DIRICHLET PROBLEM FOR STOCHASTIC PARABOLIC EQUATIONS IN SMOOTH DOMAINS

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Abstract. A second-order stochastic parabolic equation with zero Dirichlet boundary conditions is considered in a sufficiently smooth bounded domain. Existence, uniqueness, and regularity of the solution are established without assuming any compatibility relations. To control the solution near the boundary of the region, special Sobolev-type spaces with weights are introduced. To illustrate the results, two examples are considered: general linear equation with finite-dimensional noise and equation on a line segment, driven by space-time white noise.

Published in *Stochastics and Stochastics Reports*, Vol. 68, pp. 145–175, 2000.

1. Introduction

The objective of the paper is to study solvability and regularity of the solution of the Dirichlet boundary value problem for a stochastic parabolic equation in a domain $G \subset \mathbb{R}^d$ with a sufficiently smooth boundary ∂G. Suppose that $u = u(t,x)$ a a solution of

$$\begin{aligned} &du = \big(a^{ij}(t,x)u_{x^ix^j} + f(t,x,u,u_x)\big)\,dt + \Big(\sigma^{ik}(t,x)u_{x^i} + g^k(t,x,u)\Big)\,dw^k(t),\ t>0,\ x\in G,\\ &u|_{t=0} = u_0, \qquad u|_{\partial G} = 0. \end{aligned} \tag{1.1}$$

Summation over the repeated indices will be assumed throughout the paper. The number of the Wiener processes w^k can be infinite to include the Hilbert space-valued noise. Note that the usual linear equation is obtained from (1.1) by choosing appropriate f and g ($f(t,x,u,u_x) = b^i(t,x)u_{x_i} + c(t,x)u + f(t,x)$, $g^k(t,x,u) = h^k(t,x)u + g^k(t,x)$).

Simple example of a one-dimensional equation shows that, unless certain compatibility conditions are fulfilled, the second-order derivatives of the solution blow up near the boundary (see [6] for details). The general analysis of the equations with compatibility conditions was done in [2]. It was demonstrated in [6] that compatibility conditions can be avoided and the derivatives of the solution near the boundary can be controlled by considering the solution as an element of a special weighted Sobolev space. The spaces introduced in [6] correspond to the Sobolev spaces of positive integer order with power $p = 2$. When domain G is the half-space, the weighted spaces of arbitrary real order with power $p \geq 1$ were introduced in [9]. Solvability of the Dirichlet problem for (1.1) in the half-space was studied in [10] under an additional

1991 *Mathematics Subject Classification.* 60H15, 35R60.

Key words and phrases. L_p-estimates, Sobolev spaces with weights.

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assumption that the coefficients a and σ do not depend on x. The anonymous referee kindly pointed out that, in [1], Z. Brzeźniak analyzed stochastic parabolic equations in M type 2 spaces and provided an alternative approach to working in Sobolev spaces with integrability exponent $p > 2$.

In this paper the results from [10] are used to solve equation (1.1) in a bounded domain. The procedure is similar to what was done in [7] to study boundary value problems for deterministic equations. Away from the boundary, equation (1.1) is equivalent to the equation in the whole space; solvability of such equations in the spaces H^{γ}_{p} of Bessel potentials was studied in [8], and a more detailed account of the results is given in [4]. Near the boundary, equation (1.1) is transformed to the equation in the half-space so that the results from [10] can be applied. The global solution is then constructed using the partition of unity in the domain G in the same way as it is done in [7]. The boundary of the domain G is not assumed to be infinitely smooth, although the optimal regularity of G is not discussed.

Similar to [8], regularity of the solution, both in space and in time, is obtained from Sobolev-type embedding theorems for the solution space. By shifting the analysis of regularity from the particular equation to the general function space, it becomes possible to develop a unified theory of solvability for stochastic boundary value problems and strengthen many existing results.

The new function spaces are defined in Section 2. The necessary properties of the spaces, including the embedding theorems, are also given in this section. The main result about solvability of equation (1.1) is in Section 3, and the examples are presented in Section 4. Two examples are discussed: a linear equation, generalizing the main result from [6], and the one-dimensional equation driven by space-time white noise, generalizing the well known result from [15] about Hölder continuity of the solution. The proof of the main result is in Section 5. Even though the actual argument is rather long, it is just a suitable modification and combination of the methods used in [6], [7], and [8].

The following notations are used in the paper. For integer $n \geq 0$, $C^{n}(G)$ is the space of n times continuously differentiable functions on G, $C^{0}(G) = C(G)$; for $\delta \in (0,1)$, $C^{n+\delta}(G)$ is the space of n times continuously differentiable functions whose derivatives of order n are Hölder continuous of order δ. The arbitrary constant is denoted by N or $N(\cdots)$; in the second case, the value of N can depend only on the variables in parentheses. The value of N can be different in different places. Under the summation convention, summation over all repeated indices except k is carried out from 1 to d; when index k is repeated, summation is assumed over all natural numbers. A point in $\mathbb{R}^d$ is $x = (x^1, \ldots, x^d)$, $\mathbb{R}^d_+ = \{x \in \mathbb{R}^d : x^1 > 0\}$ is the half-space, $u^{(m)}_x$ denotes the generic m th order partial derivative of u with respect to x. Other notations are introduced as necessary.

2. The function spaces

Let $G \subset \mathbb{R}^d$ be a domain (open connected subset) with boundary ∂G and closure $\bar{G}$. By $B_r(x)$ we denote the open ball with radius r and center x.

Definition 2.1. (cf. [7, Definition 6.1.6]). An open connected subset G of $\mathbb{R}^d$ is called a domain of class C^ν, $\nu \geq 2$, if there exist positive numbers r_0 and M so that for every point x_0 on ∂G there is a one-to-one mapping ψ of $B_{r_0}(x_0)$ onto a domain D in $\mathbb{R}^d$ so that the following conditions are fulfilled:

(1) $\psi(B_{r_0}(x_0) \cap G) \subset \mathbb{R}^d_+$ and $\psi(x_0) = 0$;
(2) $\psi(B_{r_0}(x_0) \cap \partial G) = D \cap \{y \in \mathbb{R}^d : y^1 = 0\}$;
(3) $\|\psi\|_{C^\nu(B_{r_0}(x_0);\mathbb{R}^d)} + \|\psi^{-1}\|_{C^\nu(D;\mathbb{R}^d)} \leq M$.

Let $\rho_G(x)$, $x \in \bar{G}$, be the distance from x to the boundary of G:

$$\rho_G(x) = \mathrm{dist}\,(x, \partial G),$$

and let $\rho = \rho(x)$ be a $C^\nu(G)$ function with the following property: there exist positive numbers δ_1 and δ_2 so that

$$\delta_1\, \rho_G(x) \leq \rho(x) \leq \delta_2\, \rho_G(x)$$

for all x near the boundary of G. Such a function ρ exists if G is a bounded domain of class C^ν because in that case the distance function ρ_G is of class C^ν near the boundary of G (see [3, page 382]). This function ρ will be fixed from now on.

Next, for a bounded domain G of class C^ν, $\nu \geq 2$, we construct a **partition of unity** in G, corresponding to r_0 (cf. [7, page 81]), that is, define a collection of non-negative functions χ_m, $m = 0, \ldots, K$, with the following properties:

(1) each χ_m belongs to $C_0^\infty(\mathbb{R}^d)$;
(2) the function χ_0 is supported in the set $\{x \in G : \rho_G(x) \geq r_0/8\}$;
(3) For $m = 1, \ldots, K$, the function χ_m is supported in $B_{r_0/2}(x_m)$, where $x_m \in \partial G$;
(4) $\sum_{m=0}^K (\chi_m(x))^2 = 1$ for all $x \in G$.

For $m = 1, \ldots, K$ denote by ψ_m the corresponding diffeomorphism $B_{r_0}(x_m) \to \mathbb{R}^d$ from Definition 2.1. The operator $u \mapsto u \circ \psi_m$ will be denoted by Ψ_m.

Recall that the space H_p^γ is defined for $\gamma \in \mathbb{R}$ and $p \geq 1$ as the completion of the space $C_0^\infty(\mathbb{R}^d)$ with respect to the norm $\|\cdot\|_{H_p^\gamma} = \|\Lambda^\gamma \cdot\|_{L_p(\mathbb{R}^d)}$, where $\Lambda^\gamma f = ((1+|\xi|^2)^{\gamma/2} \hat{f}\,)\check{}$, and $\hat{}\,$, $\check{}$ are the Fourier transform and its inverse. Also, $H_p^\gamma(l_2)$ is the set of sequences $g = \{g_k,\ k \geq 1\}$ for which

$$\|g\|_{H_p^\gamma(l_2)} := \|\, \|\Lambda^\gamma g\|_{l_2}\, \|_{L_p(\mathbb{R}^d)} < \infty,$$

where $\|g\|_{l_2} = \left(\sum_{k\geq 1} |g_k|^2\right)^{1/2}$. The space $H^\gamma_{p,\theta} = H^\gamma_{p,\theta}(\mathbb{R}^d_+)$ is then defined for $p \geq 1$ and $\theta, \gamma \in \mathbb{R}$ as follows [9, Definition 2.1]: given a non-negative function $\zeta \in C_0^\infty(\mathbb{R}_+)$ satisfying $\sum_{n=-\infty}^{+\infty} |\zeta(e^{z-n})|^p \geq 1$ for all $z \in \mathbb{R}$,

$$H^\gamma_{p,\theta} = \{u \in \mathcal{D}'(\mathbb{R}^d_+) : \|u\|^p_{H^\gamma_{p,\theta}} := \sum_{n=-\infty}^{+\infty} e^{n\theta} \|u(e^n \cdot)\zeta\|^p_{H^\gamma_p} < \infty\},$$

where $\zeta(x) = \zeta(x^1)$ and $\mathcal{D}'(\mathbb{R}^d_+)$ is the set of distributions on $C_0^\infty(\mathbb{R}^d_+)$. It is shown in [9], Lemma 2.4, that the definition does not depend on the specific function ζ. The space $H^\gamma_{p,\theta}(l_2) = H^\gamma_{p,\theta}(\mathbb{R}^d_+; l_2)$ is defined similarly by replacing the norm $\|\cdot\|_{H^\gamma_p}$ with $\|\cdot\|_{H^\gamma_p(l_2)}$.

We now define the corresponding spaces of functions in a bounded domain G.

For a real number γ and a bounded domain G of the class $C^{|\gamma|+2}$, the space $H^\gamma_{p,\theta}(G)$ is defined as follows:

$$H^\gamma_{p,\theta}(G) = \left\{ u \in \mathcal{D}'(G) : \|u\|_{H^\gamma_{p,\theta}(G)} := \|u\chi_0\|_{H^\gamma_p} + \sum_{m=1}^{K} \|\eta_m \Psi_m^{-1}(u\chi_m)\|_{H^\gamma_{p,\theta}} < \infty \right\}, \tag{2.1}$$

where η_m is a $C_0^\infty(\mathbb{R}^d)$ function that is equal to 1 on $\psi_m(B_{r_0}(x_m))$.

The space $H^\gamma_{p,\theta}(G; l_2)$ is defined similarly by replacing the norms $\|\cdot\|_{H^\gamma_p}$ and $\|\cdot\|_{H^\gamma_{p,\theta}}$ with, respectively, $\|\cdot\|_{H^\gamma_p(l_2)}$ and $\|\cdot\|_{H^\gamma_{p,\theta}(l_2)}$. Direct computations show that these definitions are independent of the specific choice of functions ψ_m, χ_m, and η_m, that is, the corresponding norms defined by (2.1) are equivalent.

Note that $C_0^\infty(G) \subset H^\gamma_{p,\theta}(G)$. Indeed, if $u \in C_0^\infty(G)$, then the function $\eta_m \Psi_m^{-1}(u\chi_m)$ is compactly supported in $\mathbb{R}^d_+$ and, by Propositions 4.2.1 and 4.3.1 in [13], belongs to H^γ_p. By Remark 2.11 in [9] we conclude that $\eta_m \Psi_m^{-1}(u\chi_m) \in H^\gamma_{p,\theta}$. The argument also shows that, for given $\nu > 0$ and domain G of class $C^{\nu+2}$, the above definition of $H^\gamma_{p,\theta}(G)$ is correct only for $\gamma \in [-\nu, \nu]$.

Proposition 2.2. (Properties of the spaces $H^\gamma_{p,\theta}(G)$.) *Assume that G is a bounded domain of class $C^{\nu+2}$ and $\nu > 0$.*

1. For every $\gamma \in [-\nu, \nu]$, the space $H^\gamma_{p,\theta}(G)$ is a Banach space and, for $-\nu \le \alpha < \beta \le \nu$, $H^\beta_{p,\theta}(G) \subset H^\alpha_{p,\theta}(G)$.

2. If $\gamma = n \le \nu$ is a non-negative integer, then

$$H^\gamma_{p,\theta}(G) = \{u : u, \rho u_x, \dots, \rho^n u_x^{(n)} \in L_{p,\theta}(G)\},$$

where $L_{p,\theta}(G) = L_p(G; (\rho(x))^{\theta-d} dx)$.

3. For every α, β, γ satisfying $-\nu \le \alpha < \beta < \gamma \le \nu$ and for every $\varepsilon > 0$,

$$\|u\|_{H^\beta_{p,\theta}(G)} \le \varepsilon N(\alpha, \beta, \gamma) \|u\|_{H^\gamma_{p,\theta}(G)} + N(\alpha, \beta, \gamma, \varepsilon) \|u\|_{H^\alpha_{p,\theta}(G)}. \tag{2.2}$$

4. For every $\alpha, \gamma \in \mathbb{R}$ with $|\gamma| \le \nu$,

$$\rho^\alpha H^\gamma_{p,\theta}(G) = H^\gamma_{p,\theta-p\alpha}(G) \ \text{ and } \ \|\cdot\|_{H^\gamma_{p,\theta-p\alpha}(G)} \text{ is equivalent to } \|\rho^{-\alpha}\cdot\|_{H^\gamma_{p,\theta}(G)}. \tag{2.3}$$

5. Assume that $0<\gamma\leq\nu$ and $\gamma-d/p=k+\alpha$ for some $k=0,1,\dots$ and $\alpha\in(0,1)$. If $u\in H^{\gamma}_{p,\theta}(G)$, then

$$\rho^{m+\theta/p}u_x^{(m)}\in C(G),\ 0\leq m\leq k;\quad \|\rho^{m+\theta/p}u_x^{(m)}\|_{C(G)}\leq N(d,\gamma,p,\theta)\|u\|_{H^{\gamma}_{p,\theta}(G)};$$
$$\rho^{\gamma+\theta/p-d/p}u^{(k)}\in C^{\alpha}(G),\ \|\rho^{\gamma+\theta/p-d/p}u^{(k)}\|_{C^{\alpha}(G)}\leq N(d,\gamma,p,\theta)\|u\|_{H^{\gamma}_{p,\theta}(G)}.$$

Proof. By assumption, $\partial G\cap B_{r_0}(x_m)$ is the zero-level set of the function ψ_m^1 and the gradient of ψ^1_m does not vanish. Therefore, for $x\in B_{r_0}(x_m)\cap G$, the function ρ can be replaced with ψ^1_m. After that, Property 1 is obvious; Property 2 follows from Corollary 3.3 in [9]; Property 3, from Theorem 2.10 in [9]; Property 4, from Corollary 2.6 in [9]; Property 5, from Theorem 4.1 in [9].

Next, we describe the multipliers in the space $H^{\gamma}_{p,\theta}(G)$. For $\gamma\in\mathbb{R}$ define $\gamma'\in[0,1)$ as follows. If γ is an integer, then $\gamma'=0$; if γ is not an integer, then γ' is any number from the interval $(0,1)$ so that $|\gamma|+\gamma'$ is not an integer. It is known (see [4]) that the space of multipliers for H^{γ}_p is given by

$$B^{|\gamma|+\gamma'}=\begin{cases} L_{\infty}(\mathbb{R}^d), & \gamma=0\\ C^{n-1,1}(\mathbb{R}^d), & |\gamma|=n=1,2,\dots\\ C^{|\gamma|+\gamma'}(\mathbb{R}^d), & \text{otherwise,}\end{cases}$$

where $C^{n-1,1}(\mathbb{R}^d)$ is the set of functions from $C^{n-1}(\mathbb{R}^d)$ whose derivatives of order $n-1$ are uniformly Lipschitz continuous. In other words, if $u\in H^{\gamma}_p$ and $a\in B^{|\gamma|+\gamma'}$, then

$$\|au\|_{H^{\gamma}_p}\leq N(\gamma,d,p)\|a\|_{B^{|\gamma|+\gamma'}}\,\|u\|_{H^{\gamma}_p}.$$

For non-negative integer γ this follows by direct computation, for positive non-integer γ, from Corollary 4.2.2(ii) in [13], and for negative γ, by duality.

Similarly, if

$$B^{|\gamma|+\gamma'}(l_2)=\begin{cases} L_{\infty}(\mathbb{R}^d;l_2), & \gamma=0,\\ C^{n-1,1}(\mathbb{R}^d,l_2), & |\gamma|=n=1,2,\dots,\\ C^{|\gamma|+\gamma'}(\mathbb{R}^d;l_2), & \text{otherwise,}\end{cases}$$

then, for every $\sigma\in B^{|\gamma|+\gamma'}(l_2)$ and $u\in H^{\gamma}_p$,

$$\|\sigma u\|_{H^{\gamma}_p(l_2)}\leq N(\gamma,d,p)\|\sigma\|_{B^{|\gamma|+\gamma'}(l_2)}\,\|u\|_{H^{\gamma}_p}.$$

Let $J=(j_1,\dots,j_d)$ be a multi-index, $|J|=j_1+\cdots+j_d$, $D_i=\partial/\partial x^i$, and $D^Ju(x)=D_1^{j_1}\cdots D_d^{j_d}$. Assume that G is a domain of class C^2. For $\nu\geq0$, define the space $A^{\nu}(G)$ as follows:

(1) if $\nu=0$, then $A^{\nu}(G)=L_{\infty}(G)$;
(2) if $\nu=n=1,2,\dots$, then

$$A^{\nu}(G)=\{a:a,\rho a_x,\dots,\rho^{n-1}a_x^{(n-1)}\in L_{\infty}(G),\rho^n a_x^{(n-1)}\in C^{0,1}(G)\},$$

$$\|a\|_{A^{\nu}(G)}=\sum_{k=0}^{n-1}\max_{|J|=k}\|\rho^k D^J a\|_{L_{\infty}(G)}+\max_{|J|=n-1}\|\rho^n D^J a\|_{C^{0,1}(G)};$$

(3) if $\nu = n + \delta$, where $n = 0, 1, 2, \ldots$, $\delta \in (0, 1)$, then

$$A^{\nu}(G) = \{a : a, \rho a_x, \ldots, \rho^n a_x^{(n)} \in L_{\infty}(G),\ \rho^{\nu} a_x^{(n)} \in C^{\delta}(G)\},$$

$$\|a\|_{A^{\nu}(G)} = \sum_{k=0}^{n} \max_{|J|=k} \|\rho^k D^J a\|_{L_{\infty}(G)} + \max_{|J|=n} \|\rho^{\nu} D^J a\|_{C^{\delta}(G)}.$$

The space $A^{\nu}(G; l_2)$ is defined similarly by considering l_2-valued functions. Note that if G is a *bounded* domain of class C^{ν} and $\nu \geq 2$, then, for all $\delta > 0$, $\rho^{\delta} \in A^{\nu}(G)$.

The corresponding spaces A^{ν} and $A^{\nu}(l_2)$ of functions on $\mathbb{R}^d_+$ are defined in the same way, with x^1 used instead of $\rho(x)$.

Lemma 2.3. *Suppose that G is a bounded domain of class C^{ν}. A functions $a = a(x)$ belongs to $A^{\nu}(G)$ if and only if $a\chi_0 \in B^{\nu}$ and $\eta_m \Psi_m^{-1}(a\chi_m) \in A^{\nu}$, $m = 1, \ldots, K$ (cf. (2.1)). The norm $\|a\|_{A^{\nu}(G)}$ is equivalent to*

$$\|a\chi_0\|_{B^{\nu}} + \sum_{m=1}^{K} \|\eta_m \Psi_m^{-1}(a\chi_m)\|_{A^{\nu}}.$$

Proof. Since, for $x \in B_{r_0}(x_m) \cap G$, the function ρ can be replaced with ψ_m^1, the result follows from Lemma 6.1.8 in [7].

Theorem 2.4. (Multipliers in $H^{\gamma}_{p,\theta}(G)$.) *Assume that G is a bounded domain of class $C^{\nu+2}$, $\gamma \in \mathbb{R}$, and $|\gamma| + \gamma' \leq \nu$. Then the space $A^{|\gamma|+\gamma'}(G)$ is the space of multipliers for $H^{\gamma}_{p,\theta}(G)$: there is a constant N depending only on d, γ, p, and the domain G so that*

$$\|au\|_{H^{\gamma}_{p,\theta}(G)} \leq N \|a\|_{A^{|\gamma|+\gamma'}(G)} \|u\|_{H^{\gamma}_{p,\theta}(G)}$$

for all $a \in A^{|\gamma|+\gamma'}(G)$ and $u \in H^{\gamma}_{p,\theta}(G)$. Similarly, if $\sigma \in A^{|\gamma|+\gamma'}(G; l_2)$, then

$$\|\sigma u\|_{H^{\gamma}_{p,\theta}(G;l_2)} \leq N \|\sigma\|_{A^{|\gamma|+\gamma'}(G;l_2)} \|u\|_{H^{\gamma}_{p,\theta}(G)}.$$

The same results hold for the spaces of functions on $\mathbb{R}^d_+$.

Proof. By Lemma 2.3 it is sufficient to consider functions on $\mathbb{R}^d_+$. With no loss of generality, we can replace the function ζ in the definition of the norm $\|\cdot\|_{H^{\gamma}_{p,\theta}}$ with ζ^2. Then

$$\|au\|^p_{H^{\gamma}_{p,\theta}} = \sum_n e^{n\theta} \|\zeta^2 a(e^n \cdot) u(e^n \cdot)\|^p_{H^{\gamma}_p} \leq N \sum_n e^{n\theta} \|\zeta a(e^n \cdot)\|^p_{B^{|\gamma|+\gamma'}} \|\zeta u(e^n \cdot)\|^p_{H^{\gamma}_p},$$

and it remains to show that if $a \in A^{|\gamma|+\gamma'}$, then $\|\zeta a(e^n \cdot)\|^p_{B^{|\gamma|+\gamma'}}$ is bounded by $\|a\|_{A^{|\gamma|+\gamma'}}$ uniformly in n.

1. If $\gamma = 0$, then the result is obvious.

2. if $0 < |\gamma| \leq 1$, then, with $\delta = |\gamma| + \gamma'$,

$$\frac{|\zeta(x)a(e^n x) - \zeta(y)a(e^n y)|}{|x-y|^{\delta}} \leq \frac{|\zeta(x)(x^1)^{-\delta}| \cdot |(x^1 e^n)^{\delta} a(e^n x) - (y^1 e^n)^{\delta} a(e^n y)|}{|e^n x - e^n y|^{\delta}}$$
$$+ \frac{|(y^1)^{\delta} a(e^n y)| \cdot |(x^1)^{-\delta}\zeta(x) - (y^1)^{-\delta}\zeta(y)|}{|x-y|^{\delta}},$$

where both terms on the right are bounded by $N\|a\|_{A^{|\gamma|+\gamma'}}$ because of the assumption on a and the properties of ζ (to estimate the second term on the right note that $|a(e^n y)|$ is bounded by assumption, and $|(x^1)^{-\delta}\zeta(x) - (y^1)^{-\delta}\zeta(y)| = 0$ if both x^1 and y^1 are large).

3. If $|\gamma| > 1$, then use the above argument and observe that

$$(\zeta(x)a(e^n x))^{(k)}_x \le N \sum_{l=0}^{k} |(x^1)^{-l}\zeta_X^{(k-l)}(x)| \cdot |e^n x^1|^l \cdot |a^{(l)}_x(e^n x)| \le N \sum_{l=0}^{k} \sup_{x\in\mathbb{R}^d_+} |(x^1)^l a^{(l)}_x(x)|.$$

The proof for $A^{|\gamma|+\gamma'}(l_2)$ is similar.

Theorem 2.4 is proved.

According to the last theorem, the function $\rho^\delta,\ \delta > 0$, is a multiplier in every $H^\gamma_{p,\theta}(G)$. Together with Proposition 2.2(4), this implies the following result.

Corollary 2.5. *If $\theta_1 < \theta_2$ and G is a bounded domain, then $H^\gamma_{p,\theta_1}(G) \subset H^\gamma_{p,\theta_1}(G)$.*

We now define the spaces of stochastic processes. Fix $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$, a stochastic basis with $\mathcal{F}$ and $\mathcal{F}_0$ containing all P-null subsets of Ω; τ, a stopping time, $(\!(0,\tau]\!] = \{(\omega,t) \in \Omega\times\mathbb{R}_+ : 0 < t \le \tau(\omega)\}$; $\mathcal{P}$, the σ-algebra of predictable sets; $\{w^k,\ k \ge 1\}$, independent standard Wiener processes.

The following spaces were introduced in [8] to study parabolic equations on $\mathbb{R}^d$:

(1) $\mathbb{H}^\gamma_p(\tau) = L_p((\!(0,\tau]\!]; \mathcal{P}; H^\gamma_p)$, $\mathbb{H}^\gamma_p(\tau; l_2) = L_p((\!(0,\tau]\!]; \mathcal{P}; H^\gamma_p(l_2))$, $\mathbb{L}_p(cdots) = \mathbb{H}^0_p(\cdots)$;
(2) $\mathcal{F}^\gamma_p(\tau) = \mathbb{H}^{\gamma-1}_p(\tau) \times \mathbb{H}^\gamma_p(\tau; l_2)$;
(3) $\mathcal{H}^\gamma_p(\tau)$: the collection of processes from $\mathbb{H}^{\gamma+1}_p(\tau)$ that can be written, in the sense of distributions, as

$$u(t) = u_0 + \int_0^t f(s)ds + \int_0^t g^k(s)dw^k(s)$$

for some $u_0 \in L_p(\Omega; \mathcal{F}_0; H^{\gamma+1-2/p}_p)$ and $(f,g) \in \mathcal{F}^\gamma_p(\tau)$;

$$\|u\|^p_{\mathcal{H}^\gamma_p(\tau)} = \|u_{xx}\|^p_{\mathbb{H}^{\gamma-1}_p(\tau)} + \|(f,g)\|^p_{\mathcal{F}^\gamma_p(\tau)} + E\|u_0\|^p_{H^{\gamma+1-2/p}_p}.$$

The above definitions suggest that, to study parabolic equations in a bounded domain G, the domain must be sufficiently regular to allows, for fixed $\gamma \in \mathbb{R}$, the definition of the spaces $H^\nu_{p,\theta}(G)$ when $\nu \in [\gamma-1, \gamma+1]$. We therefore assume from now on that $\gamma \in \mathbb{R}$ is fixed and G is a bounded domain of class $C^{|\gamma|+3}$.

We now define the corresponding spaces on G:

(1) $\mathbb{H}^\gamma_{p,\theta}(\tau, G) = L_p((\!(0,\tau]\!]; \mathcal{P}; H^\gamma_{p,\theta}(G))$, $\mathbb{H}^\gamma_p(\tau, G; l_2) = L_p((\!(0,\tau]\!]; \mathcal{P}; H^\gamma_{p,\theta}(G; l_2))$;
(2) $\mathcal{F}^\gamma_{p,\theta}(\tau, G) = \mathbb{H}^{\gamma-1}_{p,\theta+p}(\tau, G) \times \mathbb{H}^\gamma_{p,\theta}(\tau, G; l_2)$, $U^\gamma_{p,\theta}(G) = L_p(\Omega; \mathcal{F}_0; H^{\gamma+1-2/p}_{p,\theta+2-p}(G))$;

(3) $\mathfrak{H}^{\gamma}_{p,\theta}(\tau,G)$: the collection of process from $\mathbb{H}^{\gamma+1}_{p,\theta-p}(\tau,G)$ that can be written, in the sense of distributions, as

$$u(t) = u_0 + \int_0^t f(s)ds + \int_0^t g^k(s)dw^k(s) \tag{2.4}$$

for some $u_0 \in U^{\gamma}_{p,\theta}(G)$, $(f,g) \in \mathcal{F}^{\gamma}_p(\tau,G)$, which means that, for every $\phi \in C_0^{\infty}(G)$, the equality

$$(u(t\wedge\tau),\phi) = (u_0,\phi) + \int_0^{t\wedge\tau} (f(s),\phi)ds + \int_0^{t\wedge\tau} (g^k(s),\phi)dw^k(s)$$

holds for all $t \geq 0$ and all ω from a set of probability 1. The norm in the space $\mathfrak{H}^{\gamma}_{p,\theta}(\tau,G)$ is defined by

$$\|u\|^p_{\mathfrak{H}^{\gamma}_{p,\theta}(\tau,G)} = \|u\|^p_{\mathbb{H}^{\gamma+1}_{p,\theta-p}(\tau,G)} + \|(f,g)\|^p_{\mathcal{F}^{\gamma}_p(\tau,G)} + \|u\|^p_{U^{\gamma}_{p,\theta}(G)}. \tag{2.5}$$

We also write $u = [f,g,u_0]$ if $u \in \mathfrak{H}^{\gamma}_{p,\theta}(\tau,G)$ and u satisfies (2.4). The corresponding spaces of processes on $\mathbb{R}^d_+$ were introduced in [10]. As before, in the case of $\mathbb{R}^d_+$ the domain G will be omitted from the argument of the spaces.

Proposition 2.6. *Assume that G is a bounded domain of class $C^{|\gamma|+3}$.*

1. Spaces $\mathbb{H}^{\gamma}_{p,\theta}(\tau,G)$, $\mathbb{H}^{\gamma}_{p,\theta}(\tau,G;l_2)$, and $\mathfrak{H}^{\gamma}_{p,\theta}(\tau,G)$ are Banach spaces.

2. The operator $\rho D_i : \mathbb{H}^{\nu}_{p,\theta}(G) \to \mathbb{H}^{\nu-1}_{p,\theta}(G)$ is a bounded linear mapping for $\gamma \leq \nu \leq \gamma+1$ and $i = 1,\dots,d$.

3. The operator $\rho : \mathbb{H}^{\nu}_{p,\theta}(G) \to \mathbb{H}^{\nu}_{p,\theta}(G)$ is a bounded linear mapping for $\gamma-1 \leq \nu \leq \gamma+1$.

4. The operator $D_i : \mathfrak{H}^{\nu}_{p,\theta}(G) \to \mathbb{H}^{\nu}_{p,\theta}(G)$ is a bounded linear mapping for $\gamma \leq \nu \leq \gamma+1$, and $i = 1,\dots,d$.

Proof. Property 1 is obvious; Property 2 follows from Theorem 2.4 and from Theorem 3.1 in [9]; Property 3, from Theorem 2.4; Property 4, from Theorem 3.6 in [9].

By definition, the space $\mathfrak{H}^{\gamma}_{p,\theta}(\tau,G)$ contains the processes that can be solutions of certain second-order parabolic equations. Indeed, it follows from Propositions 2.2(4) and 2.6(2) that D_i and D_iD_j are bounded operators from $\mathbb{H}^{\gamma+1}_{p,\theta-p}(\tau,G)$ to, respectively, $\mathbb{H}^{\gamma}_{p,\theta}(\tau,G)$ and $\mathbb{H}^{\gamma-1}_{p,\theta+p}(\tau,G)$. Assume that a^{ij} and σ^{ik} are $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^d_+)$ measurable functions and

$$\|a^{ij}(t,\cdot)\|_{A^{|\gamma-1|+\gamma'}(G)} + \|\sigma^{i\cdot}(t,\cdot)\|_{A^{|\gamma|+\gamma'}(G;l_2)} \leq N_0$$

for all $(\omega,t) \in (\!(0,\tau]\!]$ and all $i,j = 1,\dots,d$. Then there is a linear bounded map

$$u = [f_0,g_0,u_0] \in \mathfrak{H}^{\gamma}_{p,\theta}(\tau,G) \mapsto (f,g,u_0) \in \mathcal{F}^{\gamma}_{p,\theta} \times U^{\gamma}_{p,\theta}$$

defined by

$$f = f_0 - a^{ij}u_{x^ix^j},\ g^k = g^k_0 - \sigma^{ik}u_{x^i}.$$

It will be shown later, using embedding theorems, that if γ and p are sufficiently large, then u is a function that is twice continuously differentiable in G and is equal to zero on ∂G. Therefore, if f and g are defined as above and p and γ are sufficiently large, then u is a classical solution of the equation

$$\begin{aligned} du = \big(a^{ij}(t,x)u_{x^ix^j} + f(t,x)\big)\,dt + \big(\sigma^{ik}(t,x)u_{x^i} + g^k(t,x)\big)\,dw^k(t),\ t>0,\ x\in G \\ u|_{t=0} = u_0,\quad u|_{x\in\partial G} = 0. \end{aligned} \tag{2.6}$$

The result about the unique solvability of (2.6) will then state that the equation defines a linear homeomorphism between the spaces $\mathfrak{H}^\gamma_{p,\theta}(\tau,G)$ and $\mathcal{F}^\gamma_{p,\theta}\times U^\gamma_{p,\theta}$. To prove such a result, it will be necessary to impose additional conditions on the functions a and σ.

Next, we will study embedding of the space $\mathfrak{H}^\gamma_{p,\theta}(\tau,G)$ into spaces of continuous functions. For a positive real number $T>0$, a stopping time $\tau\le T$, a real number $\delta\in(0,1]$, and a Banach space X, we will use the following notations:

$$|[u]|^p_{C^\delta([0,\tau],X)} = \sup_{0\le s<t\le T}\frac{\|u(t\wedge\tau)-u(s\wedge\tau)\|^p_X}{|t-s|^{p\delta}}$$

and

$$\|u\|^p_{C^\delta([0,\tau],X)} = \sup_{0\le t\le T}\|u(t\wedge\tau)\|^p_X + |[u]|^p_{C^\delta([0,\tau],X)}.$$

It is proved in [8], Theorem 2.1, that if $u\in\mathcal{H}^\gamma_p(\tau)$, $p\ge 2$, and $\tau\le T$, then

$$E\sup_{0\le t\le T}\|u(t\wedge\tau,\cdot)\|^p_{H^\gamma_p} \le N(d,\gamma,p,T)\|u\|^p_{\mathcal{H}^\gamma_p(\tau)}, \tag{2.7}$$

and if in addition $1/p<\alpha<\beta<1/2$, then

$$E\|u\|^p_{C^{\alpha-1/p}([0,\tau],H^{\gamma+1-2\beta}_p)} \le N(\alpha,\beta,d,\gamma,p,T)\|u\|^p_{\mathcal{H}^\gamma_p(\tau)}. \tag{2.8}$$

We also know from Theorem 2.11 in [10] that

$$E\sup_{0\le t\le T}\|u(t\wedge\tau,\cdot)-u_0\|^p_{H^\gamma_{p,\theta}} \le N(d,\gamma,p,T)\|u\|^p_{\mathfrak{H}^\gamma_{p,\theta}(\tau)}. \tag{2.9}$$

The following is the corresponding embedding theorem for the space $\mathfrak{H}^\gamma_{p,\theta}(\tau,G)$.

Theorem 2.7. *Assume that G is a bounded domain of class $C^{|\gamma|+3}$, $u\in\mathfrak{H}^\gamma_{p,\theta}(\tau,G)$, $p\ge2$, $d-1<\theta<p+d-1$, and $\tau\le T$.*

1.

$$E\sup_{0\le t\le T}\|u(t\wedge\tau,\cdot)\|^p_{H^\gamma_{p,\theta}(G)} \le N\|u\|^p_{\mathfrak{H}^\gamma_{p,\theta}(\tau,G)}; \tag{2.10}$$

in particular,

$$\|u\|^p_{\mathbb{H}^\gamma_{p,\theta}(t,G)} \le N\int_0^t \|u\|^p_{\mathfrak{H}^\gamma_{p,\theta}(s,G)}ds \tag{2.11}$$

for all $t\le T$. The value of N depends only on d,γ,p,T, and the domain G.

2. If, in addition, $1/p<\alpha<\beta<1/2$, then

$$E\|u\|^p_{C^{\alpha-1/p}([0,\tau],H^{\gamma+1-2\beta}_{p,\theta-(1-2\beta)p}(G))} \le N\cdot\|u\|^p_{\mathfrak{H}^\gamma_{p,\theta}(\tau,G)}. \tag{2.12}$$

The value of N depends only on $\alpha,\beta,d,\gamma,p,T$, and the domain G.

Proof. The first part follows from (2.7), (2.9), and Corollary 2.5.

The proof of the second part requires some additional constructions. Denote by $|D|^\gamma$ the operator $f \mapsto (|\xi|^\gamma \hat{f})\check{}$, where $\hat{}$ and $\check{}$ are the Fourier transform and its inverse. Theorem 2.2.4 in [14] implies that, for $\gamma > 0$, the norm in H^γ_p is equivalent to $\|\cdot\|_{L_p} + \||D|^\gamma \cdot\|_{L_p}$.

Lemma 2.8. *Assume that $\gamma > 0$ and $f \in H^\gamma_p$. If f is compactly supported in a bounded domain G, then $\|f\|_{L_p} \le N(d,\gamma,G)\||D|^\gamma f\|_{L_p}$.*

Proof. Using the estimates from [12, Section V.1.2], we get

$$\|f\|_{L_q} \le N(\alpha,d,p)\||D|^\alpha f\|_{L_p}, \quad 1/q = 1/p - \alpha/d,$$

for every $f \in H^\alpha_p$ and $0 < \alpha < d/p$. Let $K \ge 1$ be the smallest positive integer so that $\gamma/K \le d/(2p)$, and define $p_m = p(dK/(dK - p\gamma))^m$, $m = 1, \ldots, K$. Then

$$\||D|^\gamma f\|_{L_p} \ge N\||D|^{\gamma(1-1/K)} f\|_{L_{p_1}} \ge \cdots \ge N^K \|f\|_{L_{p_K}}.$$

It remains to notice that $p < p_K \le 2^K p$, and, since f is compactly supported in G, $\|f\|_{L_p} \le N(G,p_K)\|f\|_{L_{p_K}}$.

The lemma is proved.

Lemma 2.9. *Assume that $1/p < \alpha < \beta < 1/2$. If $u \in \mathcal{H}^1_p(\tau)$ so that*

$$u(t) = u_0 + \int_0^t f(s)ds + \int_0^t g^k(s)dw^k(s),$$

then, for every $c > 0$,

$$c^{(1-2\beta)p} E|[\,|D|^{2-2\beta}u]|^p_{C^{\alpha-1/p}([0,\tau],L_p)} \le N\Big(c^{-p}\|f\|^p_{\mathbb{L}_p(\tau)} + c^p\|u\|^p_{\mathbb{H}^2_p(\tau)} + \|g\|^p_{\mathbb{H}^1_p(\tau;l_2)} + c^{-p}\|g\|^p_{\mathbb{L}_p(\tau;l_2)}\Big), \quad (2.13)$$

where N depends only on $\alpha,\beta,\gamma,d,p,T$.

Proof.

With no loss of generality assume that $u_0 = 0$. Since $\||D|^\nu \cdot\|_{L_p} \le \|\cdot\|_{H^\nu_p}$ for $\nu > 0$, we conclude from inequality (2.8) (with $\gamma = 1$), that

$$E|[\,|D|^{2-2\beta}u]|^p_{C^{\alpha-1/p}([0,\tau],L_p)} \le N \cdot \Big(\|f\|^p_{\mathbb{L}_p(\tau)} + \|u_{xx}\|^p_{\mathbb{L}_p(\tau)} + \|g_x\|^p_{\mathbb{L}_p(\tau;l_2)} + \|g\|^p_{\mathbb{L}_p(\tau;l_2)}\Big).$$

It remains to re-scale the space variable $x \to cx$ and use that $(|D|^\nu u(c\cdot))(x) = c^\nu(|D|^\nu u)(cx)$.

The lemma is proved.

Proof of inequality (2.12).

According to the definition of $H^\gamma_{p,\theta}(G)$, it is sufficient to consider the case when $G = \mathbb{R}^d_+ \cap B$, where B is a ball of sufficiently large radius $r > 0$ centered at the origin, and $u(t,x) = 0$ if $|x| > r/2$.

We know that $u_n(t,x)=\zeta(x)u(t,e^n x)$ satisfies

$$u_n(t,x)=u_{0n}+\int_0^t f_n(s,x)ds+\int_0^t g_n^k(s,x)dw^k(s),$$

where

$$u_{0n}(x)=\zeta(x)u_0(e^n x),\quad f_n(t,x)=\zeta(x)f(t,e^n x),\quad g_n(t,x)=\zeta(x)g(t,e^n x).$$

By assumption, u_n is compactly supported in G.

Next, we use Lemma 2.8, and also Lemma 2.9 with $c=e^{-n}$, to write

$$e^{np(1-2\beta)}E|[u_n]|^p_{C^{\alpha-1/p}([0,\tau],H_p^{2-2\beta})}\le N\Big(e^{np}\|f_n\|^p_{\mathbb{L}_p(\tau)} + e^{-np}\|u_n\|^p_{\mathbb{H}_p^2(\tau)}+\|g_n\|^p_{\mathbb{H}_p^1(\tau;l_2)}+e^{np}\|g_n\|^p_{\mathbb{L}_p(\tau)}\Big).$$

By replacing u_n with $\big((1+|\xi|^2)^{(\gamma-1)/2}\hat{u}_n\big)\check{}$, we rewrite this inequality as

$$e^{np(1-2\beta)}E|[u_n]|^p_{C^{\alpha-1/p}([0,\tau],H_p^{\gamma+1-2\beta})}\le N\Bigg(e^{np}\|f_n\|^p_{\mathbb{H}_p^{\gamma-1}(\tau)} + e^{-np}\|u_n\|^p_{\mathbb{H}_p^{\gamma+1}(\tau)}+\|g_n\|^p_{\mathbb{H}_p^{\gamma}(\tau;l_2)}+e^{np}\|g_n\|^p_{\mathbb{H}_p^{\gamma-1}(\tau;l_2)}\Bigg).$$

After multiplying both sides by $e^{n\theta}$ and summing over all integer n, the last inequality results in

$$E|[u]|^p_{C^{\alpha-1/p}([0,\tau],H^{\gamma+1-2\beta}_{p,\theta-(1-2\beta)p})}\le N\|u\|^p_{\mathfrak{H}^\gamma_{p,\theta}(\tau,G)}+N\|g\|^p_{\mathbb{H}^{\gamma-1}_{p,\theta+p}(\tau,G;l_2)};$$

since G is a bounded domain, the term $\|g\|^p_{\mathbb{H}^{\gamma-1}_{p,\theta+p}(\tau,G;l_2)}$ can be dropped by Corollary 2.5. By the same corollary,

$$H^{\gamma+1-2/p}_{p,\theta+2-p}(G)=\rho^{1-2/p}H^{\gamma+1-2/p}_{p,\theta}(G)\subset\rho^{1-2\beta}H^{\gamma+1-2\beta}_{p,\theta}(G)=H^{\gamma+1-2\beta}_{p,\theta-(1-2\beta)p}(G).$$

Consequently,

$$E\|u\|^p_{C^{\alpha-1/p}([0,\tau],H^{\gamma+1-2\beta}_{p,\theta-(1-2\beta)p})}\le N\Big(E\|u_0\|^p_{H^{\gamma+1-2\beta}_{p,\theta-(1-2\beta)p}(G)} + E|[u]|^p_{C^{\alpha-1/p}([0,\tau],H^{\gamma+1-2\beta}_{p,\theta-(1-2\beta)p})}\Big)\le N\|u\|^p_{\mathfrak{H}^\gamma_{p,\theta}(\tau,G)}.$$

Theorem 2.7 is proved.

Corollary 2.10. *Assume that G is a bounded domain of class $C^{|\gamma|+3}$. If $\gamma-d/p>2$, $0<\theta<p-2$, and $u\in\mathfrak{H}^\gamma_{p,\theta}(\tau,G)$, then, for all $t\ge 0$ and all ω from a set of probability 1, the function $u(t\wedge\tau,\cdot)$ is twice continuously differentiable inside G, continuous in the closure of G, and is equal to zero on the boundary of G.*

Proof. According to Theorem 2.7 and Proposition 2.2(4),

$$u\in C^{\alpha-1/p}([0,\tau];\rho^{1-2\beta}\cdot H^{\gamma+1-2\beta}_{p,\theta}(G))$$

for every $1/p < \alpha < \beta < 1/2$. By Proposition 2.2(5),

$$\rho^{1+2\beta+\theta/p} u \in C^{\alpha-1/p}([0,\tau]; C^2(G))$$

meaning that u is twice continuously differentiable in G. Also by Proposition 2.2(5),

$$\rho^{\theta/p+2\beta-1} u \in C^{\alpha-1/p}([0,\tau]; C(G)),$$

so that, choosing β sufficiently close to $1/p$ to have $\theta/p + 2\beta - 1 < 0$, we conclude that u is continuous in the closure of G and is equal to zero on ∂G.

3. Main Result

Consider the following equation:

$$\begin{aligned} du &= \left(a^{ij}(t,x)u_{x^ix^j} + f(t,x,u,u_x)\right)dt + \left(\sigma^{ik}(t,x)u_{x^i} + g^k(t,x,u)\right)dw^k(t),\ t>0,\ x \in G \\ u &= u(t,x), \qquad u|_{t=0} = u_0, \qquad u|_{\partial G} = 0. \end{aligned} \tag{3.1}$$

Fix $\gamma \in \mathbb{R}$.

Assumption 3.1. (Regularity of the domain.) The domain G is a bounded domain in $\mathbb{R}^d$ of class $C^{|\gamma|+3}$, in the sense of Definition 2.1.

Assumption 3.2. (Coercivity.) There exist positive numbers κ_1 and κ_2 so that

$$\kappa_1|\xi|^2 \le \left(a^{ij} - \frac{1}{2}\sigma^{ik}\sigma^{jk}\right)\xi^i\xi^j \le \kappa_2|\xi|^2 \tag{3.2}$$

for all $(\omega,t) \in (\!(0,\tau]\!]$, $x \in G$, and $\xi \in \mathbb{R}^d$.

Assumption 3.3. (Uniform continuity of a and σ.) For every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ so that

$$|a^{ij}(t,x) - a^{ij}(t,y)| + \|\sigma^{i\cdot}(t,x) - \sigma^{i\cdot}(t,y)\|_{l_2} \le \varepsilon$$

for all $(\omega,t) \in (\!(0,\tau]\!]$, all $x,y \in \bar{G}$ with $|x-y| < \delta_\varepsilon$, and all $i,j = 1,\dots,d$.

Assumption 3.4. (Regularity of a and σ.) The functions a^{ij} and σ^{ik} are $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^d_+)$ measurable and

$$\|a^{ij}(t,\cdot)\|_{A^{|\gamma-1|+\gamma'}(G)} + \|\sigma^{i\cdot}(t,\cdot)\|_{A^{|\gamma|+\gamma'}(G;l_2)} \le \kappa_2$$

for all $(\omega,t) \in (\!(0,\tau]\!]$ and all $i,j = 1,\dots,d$.

Assumption 3.5. (Regularity of the free terms.)

$$(f(\cdot,\cdot,0,0), g(\cdot,\cdot,0)) \in \mathcal{F}^\gamma_{p,\theta}(\tau,G),$$

and for every $\varepsilon > 0$ there exists $\mu_\varepsilon > 0$ so that

$$\begin{aligned} &\|(f(\cdot,\cdot,u,u_x) - f(\cdot,\cdot,v,v_x), g(\cdot,\cdot,u) - g(\cdot,\cdot,v))\|_{\mathcal{F}^\gamma_{p,\theta}(\tau,G)} \\ &\qquad\qquad \le \varepsilon\|u-v\|_{\mathfrak{H}^\gamma_{p,\theta}(G)} + \mu_\varepsilon\|u-v\|_{\mathbb{H}^\gamma_{p,\theta}(G)} \end{aligned}$$

for all $u,v \in \mathfrak{H}^\gamma_{p,\theta}(\tau,G)$.

Assumption 3.6. (Regularity of the initial condition.) $\quad u_0 \in U^\gamma_{p,\theta}(G)$.

Assumption 3.7. (Technical assumptions.)

(i) $p \geq 2$;

(ii) $d-1 < \theta < d+p-1$;

(iii) there exists a number $\tilde{\kappa} \in (0,1)$ so that

$$\left(\frac{\theta+1-d}{(p-1)(p+d-1-\theta)}a^{ij} - \frac{1}{2}\sigma^{ik}\sigma^{jk}\right)\xi^i\xi^j \geq \tilde{\kappa}|\xi|^2 \tag{3.3}$$

for all $(\omega,t) \in (\!(0,\tau]\!]$, $x \in G$, and $\xi \in \mathbb{R}^d$.

Definition 3.1. A process $u \in \mathfrak{H}^{\gamma}_{p,\theta}(\tau,G)$ is called a solution of (3.1) if for every $\phi \in C_0^{\infty}(G)$ the following equality holds for all $t \geq 0$ and all ω from a set of probability 1:

$$(u,\phi)(t\wedge\tau) = (u_0,\phi) + \int_0^{t\wedge\tau}\Big((a^{ij}u_{x^ix^j},\phi)(s) + (f,\phi)(s)\Big)ds + \int_0^{t\wedge\tau}(\sigma^{ik}u_{x^i} + g,\phi)(s)dw^k(s).$$

It follows from Corollary 2.10 that if a solution of (3.1) exists and γ and p are sufficiently large, then both the equation and the boundary condition are satisfied pointwise in x.

Note that Assumptions 3.1, 3.4, 3.5, and 3.6. are necessary to define the solution as an element of the space $\mathfrak{H}^{\gamma}_{p,\theta}(\tau,G)$. Assumption 3.3 is used to construct the solution by combining the local solutions using the partition of unity in G. Note that this assumption does not in general follow from Assumption 3.4; for example, the function $a(x) = \sin\ln x$, $x \in (0,1)$, belongs to every $A^n((0,1))$, but is not uniformly continuous.

It is well known that, in the case of stochastic equations, it is necessary to have $p \geq 2$ because of the Ito formula and the generalized Littlewood-Paley inequality (see [8] and [5] for details). The restriction on θ is also natural: by considering the usual heat equation, we can see that if $\theta \geq d+p-1$, then the free terms can blow up near the boundary too fast for the solution to exist, while for $\theta \leq d-1$ the solution, no matter how regular, cannot belong to the corresponding space. Condition (3.3), on the other hand, is purely technical and comes from the method used in [10] to prove solvability of the equation in the half-space. Note that (3.3) follows from (3.2) if one of the following holds:

- $\sigma \equiv 0$ or
- $p+d-2 \leq \theta < p+d-1$.

The following is the main result of the paper. The proof is given in Section 5.

Theorem 3.2. (The Main Theorem.) *Let $T > 0$ be fixed and assume that $\tau \leq T$. Then, under Assumptions 3.1–3.7, equation (3.1) has a unique solution in the space $\mathfrak{H}^{\gamma+1}_{p,\theta}(\tau,G)$ and the solution satisfies*

$$\|u\|_{\mathfrak{H}^{\gamma}_{p,\theta}(\tau,G)} \leq N\cdot\left(\|(f(\cdot,\cdot,0,0),g(\cdot,\cdot,0))\|_{\mathcal{F}^{\gamma}_{p,\theta}(\tau,G)} + \|u_0\|_{U^{\gamma}_{p,\theta}}\right). \tag{3.4}$$

The value of N depends only on $d,\gamma,\kappa_1,\kappa_2,\tilde{\kappa},p,T,\theta$, the domain G, and the functions $\delta=\delta_{\varepsilon}$ and $\mu=\mu_{\varepsilon}$.

The following is an interpretation of the result when the functions f and g^k do not depend on u. Let $u \in \mathbb{H}^{\gamma+1}_{p,\theta-p}(\tau, G)$ and functions a^{ij}, σ^{ik} satisfy Assumption 3.4. Define the operators

$$\mathcal{A}u(t,x) = a^{ij}(t,x)u_{x^ix^j}(t,x); \quad \mathcal{B}^k u(t,x) = \sigma^{ik}(t,x)u_{x^i}(t,x)$$

and write

$$(\!(\mathcal{A}, \mathcal{B})\!) u = (f, g, u_0)$$

for some $(f,g) \in \mathcal{F}^{\gamma}_{p,\theta}(\tau, G)$, $u_0 \in U^{\gamma}_{p,\theta}(G)$ if u belongs to $\mathfrak{H}^{\gamma}_{p,\theta}(\tau, G)$ and is a solution of

$$du = \left(a^{ij}(t,x)u_{x^ix^j} + f(t,x)\right) dt + \left(\sigma^{ik}(t,x)u_{x^i} + g^k(t,x)\right) dw^k(t)$$
$$u|_{t=0} = u_0, \qquad u|_{\partial G} = 0.$$

Note that if $u = [f_0, g_0, u_0] \in \mathfrak{H}^{\gamma}_{p,\theta}(\tau, G)$, then, by definition,

$$(\!(\mathcal{A}, \mathcal{B})\!) u = (f_0 - \mathcal{A}u, g - \mathcal{B}u, u_0) \tag{3.5}$$

and

$$\|(f_0 - \mathcal{A}u, g - \mathcal{B}u)\|_{\mathcal{F}^{\gamma}_{p,\theta}(\tau,G)} \le N(d,\gamma,\kappa,p)\|u\|_{\mathfrak{H}^{\gamma}_{p,\theta}(\tau,G)}.$$

This means that $(\!(\mathcal{A}, \mathcal{B})\!)$ is a bounded linear operator from $\mathfrak{H}^{\gamma}_{p,\theta}(\tau, G)$ to $\mathcal{F}^{\gamma}_{p,\theta}(\tau, G) \times U^{\gamma}_{p,\theta}(G)$. The additional conditions on a, σ, p, and θ, namely, Assumptions 3.2, 3.3, and 3.7, ensure that the operator $(\!(\mathcal{A}, \mathcal{B})\!)$ is a *linear homeomorphism* of the corresponding Banach spaces.

4. Examples

4.1. **Linear equation with finite-dimensional noise.** Consider a particular case of equation (3.1):

$$\begin{aligned} du =&(a^{ij}(t,x)u_{x^ix^j} + b^i(t,x)u_{x^i} + c(t,x)u + f(t,x))dt \\ &+ \sum_{k=1}^{d_1}(\sigma^{ik}(t,x)u_{x^i} + h^k(t,x)u(t,x) + g^k(t,x))dw^k(t), \; t>0, \; x \in G, \\ u|_{t=0} =& u_0, \qquad u|_{\partial G} = 0. \end{aligned} \tag{4.1}$$

We make the following assumptions about the functions b, c, f, h, g.

Assumption 4.1. The functions b^i, c, and h^k are $\mathcal{P} \otimes \mathcal{B}(G)$ measurable and

$$\|b^i(t,\cdot)\|_{A^{|\gamma-1|+\gamma'}(G)} + \|\rho \cdot c(t,\cdot)\|_{A^{|\gamma-1|+\gamma'}(G)} + \|h(t,\cdot)\|_{A^{|\gamma|+\gamma'}(G,l_2)} \le \kappa_2.$$

Assumption 4.2. $(f,g) \in \mathcal{F}^{\gamma}_{p,\theta}(\tau, G)$.

The next result is a generalization of Theorem 2.1 in [6].

Theorem 4.1. *Under Assumptions 3.1–3.7 with Assumptions 4.1 and 4.2 instead of Assumption 3.5, for every $\tau \le T$ equation (4.1) has a unique solution from $\mathfrak{H}^{\gamma}_{p,\theta}$ and the solution satisfies*

$$\|u\|_{\mathfrak{H}^{\gamma}_{p,\theta}(\tau,G)} \le N \cdot \left(\|(f,g)\|_{\mathcal{F}^{\gamma}_{p,\theta}(\tau,G)} + \|u_0\|_{U^{\gamma}_{p,\theta}}\right).$$

The value of N depends only on $d, \gamma, \kappa_1, \kappa_2, \tilde{\kappa}, p, T, \theta$, the domain G, and the functions $\delta = \delta_\varepsilon$ and $\mu = \mu_\varepsilon$.

Proof. By Theorem 3.2, all we need is to show that, with the functions $f(t,x,u,u_x)$, and $g(t,x,u)$ defined by

$$f(t,x,u,u_x) = b^i(t,x)u_{x^i} + c(t,x)u + f(t,x), \quad g(t,x,u) = h(t,x)u(t,x) + g(t,x),$$

Assumptions 4.1 and 4.2 imply Assumption 3.5. If $u \in \mathfrak{H}^\gamma_{p,\theta}(\tau,G)$, then Theorem 2.4 and Proposition 2.6 imply that

$$(f(\cdot,\cdot,u,u_x), g(\cdot,\cdot,u)) \in \mathcal{F}^\gamma_{p,\theta}(\tau,G).$$

Also, using Proposition 2.2(4) and Assumption 4.1, for every $u,v \in \mathfrak{H}^\gamma_{p,\theta}(\tau,G)$ we get

$$\begin{aligned}&\|(f(\cdot,\cdot,u,h_x) - f(\cdot,\cdot,v,v_x), g(\cdot,\cdot,u) - g(\cdot,\cdot,v))\|_{\mathcal{F}^\gamma_{p,\theta}(\tau,G)} \\ &\qquad = \|(b^i \cdot D_i(u-v) + c\cdot(u-v), h\cdot(u-v))\|_{\mathcal{F}^\gamma_{p,\theta}(\tau,G)} \le N\|u-v\|_{\mathbb{H}^\gamma_{p,\theta}(\tau,G)}.\end{aligned}$$

Theorem 4.1 is proved.

4.2. **Equation driven by space-time white noise.** Assume that G is an interval $I = (x_1,x_2)$, $\rho(x) = (x-x_1)(x_2-x)$, $\{\varphi_k(x),\ k\ge 1\}$ is an orthonormal basis in $L_2(I)$.

Consider the following equation:

$$\begin{aligned}du &= (a(t,x)u_{xx} + b(t,x)u_x + f(t,x,u))dt + h(t,x,u)\varphi_k(x)dw^k(t)\\ u(0,x) &= u_0(x), \quad u(t,x_1) = u(t,x_2) = 0.\end{aligned} \tag{4.2}$$

It is shown in [4, Section 7.2] that equation (4.2) is equivalent to

$$\begin{aligned}du &= (a(t,x)u_{xx} + b(t,x)u_x + f(t,x,u))dt + h(t,x,u)dB(t,x)\\ u(0,x) &= u_0(x), \quad u(t,x_1) = u(t,x_2) = 0,\end{aligned}$$

where $B(t,x)$ is the space-time white noise.

We make the following assumptions about equation (4.2).

Assumption 4.3. There exist positive numbers κ_1 and κ_2 so that $\kappa_1 \le a(t,x) \le \kappa_2$ for all $(\omega,t) \in (\!(0,\tau]\!]$ and $x\in I$.

Assumption 4.4. For every $\epsilon > 0$ there exists $\delta = \delta_\epsilon$ so that $|a(t,x) - a(t,y)| \le \delta_\epsilon$ for all $x,y \in [x_1,x_2]$ satisfying and all $(\omega,t) \in (\!(0,\tau]\!]$.

Assumption 4.5. The functions a,b are $\mathcal{P}\otimes\mathcal{B}(\mathbb{R})$ measurable and

$$\|a\|_{A^2(I)} + \|b\|_{A^1(I)} \le \kappa_2$$

for all $(\omega,t) \in (\!(0,\tau]\!]$ and $x \in I$.

Assumption 4.6. $p\ge 2$, $\quad p/2 < \theta < p$, and $\gamma \in (-1,-1/2)$.

Assumption 4.7.

$$f(\cdot,\cdot,0) \in \mathbb{H}^{\gamma-1}_{p,\theta+p}(\tau,I),\ h(\cdot,\cdot,0) \in \mathbb{L}_{p,\theta-p/2}(\tau,I),$$

and [1]

$$|\rho(x)f(t,x,y_1) - \rho(x)f(t,x,y_2)| + |h(t,x,y_1) - h(t,x,y_2)| \le \kappa_2|y_1-y_2|$$

[1] $\mathbb{L}_{p,\theta}(\tau,I) = \mathbb{H}^0_{p,\theta}(\tau,I)$

for all $(\omega, t) \in (\!(0, \tau]\!]$, $x \in I$, $y_1, y_2 \in \mathbb{R}$.

Assumption 4.8. $u_0 \in U^{\gamma}_{p,\theta}(I)$.

Theorem 4.2. *Under Assumption 4.3–4.8, if $\tau \le T$, then there is a unique solution $u \in \mathfrak{H}^{\gamma}_{p,\theta}(\tau, I)$ of equation (4.2) and*

$$\|u\|_{\mathfrak{H}^{\gamma}_{p,\theta}(\tau,I)} \le N \cdot \left(\|f(\cdot,\cdot,0)\|_{\mathbb{H}^{\gamma-1}_{p,\theta+p}(\tau,I)} + \|h(\cdot,\cdot,0)\|_{\mathbb{L}_{p,\theta-p/2}(\tau,I)} + \|u_0\|_{U^{\gamma}_{p,\theta}(I)} \right).$$

The number N depends only on $\gamma, \kappa_1, \kappa_2, \theta, p, T$, and the interval I.

Together with Theorem 2.7(2), Theorem 4.2 implies continuity of the solution in time and space. When $a = 1$ and $b = 0$, the corresponding result is obtained in [15, Chapter 3].

Corollary 4.3. *Assume that the conditions of Theorem 4.2 are fulfilled for every $\gamma \in (-1, -1/2)$, $\theta \in (p/2, p)$, and $p \ge 2$ (for example, if the functions f and h are bounded and do not depend on u, and u_0 is deterministic and belongs to $C^{\infty}_0(I)$). Then the solution $u = u(t, x)$ of (4.2) has the following properties:*

1. For every sufficiently small $\delta > 0$, $u(t, \cdot) \in C^{1/2-\delta}(I)$ with probability 1, uniformly in t, and $u(t, x_1) = u(t, x_2) = 0$ so that $u(t, x) \sim (\rho(x))^{1/2-\delta}$ near the end points;

2. For every $\delta > 0$, $u(\cdot \wedge \tau, x) \in C^{1/4-\delta}(0, T)$ with probability 1, uniformly in x.

Proof. Assume the $p > 2$ and choose α, β so that $1/p < \alpha < \beta < 1/2$. By Proposition 2.2(4) and Theorem 2.7(2),

$$u \in C^{\alpha-1/p}([0, \tau], \rho^{1-2\beta} H^{\nu-2\beta}_{p,\theta}) \quad (P\text{– a.s.})$$

for all $\nu \in (0, 1/2)$, $\theta \in (p/2, p)$, and $p \ge 2$. Assume that p is sufficiently large and $\theta > p/2 + 1$. Then we can choose ν so that

$$\theta/p - 1/p = (1 - \nu)$$

or

$$1 - 2\beta = \nu - 2\beta + \theta/p - 1/p,$$

so that, according to Proposition 2.2(5), $\rho^{1-2/p} H^{\nu-2\beta}_{p,\theta}(I) \subset C^{\nu-2\beta-1/p}(I)$ and

$$u \in C^{\alpha-1/p}\left([0, \tau], C^{\nu-2\beta-1/p}(I)\right).$$

The last inclusion implies both statements of the corollary. Indeed, to get continuity in x, choose ν sufficiently close to 1/2, p sufficiently large, and α, β sufficiently close to $1/p$; to see that $u(t, x) \sim (\rho(x))^{1/2-\delta}$, we also choose θ close to $p/2 + 1$. To get continuity in t, choose ν sufficiently close to 1/2, α and β sufficiently close to 1/4, and p sufficiently large.

Corollary 4.3 is proved.

To deduce Theorem 4.2 from Theorem 3.2, one of the problems is to show that Assumption 4.7 implies Assumption 3.5 with $g^k(t, x, u) = h(t, x, u)\varphi_k(x)$. To do so, we will need the following result.

Lemma 4.4. *Assume that $-1 < \gamma < -1/2$ and $h \in L_{p,\theta-p/2}(I)$. Define $g = \{g^k(x),\ k \ge 1\}$ so that $g^k(x) = h(x)\varphi_k(x)$. Then $g \in H^{\gamma}_{p,\theta}(I; l_2)$ and*

$$\|g\|_{H^{\gamma}_{p,\theta}(I;l_2)} \le N\|h\|_{L_{p,\theta-p/2}(I)}.$$

Proof. By the definition of the space $H_p^\gamma(I;l_2)$ it is sufficient to show that, for every function $\chi \in C_0^\infty(\mathbb{R})$ whose support contains x_1 and does not contain x_2, the function $\tilde g$ defined by $\tilde g^k(x) = \chi(x+x_1)\varphi_k(x+x_1)h(x+x_1)$ belongs to $H^\gamma_{p,\theta}(l_2)$; the analysis near the point x_2 is identical. To simplify the notations, set $x_1 = 0$.

It is known from [11] that in the case $d=1$ the spaces $H^\mu_{p,\theta}$ are generated by the corresponding powers of the operator

$$\Lambda_{p,\theta} = Q^{-1}_{p,\theta}\Lambda Q_{p,\theta},$$

where $Q_{p,\theta} : f(x) \mapsto f(e^x)e^{x\theta/p}$ and $\Lambda = \sqrt{1-d^2/dx^2}$. This means that if $R_\gamma(x,y)$ is the kernel of the operator Λ^γ, then the kernel $R_{\gamma,\theta}(x,y)$ of the operator $\Lambda^\gamma_{p,\theta}$ satisfies

$$R_{\gamma,\theta}(e^x,e^y) = R_\gamma(x,y)e^{-\theta(x-y)/p}e^{-y}. \tag{4.3}$$

By the definition of the norm in $H^\gamma_{p,\theta}(l_2)$,

$$\|\tilde g\|^p_{H^\gamma_{p,\theta}(l_2)} = \int_0^\infty \left(\sum_k \left(\int_0^{+\infty} R_{\gamma,\theta}(x,y)h(y)\varphi_k(y)\chi(y)dy\right)^2\right)^{p/2} x^{\theta-1}dx,$$

and by the Bessel inequality for orthonormal systems, the last expression is bounded by

$$\int_0^\infty \left(\int_0^{+\infty} R^2_{\gamma,\theta}(x,y)h^2(y)\chi^2(y)dy\right)^{p/2} x^{\theta-1}dx.$$

After that, relation (4.3) implies that

$$\|\tilde g\|^p_{H^\gamma_{p,\theta}(l_2)} \le \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} R^2_\gamma(x,y)|\tilde h(y)|^2dy\right)^{p/2} dx, \tag{4.4}$$

where $\tilde h(y) = e^{(\theta/p-1/2)y}h(e^y)\chi(e^y)$.

By Lemma 4.1(ii) in [8] the right-hand side of (4.4) is bounded by

$$N(\gamma,p)\|\tilde h\|^p_{L_p(\mathbb{R})}.$$

Finally, the definition of $\tilde h$ and the Hölder inequality imply that

$$\|\tilde h\|^p_{L_p(\mathbb{R})} \le N(I)\int_{x_1}^{x_2} x^{\theta-p/2-1}|h(x)|^pdx \le N(I)\|h\|^p_{L_{p,\theta-p/2}(I)}.$$

Lemma 4.4 is proved.

Remark 4.5. Note that $h \in L_{p,\theta-p/2}(I)$ if, for example, $\theta > p/2$ and $h \in L_\infty(I)$ or if $\theta \ge p/2+1$ and $h \in L_p(I)$.

Proof of Theorem 4.2.

We can now deduce Theorem 4.2 from Theorem 3.2 If $f(t,x,u,u_x) = b(t,x)u_x + f(t,x,u)$, then, by Assumptions 4.5 and 4.7,

$$\|f(\cdot,\cdot,u,u_x) - f(\cdot,\cdot,v,v_x)\|_{H^{\gamma-1}_{p,\theta+p}(\tau,I)} \le N\|u\|_{\mathbb{H}^\gamma_{p,\theta}(\tau,I)}.$$

Also, by Assumption 4.7 and Lemma 4.4, if $g^k(t,x,u) = h(t,x,u)\varphi_k(x)$, then $g(\cdot,\cdot,0) \in \mathbb{H}^\gamma_{p,\theta}(\tau, I; l_2)$ We need to show that, for every $\varepsilon > 0$, there is μ_ε so that

$$\|g(\cdot,\cdot,u) - g(\cdot,\cdot,v)\|_{\mathbb{H}^\gamma(\tau,I;l_2)} \leq \varepsilon \|u - v\|_{\mathfrak{H}^\gamma_{p,\theta}(\tau,I)} + \mu_\varepsilon \|u - v\|_{\mathbb{H}^\gamma_{p,\theta}(\tau,I)} \tag{4.5}$$

for all $u, v \in \mathfrak{H}^\gamma_{p,\theta}(\tau, I)$. By Assumption 4.7 and Lemma 4.4 we conclude that

$$\|g(\cdot,\cdot,u) - g(\cdot,\cdot,v)\|_{\mathbb{H}^\gamma_{p,\theta}(\tau,I;l_2)} \leq \|u - v\|_{\mathbb{L}_{p,\theta-p/2}(\tau,I)}.$$

For every $\varepsilon > 0$, inequality $|ab| \leq \varepsilon |a|^2 + 1/\varepsilon\, |b|^2$ implies

$$\|u - v\|^p_{\mathbb{L}_{p,\theta-p/2}(\tau,I)} \leq \varepsilon \|u - v\|^p_{\mathbb{L}_{p,\theta-p}(\tau,I)} + 1/\varepsilon \|u - v\|^p_{\mathbb{L}_{p,\theta}(\tau,I)}.$$

Next, we use (2.2) with ε^2 instead of ε to get

$$1/\varepsilon \|u - v\|^p_{\mathbb{L}_{p,\theta}(\tau,I)} \leq N(\gamma)\varepsilon \|u - v\|^p_{\mathbb{H}^{\gamma+1}_{p,\theta}(\tau,I)} + N(\gamma,\varepsilon)\|u - v\|^p_{\mathbb{H}^\gamma_{p,\theta}(\tau,I)}.$$

Finally, we note that

$$\|u - v\|^p_{\mathbb{L}_{p,\theta-p}(\tau,I)} \leq \|u - v\|^p_{\mathfrak{H}^\gamma_{p,\theta}(\tau,I)} \quad \text{and} \quad \|u - v\|^p_{\mathbb{H}^{\gamma+1}_{p,\theta}(\tau,I)} \leq N(\gamma,p,I)\|u - v\|^p_{\mathfrak{H}^\gamma_{p,\theta}(\tau,I)}.$$

Combining the above inequalities results in (4.5).

Theorem 4.2 is proved.

5. Proof of Theorem 3.2

With no loss of generality, we assume that $\tau = T$, because we can always continue the functions f and g to $\Omega \times (0,T)$ by setting them equal to zero for $\tau \leq t$.

The solution operator for equation (3.1) will be constructed by using the partition of unity in G to combine the local solution operators away from the boundary and near the boundary. A detailed description of this approach for deterministic elliptic equations can be found in [7, Sections 6.2–6.5].

The proof will proceed as follows. The local solution operator away from the boundary is constructed below in Lemma 5.1 using the results from [8]. To construct the local solution operator near the boundary, the original equation is transformed into an equation in $\mathbb{R}^d_+$. The resulting equation with "almost constant" coefficients is solved by combining the results from [10] with a perturbation result from Lemma 5.3 below. Note that after the transformation to the half-space the coefficients of the equation depend on x even if the coefficients of the original equation did not. The process of combining the local solution operators results in an integro-differential equation whose unique solvability and equivalence with the original equation are established in Propositions 5.5 and 5.6 below.

We begin by considering equation (3.1) away from the boundary. Take two functions $\chi_0, \eta_0 \in C^\infty_0(G)$, where χ_0 is the corresponding element of the partition of unity and the function η_0 satisfies $0 \leq \eta_0(x) \leq 1$ and $\eta_0(x) = 1$ on the support of χ_0. Define operators

$$(\tilde{\mathcal{A}}_0, \tilde{\mathcal{B}}_0) : \mathcal{H}^\gamma_p(T) \to \mathcal{F}^\gamma_p(T)$$

by

$$\tilde{\mathcal{A}}_0 u(t,x) = \eta_0(x)a^{ij}(t,x)u_{x^ix^j}(t,x) + (1-\eta_0(x))\Delta u(t,x),$$
$$\tilde{\mathcal{B}}_0^k u(t,x) = \eta_0(x)\sigma^{ik}(t,x)u_{x^i}(t,x),$$

where Δ is the Laplace operator. Next, define the operator

$$(\!(\tilde{\mathcal{A}}_0, \tilde{\mathcal{B}}_0)\!) : \mathcal{H}_p^\gamma(T) \to \mathcal{F}_p^\gamma(T) \times L_p(\Omega, H_p^{\gamma+1-2/p})$$

by setting (cf. (3.5)) $(\!(\tilde{\mathcal{A}}_0, \tilde{\mathcal{B}}_0)\!)u = (f_0 - \tilde{A}_0 u, g_0 - \tilde{B}_0 u, u_0)$

if $u(t) = u_0 + \int_0^t f_0(s)ds + \int_0^t g_0^k(s)dw^k(s)$ as an element of $\mathcal{H}_p^\gamma(T)$.

In other words, if $(\!(\tilde{\mathcal{A}}_0, \tilde{\mathcal{B}}_0)\!)u = (f, g, u_0)$, then u is a solution of

$$du = \left(\tilde{\mathcal{A}}_0 u(t,x) + f(t,x)\right)dt + \left(\tilde{\mathcal{B}}_0^k u(t,x) + g^k(t,x)\right)dw^k(t),\ t>0,\ x\in\mathbb{R}^d,$$
$$u|_{t=0} = u_0.$$

If $(\!(\mathcal{A}, \mathcal{B})\!)$ is the operator defined after the statement of Theorem 3.2, then

$$(\!(\mathcal{A}, \mathcal{B})\!)(u\chi_0) = (\!(\tilde{\mathcal{A}}_0, \tilde{\mathcal{B}}_0)\!)(u\chi_0)$$

for every $u \in \mathcal{H}_p^\gamma(T)$.

Lemma 5.1. *Under Assumptions 3.2, 3.3, and 3.4 the operator $(\!(\tilde{\mathcal{A}}_0, \tilde{\mathcal{B}}_0)\!)$ has a bounded inverse $\tilde{\mathcal{R}}_0$ so that*

$$\|\tilde{\mathcal{R}}_0(f,g,u_0)\|^p_{\mathcal{H}_p^\gamma(T)} \le N \cdot \left(\|(f,g)\|^p_{\mathcal{F}_p^\gamma(T)} + E\|u_0\|^p_{H_p^{\gamma+1-2/p}}\right) \tag{5.1}$$

and

$$\tilde{\mathcal{R}}_0((\!(\mathcal{A}, \mathcal{B})\!)(u\chi_0)) = u\chi_0 \tag{5.2}$$

for every $u \in \mathcal{H}_p^\gamma(T)$.

Proof. Both statements of the lemma follow from Theorem 3.2 in [8].

Next, we consider equation (3.1) in $B_{r_0}(x_0) \cap G$ for $x_0 \in \partial G$. Let ψ be the corresponding diffeomorphism (cf. Definition 2.1) $\psi : B_{r_0}(x_0) \cap G \to \mathbb{R}^d_+$, $y_0 = \psi(x_0) = 0$.

Define the following operators (cf. Section 6.2 in [7]):

$$\tilde{\mathcal{A}}v(t,y) = \tilde{a}^{ij}(t,y)v_{y^iy^j}(t,y) + \tilde{b}^i(t,y)v_{y^i}(t,y),\quad \tilde{\mathcal{B}}^k v(t,y) = \tilde{\sigma}^{ik}(t,y)v_{y^i}(t,y),$$

where

$$\tilde{a}^{ij}(t,y) = a^{i'j'}(t,x)\psi^i_{x^{i'}}(x)\psi^j_{x^{j'}}(x);\ \tilde{b}^i(t,y) = a^{i'j'}(t,x)\psi^i_{x^{i'}x^{j'}}(x);$$
$$\tilde{\sigma}^{ik}(t,y) = \sigma^{i'k}(t,x)\psi^i_{x^{i'}}(x);\ x = \psi^{-1}(y).$$

It follows that if $u(t,x) = v(t,y)$, then $(\mathcal{A}u(t,x), \mathcal{B}u(t,x)) = (\tilde{\mathcal{A}}v(t,y), \tilde{\mathcal{B}}v(t,y))$.

To solve equation (3.1) in $B_{r_0}(x_0)\cap G$, fix a function $\eta\in C_0^\infty(\mathbb{R}^d)$ so that $0\le\eta\le 1$ and $\eta(x)=1$ for $|x|\le r_0$, and define $\tilde\eta(y)=\eta(x-x_0)$,

$$\bar{\mathcal{A}}v(t,y)=\tilde\eta(y)\tilde{\mathcal{A}}v(t,y)+(1-\tilde\eta(y))\tilde a^{ij}(t,y_0)v_{y^iy^j}(t,y),$$
$$\bar{\mathcal{B}}^kv(t,y)=\tilde\eta(y)\tilde{\mathcal{B}}^kv(t,y)+(1-\tilde\eta(y))\tilde\sigma^{ik}(t,y_0)v_{y^i}(t,y),$$
$$\bar{\mathcal{A}}_0v(t,y)=\tilde a^{ij}(t,y_0)v_{y^iy^j}(t,y);\quad \bar{\mathcal{B}}_0^kv(t,y)=\tilde\sigma^{ik}(t,y_0)v_{y^i}(t,y).$$

Finally, define the operator

$$(\!|\bar{\mathcal{A}}_0,\bar{\mathcal{B}}_0|\!):\mathfrak{H}^\gamma_{p,\theta}(T)\to\mathcal{F}^\gamma_{p,\theta}(T)\times U^\gamma_{p,\theta}$$

by setting

$$(\!|\bar{\mathcal{A}}_0,\bar{\mathcal{B}}_0|\!)v=(f_0-\bar{\mathcal{A}}_0v,g_0-\bar{\mathcal{B}}_0v,v_0)$$

if $v=[f_0,g_0,v_0]$ as an element of $\mathfrak{H}^\gamma_{p,\theta}(T)$. In other words, if $(\!|\bar{\mathcal{A}}_0,\bar{\mathcal{B}}_0|\!)v=(f,g,v_0)$, then v is a solution of

$$dv=\big(\bar{\mathcal{A}}_0v(t,y)+f(t,y)\big)\,dt+\big(\bar{\mathcal{B}}_0^kv(t,y)+g^k(t,y)\big)\,dw^k(t),\ t>0,\ y\in\mathbb{R}^d_+,$$
$$v|_{t=0}=v_0,\quad v|_{y^1=0}=0.$$

Lemma 5.2. *Under Assumptions 3.2, 3.4, 3.7, and 3.1, for every $x_0\in\partial G$ the operator $(\!|\bar{\mathcal{A}}_0,\bar{\mathcal{B}}_0|\!)$ has a bounded inverse $\bar{\mathcal{R}}_0$ and the norm of $\bar{\mathcal{R}}_0$ does not depend on T.*

Proof. This follows from Theorem 3.2 in [10], because the coefficients of $\bar{\mathcal{A}}_0$ and $\bar{\mathcal{B}}_0$ do not depend on y.

Next, we establish a perturbation result.

Lemma 5.3. (cf. Theorem 5.2 in [8].) *There exists an ε_0 depending only on d,γ,κ,p,θ so that if $\varepsilon\le\varepsilon_0$ and the operators $\bar{\mathcal{A}}$, $\bar{\mathcal{B}}$ satisfy*

$$\|(\bar{\mathcal{A}}v,\bar{\mathcal{B}}v)-(\bar{\mathcal{A}}_0v,\bar{\mathcal{B}}_0v)\|_{\mathcal{F}^\gamma_{p,\theta}(T)}\le\varepsilon\|v\|_{\mathfrak{H}^\gamma_{p,\theta}(T)}+N_0\|v\|_{\mathbb{H}^\gamma_{p,\theta}(T)}$$

for some N_0 depending only on d,γ,κ,p,θ, then the operator $(\!|\bar{\mathcal{A}},\bar{\mathcal{B}}|\!)$ has a bounded inverse. If $N_0=0$, then the norm of the inverse does not depend on T.

Proof. Define the operator $\mathcal{R}$ from $\mathfrak{H}^\gamma_{p,\theta}(T)$ to itself by setting

$$\mathcal{R}v=\bar{\mathcal{R}}_0(f+(\bar{\mathcal{A}}-\bar{\mathcal{A}}_0)v,g+\bar{\mathcal{B}}-\bar{\mathcal{B}}_0)v,u_0),$$

where $\bar{\mathcal{R}}_0$ is the operator from Lemma 5.2. With this definition, v satisfies $(\!|\bar{\mathcal{A}},\bar{\mathcal{B}}|\!)v=(f,g,u_0)$ if and only if v is a fixed point of the operator $\mathcal{R}$. Therefore, it remains to show that a sufficiently high power $(\mathcal{R})^n$ of $\mathcal{R}$ is a contraction in $\mathfrak{H}^\gamma_{p,\theta}(T)$. We have

$$\|\mathcal{R}v_1-\mathcal{R}v_2\|^p_{\mathfrak{H}^\gamma_{p,\theta}(T)}\le N\|(\,(\bar{\mathcal{A}}-\bar{\mathcal{A}}_0)(v_1-v_2),(\bar{\mathcal{B}}-\bar{\mathcal{B}}_0)(v_1-v_2)\,)\|^p_{\mathcal{F}^\gamma_{p,\theta}(T)}$$
$$\le N\varepsilon^p\|v_1-v_2\|^p_{\mathfrak{H}^\gamma_{p,\theta}(T)}+N\,N_0^p\int_0^T\|v_1-v_2\|^p_{\mathfrak{H}^\gamma_{p,\theta}(s)}ds,$$

where the first inequality follows from Lemma 5.2, and the second, from the assumptions and (2.11). This completes the proof if $N_0=0$; otherwise, we iterate the last inequality as in the proof of Theorem 5.2 in [8].

Lemma 5.3 is proved.

Lemma 5.4. *For every $a \in A^{|\gamma|+\gamma'}$ there is an equivalent norm in $H^\gamma_{p,\theta}$ (also denoted by $\|\cdot\|_{H^\gamma_{p,\theta}}$) and a constant $N = N(a, d, \gamma, p)$ so that*

$$\|a\, u\|_{H^\gamma_{p,\theta}} \leq N \cdot \sup_{x\in\mathbb{R}^d_+} |a(x)| \cdot \|u\|_{H^\gamma_{p,\theta}}. \tag{5.3}$$

Similarly, for every $\sigma \in A^{|\gamma|+\gamma'}(l_2)$ there is an equivalent norm in $H^\gamma_{p,\theta}$ and a constant $N = N(d, \gamma, p, \sigma)$ so that

$$\|\sigma\, u\|_{H^\gamma_{p,\theta}(l_2)} \leq N \cdot \sup_{x\in\mathbb{R}^d_+} \|\sigma(x)\|_{l_2} \cdot \|u\|_{H^\gamma_{p,\theta}}. \tag{5.4}$$

Proof. It is known from Remark 5.2 in [8] that for every $a \in B^\gamma$ there is an equivalent norm in H^γ (defined by $\|f\|_{H^\gamma_p} = \|((m^2 + |\xi|^2)\hat{f})\check{}\|_{L_p(\mathbb{R}^d)}$ for sufficiently large m) and a constant $N = N(a, d, \gamma, p)$ so that

$$\|a\, u\|_{H^\gamma_p} \leq N \cdot \sup_{x\in\mathbb{R}^d_+} |a(x)| \cdot \|u\|_{H^\gamma_p}.$$

It remains to use this norm $\|\cdot\|_{H^\gamma_p}$ in the definition of the norm in $H^\gamma_{p,\theta}$:

$$\|a\, u\|^p_{H^\gamma_{p,\theta}} = \sum_n e^{n\theta} \|\zeta a(e^n\cdot) u(e^n\cdot)\|^p_{H^\gamma_p}.$$

The proof of the second statement is similar. Lemma 5.4 is proved.

We now use the last three lemmas to construct the local inverse of the operator $(\!(\mathcal{A}, \mathcal{B})\!)$ near the boundary of G. In view of Assumption 3.4 and Lemma 5.4, we will assume with no loss of generality that there exists a constant N so that inequality (5.3) holds for all $\tilde{a}^{ij}(t, x)$ and inequality (5.4), for all $\tilde{\sigma}^{i\cdot}(t, x)$. Note that

$$(\bar{\mathcal{A}} - \bar{\mathcal{A}}_0)v(t, y) = \tilde{\eta}(y)(\tilde{a}^{ij}(t, y) - \tilde{a}^{ij}(t, y_0))v_{y^iy^j}(t, y) + \tilde{\eta}(y)\tilde{b}^i(t, y)v_{y^i}(t, y).$$

Consequently, by Assumptions 3.3 and 3.1, for every $\varepsilon > 0$, we can choose r_0 in Definition 2.1 so that

$$\|(\bar{\mathcal{A}} - \bar{\mathcal{A}}_0)v\|_{\mathbb{H}^{\gamma-1}_{p,\theta+p}(T)} \leq N \cdot \Big(\varepsilon\|v_y\|_{\mathbb{H}^\gamma_{p,\theta}(T)} + \|v\|_{\mathbb{H}^\gamma_{p,\theta}(T)}\Big)$$

and

$$\|(\bar{\mathcal{B}} - \bar{\mathcal{B}}_0)v)\|_{\mathbb{H}^\gamma_{p,\theta}(T;l_2)} \leq N\varepsilon\|v_y\|_{\mathbb{H}^\gamma_{p,\theta}(T)}$$

with N independent of T. As a result,

$$\|((\bar{\mathcal{A}} - \bar{\mathcal{A}}_0)v, (\bar{\mathcal{B}} - \bar{\mathcal{B}}_0)v)\|_{\mathcal{F}^\gamma_{p,\theta}(T)} \leq N\varepsilon\|v\|_{\mathfrak{H}^\gamma_{p,\theta}(T)} + N\|v\|_{\mathbb{H}^\gamma_{p,\theta}(T)}.$$

We now choose ε so that $N\varepsilon \leq \varepsilon_0$, where ε_0 is as in Theorem 5.3; the corresponding value of r_0 will be denoted by r_0^*. Then by Theorem 5.3 the operator $(\!(\bar{\mathcal{A}}, \bar{\mathcal{B}})\!)$ has a bounded inverse $\bar{\mathcal{R}}$.

Define the operator S_{x_0} (*local solution operator near the boundary*) by

$$S_{x_0}(f, g, u_0) = \Psi\bar{\mathcal{R}}(\tilde{f}, \tilde{g}, \tilde{u}_0),$$

where

$$\tilde{f}(t, y) = \tilde{\eta}(y)f(t, x),\ \tilde{g}(t, y) = \tilde{\eta}(y)g(t, x),\ \tilde{u}_0(y) = \tilde{\eta}(y)u_0(x),\ x = \psi^{-1}(y).$$

Properties of the operator S_{x_0} :

$$\|S_{x_0}(f,g,u_0)\|_{\mathfrak{H}^\gamma_{p,\theta}(T,G\cap B_{r_0^*/2}(x_0))} \le N\cdot\left(\|(f,g)\|_{\mathcal{F}^\gamma_{p,\theta}(T,G)} + \|u_0\|_{U^\gamma_{p,\theta}(G)}\right); \tag{5.5}$$

$$S_{x_0}((\!|\mathcal{A},\mathcal{B}|\!)(\chi u)) = \chi u \tag{5.6}$$

for all $u\in\mathfrak{H}^\gamma_{p,\theta}(T,G)$ and $\chi\in C_0^\infty(B_{r_0^*/2}(x_0))$.

These properties follow from the definition of S_{x_0} and from Lemma 5.3.

Now we can construct the global solution operator. Let $\chi_0,\chi_1,\dots,\chi_K$ be a partition of unity in G, corresponding to $r_0=r_0^*$; in particular, for $m=1,\dots,K$, the function χ_m is supported in $B_{r_0^*/2}(x_m)$ for some $x_m\in\partial G$. Define $S_0=\tilde{\mathcal{R}}_0$, the solution operator from Lemma 5.1, and $S_m=S_{x_m}$, $m=1,\dots,K$, the corresponding local solution operators near the boundary.

Proposition 5.5. (cf. Lemma 6.4.1 in [7].) *If* $u\in\mathfrak{H}^\gamma_{p,\theta}(T,G)$ *and*

$$(\!|\mathcal{A},\mathcal{B}|\!)u = (f,g,u_0), \tag{5.7}$$

then

$$u=\sum_{m=0}^K \chi_m S_m(\chi_m f - \hat{\mathcal{A}}_m u, \chi_m g - \hat{\mathcal{B}}_m u, \chi_m u_0), \tag{5.8}$$

where

$$\hat{\mathcal{A}}_m u(t,x) = \mathcal{A}(\chi_m u) - \chi_m\mathcal{A}u = 2a^{ij}(t,x)(\chi_m(x))_{x^i}u_{x^j}(t,x) + a^{ij}(t,x)(\chi_m(x))_{x^ix^j}u(t,x),$$

$$\hat{\mathcal{B}}^k_m u(t,x) = \mathcal{B}^k(\chi_m u) - \chi_m\mathcal{B}^k u = \sigma^{ik}(t,x)(\chi_m(x))_{x^i}u(t,x).$$

Proof. It follows from (5.2) and (5.6) that $\chi_m u = S_m(\,(\!|\mathcal{A},\mathcal{B}|\!)\,(\chi_m u)),\quad 0\le m\le K$, and also, by the definition of the operator $(\!|\mathcal{A},\mathcal{B}|\!)$,

$$\mathcal{A}(\chi_m u) = \chi_m f - \hat{\mathcal{A}}_m u,\ \ \mathcal{B}(\chi_m u) = \chi_m g - \hat{\mathcal{B}}_m u.$$

Proposition 5.5 is proved.

Proposition 5.6. (cf. Section 6.5 in [7].)

1. For every $(f,g)\in\mathcal{F}^\gamma_{p,\theta}(T,G)$ *and* $u_0\in U^\gamma_{p,\theta}(G)$ *there is a unique solution* $u\in\mathfrak{H}^\gamma_{p,\theta}(T,G)$ *of equation (5.8). This solution satisfies*

$$\|u\|_{\mathfrak{H}^\gamma_{p,\theta}(T,G)} \le N\cdot\left(\|(f,g)\|_{\mathcal{F}^\gamma_{p,\theta}(T,G)} + \|u_0\|_{U^\gamma_{p,\theta}(G)}\right). \tag{5.9}$$

2. If $u\in\mathfrak{H}^\gamma_{p,\theta}(T,G)$ *is a solution of (5.8), then* u *also satisfies (5.7).*

Proof. 1. To prove existence and uniqueness, it is enough to show that a sufficiently high power of the linear operator

$$\hat{S}: u\mapsto \sum_{m=0}^K \chi_m S_m(\hat{\mathcal{A}}_m u, \hat{\mathcal{B}}_m u, 0)$$

is a contraction in $\mathfrak{H}^\gamma_{p,\theta}(T,G)$. To this end note that

$$\|\hat{S}u\|^p_{\mathfrak{H}^\gamma_{p,\theta}(T,G)} \le N\|u\|^p_{\mathbb{H}^\gamma_{p,\theta}(T,G)} \le N\int_0^T \|u\|^p_{\mathfrak{H}^\gamma_{p,\theta}(t,G)}dt,$$

where the first inequality follows from the definitions of the operators $\hat{\mathcal{A}}_m$ and $\hat{\mathcal{B}}_m$ and the properties of the spaces, while the second inequality is (2.11). The result then follows after iteration.

The unique solution of (5.8) then satisfies

$$\|u\|^p_{\mathfrak{H}^\gamma_{p,\theta}(T,G)} \le N \cdot \left(\|(f,g)\|^p_{\mathcal{F}^\gamma_{p,\theta}(T,G)} + \|u_0\|^p_{U^\gamma_{p,\theta}(G)} + \int_0^T \|u\|^p_{\mathfrak{H}^\gamma_{p,\theta}(t,G)} dt \right)$$

so that (5.9) follows by the Gronwall inequality.

2. Assume that $u \in \mathfrak{H}^\gamma_{p,\theta}(T,G)$ is a solution of (5.8). Then, as an element of $\mathfrak{H}^\gamma_{p,\theta}(T,G)$, the function u satisfies $(\!|\mathcal{A},\mathcal{B}|\!)u = (f_0, g_0, u_0)$ with some $(f_0,g_0) \in \mathcal{F}^\gamma_{p,\theta}(T,G)$ and the same u_0 (by construction), and we have to show that $f = f_0$, $g = g_0$. By Proposition 5.5 the function u satisfies

$$u = \sum_{m=0}^K \chi_m S_m(\chi_m f_0 - \hat{\mathcal{A}}_m u, \chi_m g_0 - \hat{\mathcal{B}}_m u, \chi_m u_0). \tag{5.10}$$

Therefore, if $\hat{f} = f - f_0$, $\hat{g} = g - g_0$, then, by comparing (5.10) with (5.8) we get

$$\sum_{m=0}^K \chi_m S_m(\chi_m \hat{f}, \chi_m \hat{g}, 0) = 0,$$

and, after applying the operator $(\!|\mathcal{A},\mathcal{B}|\!)$ and using (5.2) and (5.6),

$$\hat{f} = \sum_{m=0}^K \hat{\mathcal{A}}_m S_m(\chi_m \hat{f}, \chi_m \hat{g}, 0),\ \ \hat{g} = \sum_{m=0}^K \hat{\mathcal{B}}_m S_m(\chi_m \hat{f}, \chi_m \hat{g}, 0).$$

To conclude that $\hat{f} = 0$, and $\hat{g} = 0$, it is enough to show that a sufficiently high power of the linear operator

$$\hat{S} : (f,g) \mapsto \left(\sum_{m=0}^K \hat{\mathcal{A}}_m S_m(\chi_m f, \chi_m g, 0), \sum_{m=0}^K \hat{\mathcal{B}}_m S_m(\chi_m f, \chi_m g, 0) \right)$$

is a contraction in $\mathcal{F}^\gamma_{p,\theta}(T,G)$. Clearly,

$$\begin{aligned}\|\hat{S}(f,g)\|^p_{\mathcal{F}^\gamma_{p,\theta}(T,G)} &\le N \sum_{m=1}^K \|S_m(\chi_m f, \chi_m g, 0)\|^p_{\mathbb{H}^\gamma_{p,\theta}(T,G)} \\ &\le N \sum_{m=1}^K \int_0^T \|S_m(\chi_m f, \chi_m g, 0)\|^p_{\mathfrak{H}^\gamma_{p,\theta}(t,G)} dt \le N \int_0^T \|(f,g)\|^p_{\mathcal{F}^\gamma_{p,\theta}(t,G)} dt,\end{aligned}$$

and the result follows after iteration.

Proposition 5.6 is proved.

We can now finish the proof of Theorem 3.2. It follows from Propositions 5.5 and 5.6 that the operator $(\!|\mathcal{A},\mathcal{B}|\!)$ has a bounded inverse $\mathcal{R}$. Therefore, every solution of (3.1) satisfies

$$u = \mathcal{R}(f(u), g(u), u_0),$$

where for simplicity the dependence of f and g on t, x is omitted. To prove Theorem 3.2, that is, to show that equation (3.1) has a unique solution in $\mathfrak{H}^{\gamma}_{p,\theta}(T,G)$ and the solution satisfies (3.4), it is enough to show that a sufficiently high power of the operator

$$S : u \mapsto \mathcal{R}(f(u,u_x), g(u), u_0)$$

is a contraction in $\mathfrak{H}^{\gamma}_{p,\theta}(T,G)$. We have

$$\begin{aligned} \|Su - Sv\|^p_{\mathfrak{H}^{\gamma}_{p,\theta}(T,G)} &= \|\mathcal{R}(f(u,u_x) - f(v,v_x), g(u) - g(v), 0)\|^p_{\mathfrak{H}^{\gamma}_{p,\theta}(T,G)} \\ &\leq N\|(f(u,u_x) - f(v,v_x), g(u) - g(v))\|^p_{\mathcal{F}^{\gamma}_{p,\theta}(T,G)} \\ &\leq N\varepsilon\|u - v\|^p_{\mathfrak{H}^{\gamma}_{p,\theta}(T,G)} + N\mu_{\varepsilon}\|u - v\|^p_{\mathbb{H}^{\gamma}_{p,\theta}(T,G)}, \end{aligned}$$

where the first inequality follows from (5.9) and the second, from Assumption 3.5. Therefore,

$$\|Su_1 - Su_2\|^p_{\mathfrak{H}^{\gamma}_{p,\theta}(T,G)} \leq N\varepsilon\|u_1 - u_2\|^p_{\mathfrak{H}^{\gamma}_{p,\theta}(T,G)} + N\int_0^T \|u_1 - u_2\|^p_{\mathfrak{H}^{\gamma}_{p,\theta}(t,G)} dt,$$

and it remains to iterate the last inequality with ε sufficiently small.

Theorem 3.2 is proved.

Acknowledgment. I wish to thank Professor N. V. Krylov who carefully read the earlier version of the paper and made many helpful suggestions.

References

[1] Z. Brzeźniak. Stochastic partial differential equations in M-type 2 Banach spaces. *Potential Anal.*, 4(1):1–45, 1995,

[2] F. Flandoli. *Regularity theory and stochastic flows for parabolic SPDEs*. Gordon and Breach Science Publishers, Stochastics Monographs, v. 9, 1995.

[3] D. Gilbarg and N. S. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Springer, Berlin, 1977.

[4] N. V. Krylov. An analytic approach to SPDEs. In: B. L. Rozovskii and R. Carmona, editors, *Stochastic Partial Differential Equations. Six Perspectives*, Mathematical Surveys and Monographs, pages 185–242, AMS, 1999.

[5] N. V. Krylov. A generalization of the Littlewood-Paley inequality and some other results related to stochastic partial differential equations. *Ulam Quarterly*, 2:16–26, 1994. On the Web at ftp://ftp.math.ufl.edu/pub/ulam/volume2/issue4/.

[6] N. V. Krylov. A W^n_2-Theory of the Dirichlet Problem for SPDEs in General Smooth Domains. *Probab. Theory Relat. Fields*, 98:389–421, 1994.

[7] N. V. Krylov. *Lectures on Elliptic and Parabolic Equations in Hölder Spaces*. American Mathematical Society, Graduate Studies in Mathematics, v. 12, Providence, RI, 1996.

[8] N. V. Krylov. On L_p-Theory of Stochastic Partial Differential Equations in the Whole Space. *SIAM J. Math. Anal.*, 27(2):313–340, 1996.

[9] N. V. Krylov. Weighted Sobolev Spaces and Laplace's and the Heat Equations in a Half Space. *Communications in PDE*, 1999. To appear.

[10] N. V. Krylov and S. V. Lototsky. A Sobolev Space Theory of SPDEs with Constant Coefficients in a Half Space. *SIAM J. Math. Anal.*, 1999. To appear.

[11] N. V. Krylov and S. V. Lototsky. A Sobolev Space Theory of SPDEs with Constant Coefficients on a Half Line. *SIAM J. Math. Anal.*, 30(2): 298–325, 1999. Also available at http://www.siam.org/journals/sima/30-2/32690.html,

[12] E. Stein. *Singular Integrals and Differentiability Properties of Functions.* Princeton University Press, Princeton, NJ, 1970.

[13] H. Triebel. *Theory of Function Spaces II.* Birkhauser, Basel, 1992.

[14] H. Triebel. *Interpolation Theory, Function Spaces, Differential Operators.* Johann Amrosius Barth, Heidelberg, 1995.

[15] J. B. Walsh. An Introduction to Stochastic Partial Differential Equations. In P. L. Hennequin, editor, *Ecole d'été de Probabilités de Saint-Flour, XIV, Lecture Notes in Mathematics*, volume 1180, pages 265–439, Springer, Berlin, 1984.