n x)/\sum_k \zeta(e^k x)$, and k_0 is the smallest positive integer for which $b_1 > e^{-k_0}$, $b_2 < e^{k_0}$.

3. Pointwise multipliers, change of variables, and localization

A function $a = a(x)$ is a *pointwise multiplier* in a liner normed function space X if the operation of multiplication by a is defined and continuous in X.

To describe the pointwise multipliers in the space $H^{\gamma}_{p,\theta}(G)$, we need some preliminary constructions. For $\gamma \in \mathbb{R}$ define $\gamma' \in [0,1)$ as follows. If γ is an integer, then $\gamma' = 0$; if γ is not an integer, then γ' is any number from the interval $(0,1)$ so that $|\gamma| + \gamma'$ is not an integer. The space of pointwise multipliers in H^{γ}_p is given by

$$B^{|\gamma|+\gamma'} = \begin{cases} L_{\infty}(\mathbb{R}^d), & \gamma = 0 \\ C^{n-1,1}(\mathbb{R}^d), & |\gamma| = n = 1, 2, \dots \\ C^{|\gamma|+\gamma'}(\mathbb{R}^d), & \text{otherwise,} \end{cases}$$

where $C^{n-1,1}(\mathbb{R}^d)$ is the set of functions from $C^{n-1}(\mathbb{R}^d)$ whose derivatives of order $n-1$ are uniformly Lipschitz continuous. In other words, if $u \in H^{\gamma}_p$ and $a \in B^{|\gamma|+\gamma'}$, then

$$\|au\|_{H^{\gamma}_p} \le N(\gamma, d, p)\|a\|_{B^{|\gamma|+\gamma'}} \|u\|_{H^{\gamma}_p}.$$

For non-negative integer γ this follows by direct computation, for positive non-integer γ, from Corollary 4.2.2(ii) in [9], and for negative γ, by duality.

For $\nu \ge 0$, define the space $A^{\nu}(G)$ as follows:

(1) if $\nu = 0$, then $A^{\nu}(G) = L_{\infty}(G)$;

(2) if $\nu = m = 1, 2, \dots$, then

$$A^{\nu}(G) = \{a : a, \rho_G Da, \dots, \rho_G^{m-1} D^{m-1} a \in L_{\infty}(G), \rho_G^m D^{m-1} a \in C^{0,1}(G)\},$$

$$\|a\|_{A^{\nu}(G)} = \sum_{k=0}^{m-1} \|\rho_G^k D^k a\|_{L_{\infty}(G)} + \|\rho_G^m D^m a\|_{C^{0,1}(G)};$$

(3) if $\nu = m + \delta$, where $m = 0, 1, 2, \dots$, $\delta \in (0,1)$, then

$$A^{\nu}(G) = \{a : a, \rho_G Da, \dots, \rho_G^m D^m a \in L_{\infty}(G),\ \rho_G^{\nu} D^m a \in C^{\delta}(G)\},$$

$$\|a\|_{A^{\nu}(G)} = \sum_{k=0}^{m} \|\rho_G^k D^m a\|_{L_{\infty}(G)} + \|\rho_G^{\nu} D^m a\|_{C^{\delta}(G)}.$$

Note that, for every $a \in A^{\nu}(G)$ and $n \in \mathbb{Z}$,

$$\|\zeta_{-n}(2^n \cdot) a(2^n \cdot)\|_{B^{\nu}} \le N \|a\|_{A^{\nu}(G)} \tag{3.1}$$

with N independent of n.

Theorem 3.1. *If $a \in A^{|\gamma|+\gamma'}(G)$, then*

$$\|au\|_{H^{\gamma}_{p,\theta}(G)} \leq N(d,\gamma,p)\|a\|_{A^{|\gamma|+\gamma'}(G)} \cdot \|u\|_{H^{\gamma}_{p,\theta}(G)}.$$

Proof. We have to show that $\|\eta_{-n}(2^n\cdot)a(2^n\cdot)\|_{B^{|\gamma|+\gamma'}} \leq N\|a\|_{A^{|\gamma|+\gamma'}(G)}$ with constant N independent of n. The result is obvious for $\gamma = 0$; for $|\gamma| \in (0,1]$ it follows from the inequality (with $\delta = |\gamma| + \gamma'$)

$$|\eta_{-n}(x)a(x) - \eta_{-n}(y)a(y)| \leq \eta_{-n}(x)\rho_G^{-\delta}(x)|a(x)\rho_G^{\delta}(x) - a(y)\rho_G^{\delta}(y)|$$
$$+ |a(y)|\,|\eta_{-n}(x) - \eta_{-n}(y)| + \eta_{-n}(x)\rho_G^{-\delta}(x)|a(y)|\,|\rho_G^{\delta}(x) - \rho_G^{\delta}(y)|$$

and the observation that both $2^n\eta_{-n}$ and ρ_G are uniformly Lipschitz continuous. If $|\gamma| > 1$, we apply the same arguments to the corresponding derivatives.

Next, we study the following question: for what mappings $\psi : G_1 \to G_2$ is the operator $u(\cdot) \mapsto u(\psi(\cdot))$ continuous from $H^{\gamma}_{p,\theta}(G_2)$ to $H^{\gamma}_{p,\theta}(G_1)$?

Theorem 3.2. *Suppose that G_1 and G_2 are domains with non-empty boundaries and $\psi : G_1 \to G_2$ is a C^1-diffeomorphism so that $\psi(\partial G_1) = \partial G_2$. For a positive integer m define $\nu = \max(m-1, 0)$. If $D\psi \in A^{\nu}(G_1)$, then, for every $\gamma \in [-\nu, m]$ and $u \in H^{\gamma}_{p,\theta}(G_2)$,*

$$\|u(\psi(\cdot))\|_{H^{\gamma}_{p,\theta}(G_1)} \leq N\|u\|_{H^{\gamma}_{p,\theta}(G_2)}$$

with N independent of u.

Proof. Denote by ϕ the inverse of ψ. If $\gamma = 0$, then

$$\|u(\psi(\cdot))\|^p_{H^{\gamma}_{p,\theta}(G_1)} = \int_{G_2} |u(y)|^p \rho_{G_1}^{\theta-d}(\phi(y))|D\phi(y)|dy$$

and the result follows because uniform Lipschitz continuity of ρ_{G_i}, ψ, and ϕ implies that the ratio $\rho_{G_1}(\phi(x))/\rho_{G_2}(x)$ is uniformly bounded from above and below. If $\gamma = m$, the computation is similar. After that, for $\gamma \in (0,m)$, the result follows by interpolation, and for $\gamma \in [-\nu, 0)$, by duality.

The last result in this section is about localization. It answers the following question: for what collections of $C^\infty(G)$ functions $\{\xi_k, k = 1, 2, \dots\}$ are the values of $\|u\|^p_{H^\gamma_{p,\theta}(G)}$ and $\sum_n \|u\zeta_n\|^p_{H^\gamma_{p,\theta}(G)}$ comparable? To begin with, let us recall the corresponding theorem for H^γ_p.

Theorem 3.3. ([4, Lemma 6.7].) *If $\{\xi_k,\ k = 0, 1, \dots,\}$ is a collection of $C^\infty(\mathbb{R}^d)$ functions so that $\sup_x \sum_k |D^m \xi_k(x)| \le M(m), m \ge 0$, then $\sum_{k\ge 0} \|\xi_k v\|^p_{H^\gamma_p} \le N\|v\|^p_{H^\gamma_p}$ with N independent of v. If in addition $\inf_x \sum_k |\xi_k(x)|^p \ge \delta$ then the reverse inequality also holds: $\|v\|^p_{H^\gamma_p} \le N \sum_{k\ge 0} \|\xi_k v\|^p_{H^\gamma_p}$ with N independent of v.*

The following is the analogous result for $H^\gamma_{p,\theta}(G)$.

Theorem 3.4. *Suppose that $\{\chi_k,\ k \ge 1\}$ is a collection of $C^\infty(G)$ functions so that $\sup_{x\in G} \sum_k \rho^m_G(x)|D^m \chi_k(x)| \le N(m), m \ge 0$. Then $\sum_k \|u\chi_k\|^p_{H^\gamma_{p,\theta}(G)} \le N\|u\|^p_{H^\gamma_{p,\theta}(G)}$. If, in addition, $\inf_{x\in G} \sum_k |\chi_k(x)|^p \ge \delta$ for some $\delta > 0$, then $\|u\|^p_{H^\gamma_{p,\theta}(G)} \le N \sum_k \|u\chi_k\|^p_{H^\gamma_{p,\theta}(G)}$.*

Proof. With $\hat\chi_{0,n} = 1 - \eta_n$, $\hat\chi_{k,n}(x) = \chi_k(x)\eta_{-n}(x),\ k \ge 1$, we find

$$\sum_{k\ge 1} \|u\chi_k\|^p_{H^\gamma_{p,\theta}(G)} = \sum_{n\in\mathbb{Z}} \sum_{k\ge 0} 2^{n\theta} \|\hat\chi_{k,n}(2^n\cdot)\zeta_{-n}(2^n\cdot)u(2^n\cdot)\|^p_{H^\gamma_p}.$$

Both statements of the theorem now follow from Theorem 3.3.

Example. (cf. [7, Section 2].) Let G be a bounded domain of class $C^{|\gamma|+2}$ with a partition of unity $\chi_0 \in C^\infty_0(G), \chi_1, \dots \chi_K \in C^\infty_0(\mathbb{R}^d)$ and the corresponding diffeomorphism $\psi_1, \dots, \psi_K$ that stretch the boundary inside the support of $\chi_1, \dots, \chi_K$ (see, for example, Chapter 6 of [2] for details). Then an equivalent norm in $H^\gamma_{p,\theta}(G)$ is given by

$$\|u\|_{H^\gamma_{p,\theta}(G)} = \|u\chi_0\|_{H^\gamma_p} + \sum_{m=1}^{K} \|u(\psi_m^{-1}(\cdot))\chi_m(\psi_m^{-1}(\cdot))\|_{H^\gamma_{p,\theta}(\mathbb{R}^d_+)}.$$

Indeed, writing $\sim$ to denote the equivalent norms, we deduce from Proposition 2.2(1) and Theorems 3.2 and 3.4 that

$$\|u\|_{H^{\gamma}_{p,\theta}(G)} \sim \sum_{m=0}^{K} \|u\chi_m\|_{H^{\gamma}_{p,\theta}(G)} \sim \|u\chi_0\|_{H^{\gamma}_{p}} + \sum_{m=1}^{K} \|u(\psi_m^{-1}(\cdot))\chi_m(\psi_m^{-1}(\cdot))\|_{H^{\gamma}_{p,\theta}(\mathbb{R}^d_+)}.$$

4. Further properties of the spaces $H^{\gamma}_{p,\theta}(G)$

Let $\rho = \rho(x)$ be a $C^{\infty}(G)$ function so that $N_1\rho_G(x) \le \rho(x) \le N_2\rho_G(x)$ and $|\rho_G^m(x)D^{m+1}\rho(x)| \le N(m)$ for all $x \in G$ and for every $m = 0, 1, \ldots$. In particular, $\rho(x) = 0$ on ∂G and all the first-order partial derivatives of ρ are pointwise multipliers in every $H^{\gamma}_{p,\theta}(G)$. An example of the function ρ is

$$\rho(x) = \sum_{n\in\mathbb{Z}} 2^{-n}\zeta_n(x),$$

where the functions ζ_n are as in Section 2 with $c = 2$.

Theorem 4.1. *1. The following conditions are equivalent:*

- $u \in H^{\gamma}_{p,\theta}(G)$;
- $u \in H^{\gamma-1}_{p,\theta}(G)$ *and* $\rho Du \in H^{\gamma-1}_{p,\theta}(G)$;
- $u \in H^{\gamma-1}_{p,\theta}(G)$ *and* $D(\rho u) \in H^{\gamma-1}_{p,\theta}(G)$.

In addition, under either of these conditions, the norm $\|u\|_{H^{\gamma}_{p,\theta}(G)}$ *can be replaced by* $\|u\|_{H^{\gamma-1}_{p,\theta}(G)} + \|\rho Du\|_{H^{\gamma-1}_{p,\theta}(G)}$ *or by* $\|u\|_{H^{\gamma-1}_{p,\theta}(G)} + \|D(\rho u)\|_{H^{\gamma-1}_{p,\theta}(G)}$.

2. For every $\nu, \gamma \in \mathbb{R}$,

$$\rho^{\nu} H^{\gamma}_{p,\theta}(G) = H^{\gamma}_{p,\theta-p\nu}(G) \quad \text{and} \quad \|\cdot\|_{H^{\gamma}_{p,\theta-p\nu}(G)} \text{ is equivalent to } \|\rho^{-\nu}\cdot\|_{H^{\gamma}_{p,\theta}(G)}. \tag{4.1}$$

Proof. It is sufficient to repeat the arguments from the proofs of, respectively, Theorem 3.1 and Corollary 2.6 in [3].

Corollary 4.2. *1. If* $u \in H^{\gamma}_{p,\theta}(G)$, *then*

$$Du \in H^{\gamma-1}_{p,\theta+p}(G) \quad \text{and} \quad \|Du\|_{H^{\gamma-1}_{p,\theta+p}(G)} \le N(d,\gamma,p,\theta)\|u\|_{H^{\gamma}_{p,\theta}}(G).$$

2. If ρ_G is a bounded function (for example, if G is a bounded domain), then $H^\gamma_{p,\theta_1}(G) \subset H^\gamma_{p,\theta_2}(G)$ for $\theta_1 < \theta_2$ and $H^\gamma_p(G) \subset H^\gamma_{p,\theta}(G)$ for $\theta \ge d$.

Recall the following notations for continuous functions u in G:

$$\|u\|_{C(G)} = \sup_{x\in G} |u(x)|,\ [u]_{C^\nu(G)} = \sup_{x,y\in G} \frac{|u(x)-u(y)|}{|x-y|^\nu},\ \nu \in (0,1).$$

Theorem 4.3. *Assume that $\gamma - d/p = k + \nu$ for some $k = 0, 1, \dots$ and $\nu \in (0,1)$. If $u \in H^\gamma_{p,\theta}(G)$, then*

$$\sum_{k=0}^{m} \|\rho^{k+\theta/p} D^k u\|_{C(G)} + [\rho^{m+\nu+\theta/p} D^m u]_{C^\nu(G)} \le N(d,\gamma,p,\theta)\|u\|_{H^\gamma_{p,\theta}(G)}.$$

Proof. It is sufficient to repeat the arguments from the proof of Theorem 4.1 in [3].

Note that if $u \in H^\gamma_{p,\theta}(G)$ with $\gamma > 1 + d/p$ and $\theta < 0$, then, by Theorem 4.3, u is continuously differentiable in G and is equal to zero on the boundary of G. This is one reason why the spaces $H^\gamma_{p,\theta}(G)$ can be considered as a generalization of the usual Sobolev spaces with zero boundary conditions.

5. Degenerate elliptic equations in general domains

Throughout this section, $G \subset \mathbb{R}^d$ is a domain with a non-empty boundary but otherwise arbitrary, and ρ is the function introduced at the beginning of Section 4. Consider a second-order elliptic differential operator

$$\mathcal{L} = a^{ij}(x) D_i D_j + \frac{b^i(x)}{\rho(x)} D_i - \frac{c(x)}{\rho^2(x)},$$

where $D_i = \partial/\partial x_i$ and summation over the repeated indices is assumed. A related but somewhat different operator is studied in Section 6 of [10]. The objective of this section is to study solvability in $H^\gamma_{p,\theta}(G)$ of the equation $\mathcal{L}u = f$. It follows from Theorem 4.3 that, for appropriate θ and γ, the solution of the equation will also be a classical solution of the Dirichlet problem

$\mathcal{L}u = f,\ u|_{\partial G} = 0$. The values of $\gamma \in \mathbb{R}, 1 < p < \infty$, and $\theta \in \mathbb{R}$ will be fixed throughout the section.

The following assumptions are made.

Assumption 5.1. Uniform ellipticity: there exist $\kappa_1, \kappa_2 > 0$ so that, for all $x \in G$ and $\xi \in \mathbb{R}^d$, $\kappa_1|\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \kappa_2|\xi|^2$.

Assumption 5.2. Regularity of the coefficients:

$$\|a\|_{A^{\nu_1}(G)} + \|b\|_{A^{\nu_2}(G)} + \|c\|_{A^{|\gamma+1|+\gamma'}(G)} \leq \kappa_2,$$

where $\nu_1 = \max(2, |\gamma-1|+\gamma')$, $\nu_2 = \max(1, |\gamma|+\gamma')$. (See beginning of Section 3 for the definition of γ'.)

Note that under assumption 5.2 the operator $\mathcal{L}$ is bounded from $H^{\gamma+1}_{p,\theta-p}(G)$ to $H^{\gamma-1}_{p,\theta+p}(G)$. Therefore, we say that $u \in H^{\gamma+1}_{p,\theta-p}(G)$ is a solution of $\mathcal{L}u = f$ with $f \in H^{\gamma-1}_{p,\theta+p}(G)$ if the equality $\mathcal{L}u = f$ holds in $H^{\gamma-1}_{p,\theta+p}(G)$.

Theorem 5.1. *Under Assumptions 5.1 and 5.2, there exists a $c_0 > 0$ depending only on d, p, θ, the function ρ, and the coefficients a, b so that, for every $f \in H^{\gamma-1}_{p,\theta+p}(G)$ and every $c(x)$ satisfying $c(x) \geq c_0$, the equation $\mathcal{L}u = f$ has a unique solution $u \in H^{\gamma+1}_{p,\theta-p}(G)$ and $\|u\|_{H^{\gamma+1}_{p,\theta-p}(G)} \leq N\|f\|_{H^{\gamma-1}_{p,\theta+p}(G)}$ with the constant N depending only on d, γ, p, θ, the function ρ, and the coefficients a, b, c.*

To prove Theorem 5.1, we first establish the necessary a priori estimates, then prove the theorem for some special operator $\mathcal{L}$, and finally use the method of continuity to extend the result to more general operators.

Lemma 5.2. *If $u \in H^{\gamma+1}_{p,\theta-p}(G)$ and Assumptions 5.1 and 5.2 hold, then*

$$\|u\|_{H^{\gamma+1}_{p,\theta-p}(G)} \leq N\left(\|\mathcal{L}u\|_{H^{\gamma-1}_{p,\theta+p}(G)} + \|u\|_{H^{\gamma-1}_{p,\theta-p}(G)}\right)$$

with N independent of u.

Proof. Assume first that $b = c = 0$. Define $u_n(x) = \zeta_{-n}(2^n x)u(2^n x)$ and the operator

$$\mathcal{A}_n = (a^{ij}(2^n x)\eta_{-n}(2^n x) + (1 - \eta_{-n}(2^n x)\delta^{ij}))D_{ij},$$

where η is as in Remark 2.3. Clearly, $\|u_n\|_{H_p^{\gamma+1}} \leq N\left(\|\mathcal{A}u_n\|_{H_p^{\gamma-1}} + \|u\|_{H_p^{\gamma-1}}\right)$, and, by (3.1), N is independent of n. On the other hand,

$$\mathcal{A}_n u_n(x) = 2^{2n}\left(\zeta_{-n}\mathcal{L}u + 2a^{ij}D_i\zeta_{-n}D_ju + a^{ij}uD_{ij}\zeta_{-n}\right)(2^n x).$$

It remains to use the inequalities $\|Du\|_{H_p^{\gamma-1}} \leq N\|u\|_{H_p^{\gamma}} \leq \epsilon\|u\|_{H_p^{\gamma+1}} + N\epsilon^{-1}\|u\|_{H_p^{\gamma-1}}$ with sufficiently small ϵ, and then sum up the corresponding terms according to (2.1).

If b, c are not zero, then

$$\|a^{ij}D^{ij}u\|_{H^{\gamma-1}_{p,\theta+p}(G)} \leq \|\mathcal{L}u\|_{H^{\gamma-1}_{p,\theta+p}(G)} + N\|u\|_{H^{\gamma}_{p,\theta-p}(G)} + N\|u\|_{H^{\gamma-1}_{p,\theta-p}(G)},$$

and the result follows from the interpolation inequality (2.4).

Lemma 5.3. *If Assumptions 5.1 and 5.2 hold, then there exists a $c_0 > 0$ depending on d, p, θ, the function ρ, and the coefficients a, b, so that, for every $c(x)$ satisfying $c(x) \geq c_0$ and every $u \in L_{p,\theta}(G)$,*

$$\|u\|_{L_{p,\theta}(G)} \leq N\|\rho^2\mathcal{L}u\|_{L_{p,\theta}(G)}$$

with N independent of u.

Proof. It is enough to consider $u \in C_0^\infty(G)$. Writing $f = -\rho^2\mathcal{L}u$, we multiply both sides by $|u|^{p-2}u\rho^{\theta-d}$ and integrate by parts similar to the proof of Theorem 3.16 in [3]. The result is

$$\int_G f|u|^{p-2}u\rho^{\theta-d}dx = \int_G \Big(c(x) + h(x)\Big)|u|^p\rho^{\theta-d}dx,$$

where $|h(x)| \leq N_h$ and N_h depends on d, p, θ, and $\|a\|_{A^2(G)} + \|b\|_{A^1(G)} + \|D\rho\|_{A^1(G)}$. It remains to take $c_0 = 2N_h$ and use the Hölder inequality.

It follows from Lemmas 5.2 and 5.3 that if $c(x) \geq c_0$ and $\gamma \geq 1$, then

$$\|u\|_{H^{\gamma+1}_{p,\theta-p}(G)} \leq N\|\mathcal{L}u\|_{H^{\gamma-1}_{p,\theta+p}(G)}. \tag{5.1}$$

Lemma 5.4. *There exists a $\bar{c} > 0$ depending on p, θ, γ, and the function ρ so that the operator $\rho^2(x)\Delta - \bar{c}$ is a homeomorphism from $H^{\gamma+1}_{p,\theta}(G)$ to $H^{\gamma-1}_{p,\theta}(G)$.*

Proof. Keeping in mind that $\rho \in C^{0,1}(G)$ and $\rho(x) = 0$ on ∂G, let $\bar{\rho}$ be a $C^{0,1}(\mathbb{R}^d)$ extension of ρ so that $\bar{\rho} \in C^{\infty}(G - \partial G)$. Consider a family of diffusion processes $(X_t^x,\ x \in \mathbb{R}^d, t \geq 0)$ defined by

$$X_t^x = x + \sqrt{2}\int_0^t \bar{\rho}(X_s^x)dW_s,$$

where $(W_t, t \geq 0)$ is a standard d-dimensional Wiener process on some probability space $(\Omega, \mathcal{F}, P)$ (see, for example, Chapter V of [1] or Chapter I of [8]). Note that, by uniqueness, $X_t^x = x$ if $x \in \partial G$, and $X_t^x \in G$ for all $t > 0$ as long as $x \in G$. Theorems (3.3) and (3.9) from Chapter I of [8] imply that, with probability one, both DX_t^x and its inverse are in $C(G)$ for all $t \geq 0$. Further analysis shows that, for every $p > 1$ and every positive integer m,

$$E\|DX_t^x\|^p_{A^m(G)} + E\|D(X_t^x)^{-1}\|^p_{A^m(G)} \leq N_1 e^{N_2 t} \tag{5.2}$$

with constants N_1 and N_2 depending on p, m.

Assume that $f \in C_0^{\infty}(G)$ and define

$$u(x) = -E\int_0^{\infty} f(X_t^x)e^{-\bar{c}t}dt.$$

By Theorem 5.8.5 in [1], there exists a $c_1 > 0$ depending only on d and $\bar{\rho}$ so that, for $\bar{c} > c_1$, the function u is twice continuously differentiable in G and $\bar{\rho}^2(x)\Delta u(x) - \bar{c}u(x) = f(x)$ for all $x \in G$. On the other hand, after repeating the proof of Theorem 3.2 and using (5.2), we conclude that there exists a c_2 depending on $d, \gamma, \bar{\rho}$ so that, for $\bar{c} > c_2$ and for every $\gamma \in \mathbb{R}$, the function u belongs to $H^{\nu}_{p,\theta}(G)$ and

$$\|u\|_{H^{\gamma}_{p,\theta}(G)} \leq N\|f\|_{H^{\gamma}_{p,\theta}(G)}.$$

The statement of Lemma 5.4 now follows.

Proof of Theorem 5.1. Take $\bar{c}$ as in Lemma 5.4 and define the operators $\mathcal{L}_0 = \Delta - \bar{c}/\rho^2(x)$ and $\bar{\mathcal{L}}_0 = \rho^2(x)\Delta - \bar{c}$. Lemmas 5.4 and Theorem 4.1(2) imply that, for all $\gamma, \theta \in \mathbb{R}$ and $1 < p < \infty$, these operators are homeomorphisms from $H^{\gamma+1}_{p,\theta-p}(G)$ to, respectively, $H^{\gamma-1}_{p,\theta+p}(G)$ and $H^{\gamma-1}_{p,\theta-p}(G)$.

Assume first that $\gamma \geq 1$. Then a priory estimate (5.1) and the method of continuity (using the operators $\lambda\mathcal{L}+(1-\lambda)\mathcal{L}_0$, $0 \leq \lambda \leq 1$) imply the conclusion of the theorem.

If $\nu < 1$, then assume first that $0 \leq \nu < 1$. For $f \in H^{\nu-1}_{p,\theta+p}(G)$, define $u = \bar{\mathcal{L}}_0\mathcal{L}^{-1}(\bar{\mathcal{L}}_0^{-1}f) - \mathcal{L}^{-1}(\bar{f})$, where $\bar{f} = (\mathcal{L}\bar{\mathcal{L}}_0 - \bar{\mathcal{L}}_0\mathcal{L})\mathcal{L}^{-1}(\bar{\mathcal{L}}_0^{-1}f)$. Direct computations show that

- $\bar{f} \in H^{\nu}_{p,\theta+p}(G)$ and $\|\bar{f}\|_{H^{\nu}_{p,\theta+p}(G)} \leq N\|f\|_{H^{\nu-1}_{p,\theta+p}(G)}$;
- u is well defined, $u \in H^{\nu+1}_{p,\theta-p}(G)$, $\|u\|_{H^{\nu+1}_{p,\theta-p}(G)} \leq N\|f\|_{H^{\nu-1}_{p,\theta+p}(G)}$, and $\mathcal{L}u = f$.

This process can be repeated as many time as necessary. Theorem 5.1 is proved.

Remark 5.5. It follows from Theorem 4.3 that, if the conditions of Theorem 5.1 hold with $\gamma > d/p + 2$ and $\theta < p$, then the function u is the classical solution of

$$a^{ij}(x)D_{ij}u + \frac{b^i(x)}{\rho(x)}D_iu - \frac{c(x)}{\rho^2(x)}u = f, \ x \in G; \quad u|_{\partial G} = 0.$$

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References

[1] N. V. Krylov. *Introduction to the Theory of Diffusion Processes.* American Mathematical Society, Providence, RI, 1995.

[2] N. V. Krylov. *Lectures on Elliptic and Parabolic Equations in Hölder Spaces.* American Mathematical Society, Graduate Studies in Mathematics, v. 12, Providence, RI, 1996.

[3] N. V. Krylov. Weighted Sobolev Spaces and Laplace Equation and the Heat Equations in a Half Space. *Comm. Partial Differential Equations,* **24** (1999), no. 9–10, 1611–1653.

[4] N. V. Krylov. An analytic approach to SPDEs. In B. L. Rozovskii and R. Carmona, editors, *Stochastic Partial Differential Equations. Six Perspectives, Mathematical Surveys and Monographs*, pages 185–242. AMS, 1999.

[5] N. V. Krylov and S. V. Lototsky. A Sobolev Space Theory of SPDEs with Constant Coefficients in a Half Space. *SIAM J. Math. Anal.* **31** (2000), no. 1, 19–33

[6] J. L. Lions and E. Magenes. *Non-homogeneous Boundary Value Problems and Applications,I.* Springer-Verlag, Berlin, 1972.

[7] S. V. Lototsky. Dirichlet Problem for Stochastic Parabolic Equations in Smooth Domains. *Stochastics Stochastics Rep.* **68** (1999), no. 1–2, 145–175.

[8] D. W. Stroock. *Topics in Stochastic Differential Equations.* Tata Institute of Fundamental Research, Bombay, India, 1982.

[9] H. Triebel. *Theory of Function Spaces II.* Birkhauser, Basel, 1992.

[10] H. Triebel. *Interpolation Theory, Function Spaces, Differential Operators.* Johann Amrosius Barth, Heidelberg, 1995.