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A SOBOLEV SPACE THEORY OF SPDEs WITH CONSTANT COEFFICIENTS IN A HALF SPACE*

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Abstract. Equations of the form $du = (a^{ij}u_{x^ix^j} + D_i f^i)\,dt + \sum_k(\sigma^{ik}u_{x^i} + g^k)\,dw_t^k$ are considered for $t > 0$ and $x \in \mathbb{R}^d_+$. The unique solvability of these equations is proved in weighted Sobolev spaces with fractional positive or negative derivatives, summable to the power $p \in [2, \infty)$.

Key words. stochastic partial differential equations, Sobolev spaces with weights

AMS subject classifications. 60H15, 35R60

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Introduction. The main goal of this article is to extend the results of [6] to multidimensional cases. We are dealing with the equation

$$du = (a^{ij}u_{x^ix^j} + f^i_{x^i})\,dt + (\sigma^{ik}u_{x^i} + g^k)\,dw_t^k$$

given for $t \geq 0$ and $x \in \mathbb{R}^d_+ := \{x = (x^1, x') : x^1 > 0, x' \in \mathbb{R}^{d-1}\}$. Here w_t^k are independent one-dimensional Wiener processes, i and j run from 1 to d, k runs through $\{1, 2, \dots\}$ with the summation convention being enforced, and f^i and g^k are some given functions of (ω, t, x) defined for $i = 1, \dots, d$ and $k \geq 1$. The functions a^{ij} and σ^{ik} are assumed to depend only on ω and t, and in this sense we consider equations with "constant" coefficients. Without loss of generality we also assume that $a^{ij} = a^{ji}$.

As in [6], let us mention that such equations with a finite number of the processes w_t^k appear, for instance, in nonlinear filtering problems for partially observable diffusions (see [8]). Considering infinitely many w_t^k turns out to be instrumental in treating equations for measure valued processes, for instance, driven by space-time white noise (see [3] or [4]).

Our main goal is to prove the solvability of such equations in spaces similar to Sobolev spaces, in which derivatives are understood as generalized functions, the number of derivatives may be fractional or negative, and underlying power of summability is $p \in [2, \infty)$.

The motivation for this goal is explained in detail in [3] or [4], where an L_p-theory is developed for the equations in the whole space. We mention only that if $p = 2$, the theory was developed long ago and an account of it can be found, for instance, in [8]. The case of equations in domains is also treated in [8]. However, the solvability is proved only in spaces W_2^1 of functions having one generalized derivative in x square summable in (ω, t, x). It turns out that going to better smoothness of solutions is not possible in spaces W_2^n and one needs to consider Sobolev spaces with weights, allowing derivatives to blow up near the boundary. The theory of solvability in Hilbert spaces like W_2^n with weights is developed in [1] and [7], where n is an integer. Here we show

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what happens if one takes a fractional or negative number of derivatives and replaces 2 with any $p \geq 2$. By the way, according to [2], it is not possible to take $p < 2$ when a stochastic term is present in the equation.

One of the main difficulties in developing the theory presented below was finding the right spaces where to look for solutions. In the one-dimensional case $\mathbb{R}^d_+ = \mathbb{R}_+$ they have been found in [6]. It turns out that there are many multidimensional counterparts of spaces from [6]. The one which looks the most natural is to apply weights only to derivatives with respect to x^1. Indeed, why should we allow the derivatives with respect to tangential variables blow up near $x^1 = 0$? The equation is translation invariant with respect to x', isn't it? However, in such spaces it is impossible to solve equations with variable coefficients in smooth domains unless the coefficients not only are smooth with respect to x but also behave in a very restrictive way as x approaches the boundary. And, of course, considering equations with constant coefficients in half spaces aims at equations with variable coefficients in smooth domains.

This shows that one cannot just imitate the original definition of Sobolev spaces with weights $H^\gamma_{p,\theta}$ from [6]. However, it turns out that one can very naturally generalize to the multidimensional case an equivalent definition, looking much more complex, which is discovered in [6] and stated there as Theorem 1.11 (see Definition 1.1 below).

This article is organized as follows. In section 1 we present some definitions and facts from [5] on the basis of which, in section 2, we introduce the stochastic Banach spaces in which we are going to solve our equations. Our main result is given and proved in section 3. One auxiliary result used in section 3 is proved in section 4.

We finish the introduction with some notation. Everywhere, apart from section 1, we assume that $p \in [2, \infty)$. By $C^n_0(D)$ we denote the set of all n times continuously differentiable (real-valued) functions with compact support belonging to D. We denote

$$D_i = \partial/\partial x^i, \quad Du = u_x = (D_1 u, \dots, D_d u).$$

For a multi-index $\alpha = (\alpha_1, \dots, \alpha_d)$, where α_i's are nonnegative integers, denote

$$D^\alpha = D_1^{\alpha_1} \cdots D_d^{\alpha_d}, \quad |\alpha| = \alpha_1 + \dots + \alpha_d.$$

By $H^\gamma_p = H^\gamma_p(\mathbb{R}^d)$ we denote the space of Bessel potentials ($= (1-\Delta)^{-\gamma/2} L_p$) with norm $||\cdot||_{\gamma,p}$ (see [9]). For $\gamma = 0$, we have $H^0_p = L_p$ and we denote $||\cdot||_p = ||\cdot||_{0,p}$.

Any function given on $\mathbb{R}_+ := \mathbb{R}^1_+$ is also considered as a function on $\mathbb{R}^d_+$ independent of x'. Define M^α as the operator of multiplying by $(x^1)^\alpha$, $M = M^1$.

Finally, by $\mathcal{D}(R^d_+)$ we denote the space of all distributions on $\mathbb{R}^d_+$ that is of continuous linear functionals on $C^\infty_0(\mathbb{R}^d_+)$.

1. Sobolev spaces with weights. Here we collect some definitions and facts from [5].

DEFINITION 1.1. *Take and fix a nonnegative function $\zeta \in C^\infty_0(\mathbb{R}_+)$ such that*

$$\sum_{n=-\infty}^{\infty} \zeta^p(e^{x-n}) \geq 1 \quad \textit{for all} \;\; x \in \mathbb{R}. \tag{1.1}$$

For $\gamma, \theta \in \mathbb{R}$, and $p \in (1, \infty)$ let $H^\gamma_{p,\theta}$ be the set of all distributions u on $\mathbb{R}^d_+$ such that

$$||u||^p_{\gamma,p,\theta} := \sum_{n=-\infty}^{\infty} e^{n\theta} ||u(e^n \cdot)\zeta||^p_{\gamma,p} = \sum_{n=-\infty}^{\infty} e^{n\theta} ||(1-\Delta)^{\gamma/2}(u(e^n \cdot)\zeta)||^p_p < \infty. \tag{1.2}$$

Denote $L_{p,\theta} = H^0_{p,\theta}$.

In the same way, for any separable Banach space X, *we introduce the spaces* $H^\gamma_{p,\theta}(X)$ *of* X*-valued functions by replacing* $(1-\Delta)^{\gamma/2}(u(e^n\cdot)\zeta)$ *in* (1.2) *with* $|(1-\Delta)^{\gamma/2}(u(e^n\cdot)\zeta)|_X$.

LEMMA 1.2. (i) *The spaces* $H^\gamma_{p,\theta}$ *are Banach spaces and the space* $C_0^\infty(\mathbb{R}^d_+)$ *is dense in* $H^\gamma_{p,\theta}$.

(ii) *For different* ζ *satisfying* (1.1), *we get the same spaces* $H^\gamma_{p,\theta}$ *with equivalent norms. Furthermore, if* $\eta \in C_0^\infty(\mathbb{R}^d_+)$, *then for any* $u \in \mathcal{D}(\mathbb{R}^d_+)$ *and* γ, θ, p *we have*

$$\sum_{n=-\infty}^{\infty} e^{n\theta}||u(e^n\cdot)\eta||^p_{\gamma,p} \le N \sum_{n=-\infty}^{\infty} e^{n\theta}||u(e^n\cdot)\zeta||^p_{\gamma,p},$$

where N *depends only on* γ, θ, p, η, d *(and* ζ*).*

(iii) *Let* $\alpha \in \mathbb{R}$. *We have* $u \in H^\gamma_{p,\theta}$ *if and only if* $u = M^\alpha v$ *with* $v \in H^\gamma_{p,\theta+\alpha p}$. *Hence,*

$$M^\alpha H^\gamma_{p,\theta+\alpha p} = H^\gamma_{p,\theta}.$$

In addition,

$$||u||_{\gamma,p,\theta} \le N||M^{-\alpha}u||_{\gamma,p,\theta+\alpha p} \le N||u||_{\gamma,p,\theta},$$

where N *are independent of* u.

(iv) *The space* $L_{p,\theta}$ *coincides with the space of functions summable to the power* p *over* $\mathbb{R}^d_+$ *with respect to the measure* $(x^1)^{\theta-d}\,dx$.

(v) *If* γ *is a nonnegative integer, then the space* $H^\gamma_{p,\theta}$ *is*

$$\{u : u, x^1u_x, \dots, (x^1)^{|\alpha|}D^\alpha u \in L_{p,\theta} \quad \textit{for all} \;\; \alpha : |\alpha| \le \gamma\}$$

with a natural norm.

The spaces $H^\gamma_{p,\theta}$ are introduced and studied in [5] for all $\theta \in \mathbb{R}$. However, below in this article we always suppose that $d-1 < \theta < p+d-1$. For this range of θ, the following results, again borrowed from [5], are true.

LEMMA 1.3. *Let* $d-1 < \theta < p+d-1$.

(i) *The following conditions are equivalent:*

(a) $u \in H^\gamma_{p,\theta}$,

(b) $u \in H^{\gamma-1}_{p,\theta}$ *and* $Mu_x \in H^{\gamma-1}_{p,\theta}$,

(c) $u \in H^{\gamma-1}_{p,\theta}$ *and* $(Mu)_x \in H^{\gamma-1}_{p,\theta}$.

In addition, under either of these three conditions for some constants $N = N(\gamma, p, \theta, d)$ *we have*

$$||u||_{\gamma,p,\theta} \le N||Mu_x||_{\gamma-1,p,\theta} \le N||u||_{\gamma,p,\theta},$$

$$||u||_{\gamma,p,\theta} \le N||(Mu)_x||_{\gamma-1,p,\theta} \le N||u||_{\gamma,p,\theta}.$$

(ii) *We have* $M^{-1}u \in H^\gamma_{p,\theta}$ *if and only if*

$$u_x \in H^{\gamma-1}_{p,\theta} \quad \textit{and} \quad M^{-1}u \in \bigcup_\mu H^\mu_{p,\theta}.$$

Moreover, there exist constants $N = N(d,\gamma,\mu,\theta,p)$ such that, for any $\mu \le \gamma$ and $M^{-1}u \in H^{\gamma}_{p,\theta}$, we have

$$||M^{-1}u||_{\gamma,p,\theta} \le N||u_x||_{\gamma-1,p,\theta} \le N||M^{-1}u||_{\gamma,p,\theta}.$$

(iii) *The operator $\mathcal{L} := M^2\Delta + 2MD_1$ is a bounded operator from $H^{\gamma}_{p,\theta}$ onto $H^{\gamma-2}_{p,\theta}$ and its inverse is also bounded.*

(iv) *There is a bounded operator*

$$Q : u \in H^{\gamma}_{p,\theta} \to Qu = (Q_1 u, \dots, Q_d u) \in (H^{\gamma+1}_{p,\theta})^d$$

such that, for any $u \in H^{\gamma}_{p,\theta}$, we have $u = MD_iQ_iu$.

2. Stochastic Banach spaces on $\mathbb{R}^d_+$. Let $(\Omega,\mathcal{F},P)$ be a complete probability space, $(\mathcal{F}_t, t \ge 0)$ be an increasing filtration of σ-fields $\mathcal{F}_t \subset \mathcal{F}$ containing all P-null subsets of Ω, and $\mathcal{P}$ be the predictable σ-field generated by $(\mathcal{F}_t, t \ge 0)$. Let $\{w^k_t; k = 1,2,\dots\}$ be a family of independent one-dimensional $\mathcal{F}_t$-adapted Wiener processes defined on $(\Omega,\mathcal{F},P)$. We are going to use the Banach spaces $\mathbb{H}^{\gamma}_p(\tau)$, $\mathbb{H}^{\gamma}_p(\tau,l_2)$, and $\mathcal{H}^{\gamma}_p(\tau)$ introduced in [3] or [4].

Throughout the remaining part of the paper we assume that

$$d-1 < \theta < p+d-1.$$

Definition 2.1. *Let τ be a stopping time and f and g^k, $k = 1,2,\dots$, be $\mathcal{D}(\mathbb{R}^d_+)$-valued $\mathcal{P}$-measurable functions defined on $(\!(0,\tau]\!]$. We write $f \in \mathbb{H}^{\gamma}_{p,\theta}(\tau)$ and $g \in \mathbb{H}^{\gamma}_{p,\theta}(\tau,l_2)$ if and only if $f \in L_p((\!(0,\tau]\!]; H^{\gamma}_{p,\theta})$ and $g \in L_p((\!(0,\tau]\!]; H^{\gamma}_{p,\theta}(l_2))$, respectively. We also denote*

$$\mathbb{H}^{\gamma}_{p,\theta} = \mathbb{H}^{\gamma}_{p,\theta}(\infty), \quad \mathbb{H}^{\gamma}_{p,\theta}(l_2) = \mathbb{H}^{\gamma}_{p,\theta}(\infty,l_2), \quad \mathbb{L}_{\dots}\dots = \mathbb{H}^0_{\dots}\dots.$$

In the case $f \in \mathbb{H}^{\gamma}_{p,\theta}(\tau)$ and $g \in \mathbb{H}^{\gamma+1}_{p,\theta}(\tau,l_2)$ we write $(f,g) \in \mathcal{F}^{\gamma}_{p,\theta}(\tau)$ and define

$$||f||_{\mathbb{H}^{\gamma}_{p,\theta}(\tau)} = E\int_0^{\tau} ||f(t)||^p_{\gamma,p,\theta}\,dt, \quad ||g||_{\mathbb{H}^{\gamma}_{p,\theta}(\tau,l_2)} = E\int_0^{\tau} ||g(t)||^p_{H^{\gamma}_{p,\theta}(l_2)}\,dt,$$

$$||(f,g)||_{\mathcal{F}^{\gamma}_{p,\theta}(\tau)} = ||f||_{\mathbb{H}^{\gamma}_{p,\theta}(\tau)} + ||g||_{\mathbb{H}^{\gamma+1}_{p,\theta}(\tau,l_2)}.$$

Finally, we introduce spaces of initial data. We write $u_0 \in U^{\gamma}_{p,\theta}$ if and only if $M^{2/p-1}u(0,\cdot) \in L_p(\Omega,\mathcal{F}_0,H^{\gamma-2/p}_{p,\theta})$ (or by Lemma 1.2, part (iii)*, if and only if $u(0,\cdot) \in L_p(\Omega,\mathcal{F}_0,H^{\gamma-2/p}_{p,\theta+2-p})$) and denote*

$$||u(0,\cdot)||^p_{U^{\gamma}_{p,\theta}} = E||M^{2/p-1}u(0,\cdot)||^p_{\gamma-2/p,p,\theta}.$$

Definition 2.2. *For a $\mathcal{D}(\mathbb{R}^d_+)$-valued function u defined on $\Omega \times ([0,\tau] \cap [0,\infty))$ with $u(0,\cdot) \in U^{\gamma}_{p,\theta}$, we write $u \in \mathfrak{H}^{\gamma}_{p,\theta}(\tau)$ if and only if $M^{-1}u \in \mathbb{H}^{\gamma}_{p,\theta}(\tau)$ and there exists $(f,g) \in \mathcal{F}^{\gamma-2}_{p,\theta}(\tau)$ such that, for any $\phi \in C^{\infty}_0(\mathbb{R}^d_+)$, with probability one, we have*

$$(u(t,\cdot),\phi) = (u(0,\cdot),\phi) + \int_0^t (M^{-1}f(s,\cdot),\phi)\,ds + \sum_{k=1}^{\infty}\int_0^t (g^k(s,\cdot),\phi)\,dw^k_s \tag{2.1}$$

for all $t \in [0,\tau] \cap [0,\infty)$. *In this situation we also write* $M^{-1}f = \tilde{\mathbb{D}}u$, $g = \tilde{\mathbb{S}}u$,

$$du = M^{-1}f\,dt + g^k\,dw_t^k$$

and we define $\mathfrak{H}^{\gamma}_{p,\theta,0}(\tau) = \mathfrak{H}^{\gamma}_{p,\theta}(\tau) \cap \{u : u(0,\cdot) = 0\}$,

$$||u||^p_{\mathfrak{H}^{\gamma}_{p,\theta}(\tau)} = ||u_x||^p_{\mathbb{H}^{\gamma-1}_{p,\theta}(\tau)} + ||(f,g)||^p_{\mathcal{F}^{\gamma-2}_{p,\theta}(\tau)} + ||u(0,\cdot)||^p_{U^{\gamma}_{p,\theta}}. \tag{2.2}$$

As always, we drop τ *in* $\mathfrak{H}^{\gamma}_{p,\theta}(\tau)$ *and* $\mathcal{F}^{\gamma}_{p,\theta}(\tau)$ *if* $\tau = \infty$.

Remark 2.3. If $u \in \mathfrak{H}^{\gamma}_{p,\theta}(\tau)$ and $\phi(x) = \phi(x^1)$ with $\phi \in C_0^{\infty}(\mathbb{R}_+)$, then ϕu lies in $\mathcal{H}^{\gamma}_p(\tau)$. By Theorem 2.7 of [4] this implies that if $u \in \mathfrak{H}^{\gamma}_{p,\theta}(\tau)$ and $||u||_{\mathfrak{H}^{\gamma}_{p,\theta}(\tau)} = 0$, then u is indistinguishable from zero.

Of course, we identify elements of $\mathfrak{H}^{\gamma}_{p,\theta}(\tau)$ which are indistinguishable.

Remark 2.4 (cf. Remark 2.3 in [4]). Given $u \in \mathfrak{H}^{\gamma}_{p,\theta}(\tau)$, there exists only one couple of functions f and g in Definition 2.2. Therefore, the notations $M^{-1}f = \tilde{\mathbb{D}}u$, $g = \tilde{\mathbb{S}}u$, and (2.2) make sense.

It is also worth noting that the last series in (2.1) converges uniformly in t on each interval $[0, \tau \wedge T]$, $T \in (0,\infty)$, in probability.

Remark 2.5. It follows from Lemma 1.3 part (ii) that the condition $M^{-1}u \in \mathbb{H}^{\gamma}_{p,\theta}(\tau)$ can be replaced with

$$M^{-1}u \in \bigcup_{\mu} \bigcap_{T>0} \mathbb{H}^{\mu}_{p,\theta}(\tau \wedge T) \quad \text{and} \quad u_x \in \mathbb{H}^{\gamma-1}_{p,\theta}(\tau).$$

Also in (2.2), replacing the norm $||u_x||_{\mathbb{H}^{\gamma-1}_{p,\theta}(\tau)}$ with $||M^{-1}u||_{\mathbb{H}^{\gamma}_{p,\theta}(\tau)}$ leads to an equivalent norm.

Remark 2.6. In the same way as in Remark 2.6 of [6] one proves that the spaces $\mathfrak{H}^{\gamma}_{p,\theta}(\tau)$ and $\mathfrak{H}^{\gamma}_{p,\theta,0}(\tau)$ are Banach spaces.

Remark 2.7. The term $M^{-1}f$ in (2.1) can be replaced with $D_i f^i$ for $f^i := Q_i f \in \mathbb{H}^{\gamma-1}_{p,\theta}(\tau)$, $i = 1,\dots,d$ (see Lemma 1.3), and the norm $||f||_{\mathbb{H}^{\gamma-2}_{p,\theta}(\tau)}$ (participating in (2.2)) with $\sum_i ||f^i||_{\mathbb{H}^{\gamma-1}_{p,\theta}(\tau)}$, the latter leading to an equivalent norm.

Remark 2.8. If $u \in \mathfrak{H}^{\gamma}_{p,\theta}(\tau)$, then $MD_iu \in \mathfrak{H}^{\gamma-1}_{p,\theta}(\tau)$ for $i = 1,\dots d$, and

$$||MDu||_{\mathfrak{H}^{\gamma-1}_{p,\theta}(\tau)} \le N(\gamma,\theta,p,d)||u||_{\mathfrak{H}^{\gamma}_{p,\theta}(\tau)}.$$

Indeed, by Lemma 1.3, $M^{-1}(MD_iu) = D_iu \in \mathbb{H}^{\gamma-1}_{p,\theta}(\tau)$ and by Remark 2.7, $du = D_jf^j\,dt + g^k\,dw_t^k$ with $f^j \in \mathbb{H}^{\gamma-1}_{p,\theta}(\tau)$ and $g \in \mathbb{H}^{\gamma-1}_{p,\theta}(\tau, l_2)$, so that

$$d(MD_iu) = M^{-1}M^2D_iD_jf^j\,dt + MD_ig^k\,dw_t^k,$$

where $M^2D_iD_jf^j = MD_iMD_jf^j - \delta^{1i}MD_jf^j$. By Lemma 1.3

$$||M^2D_iD_jf^j||_{\mathbb{H}^{\gamma-3}_{p,\theta}(\tau)} \le N\sum_j ||f^j||_{\mathbb{H}^{\gamma-1}_{p,\theta}(\tau)} \le N||f||_{\mathbb{H}^{\gamma-2}_{p,\theta}(\tau)},$$

$$||MD_ig||_{\mathbb{H}^{\gamma-2}_{p,\theta}(\tau,l_2)} \le N||g||_{\mathbb{H}^{\gamma-1}_{p,\theta}(\tau,l_2)},$$

$$\begin{aligned} &||M^{2/p-1}MD_iu(0,\cdot)||_{\gamma-1-2/p,p,\theta} \\ &\quad = ||MD_i(M^{2/p-1}u(0,\cdot)) - \delta^{1i}(2/p-1)M^{2/p-1}u(0,\cdot)||_{\gamma-1-2/p,p,\theta} \\ &\quad \le N||M^{2/p-1}u(0,\cdot)||_{\gamma-2/p,p,\theta}. \end{aligned}$$

THEOREM 2.9. *For any nonnegative integer $n \geq \gamma$, the set*

$$\mathfrak{H}^n_{p,\theta}(\tau) \bigcap \bigcup_{k=1}^{\infty} \bigcap_{T\in(0,\infty)} L_p(\Omega, C([0,\tau\wedge T], C^n_0(G_k))), \tag{2.3}$$

where $G_k = (1/k, k) \times \{|x'| < k\}$, is everywhere dense in $\mathfrak{H}^\gamma_{p,\theta}(\tau)$.

Proof. Corollary 1.20 of [5] states that there exists a sequence of functions $\eta_k \in C^\infty_0(\mathbb{R}_+)$ vanishing near zero and infinity and such that, for any $u \in H^\gamma_{p,\theta}$, we have

$$||\eta_k u||_{\gamma,p,\theta} \leq N||u||_{\gamma,p,\theta}, \quad ||\eta_k u - u||_{\gamma,p,\theta} \to 0$$

as $k \to \infty$, where N is independent of k and u. Obviously, if $u \in \mathfrak{H}^\gamma_{p,\theta}(\tau)$, then $\eta_k u \in \mathfrak{H}^\gamma_{p,\theta}(\tau)$ and by Remark 2.5 and the above result of [5] we get that $\eta_k u \to u$ in $\mathfrak{H}^\gamma_{p,\theta}(\tau)$.

To prove the theorem it remains to show only that any $u \in \mathfrak{H}^\gamma_{p,\theta}(\tau)$, vanishing outside some G_k, can be approximated by elements of set (2.3). To do this, notice that for such u its $\mathfrak{H}^\gamma_{p,\theta}(\tau)$-norm is equivalent to $\mathcal{H}^\gamma_p(\tau)$-norm. Next, take a function $\xi \in C^\infty_0(\mathbb{R}^d_+)$ with unit integral and for $\varepsilon > 0$ define $\xi_\varepsilon(x) = \varepsilon^{-d}\xi(x/\varepsilon)$, $u^{(\varepsilon)}(t,x) := \xi_\varepsilon(x) * u(t,x)$. It is easy to check that for ε small enough (for instance, such that $u^{(\varepsilon)}$ vanishes when x^1 is close to zero or infinity), we have $u^{(\varepsilon)} \in \mathfrak{H}^n_{p,\theta}(\tau)$ and $u^{(\varepsilon)} \in \mathcal{H}^n_p(\tau)$ for all n. In addition, by well-known properties of mollified functions, $u^{(\varepsilon)}$ converge to u in $\mathcal{H}^\gamma_p(\tau)$- and $\mathfrak{H}^\gamma_{p,\theta}(\tau)$-norm as $\varepsilon \downarrow 0$. Of course, $u^{(\varepsilon)}(t,x)$ is infinitely differentiable with respect to x.

Finally, since $u \in \mathcal{H}^\gamma_p(\tau)$, by Theorems 7.1 and 7.2 of [4] we have

$$u \in L_p(\Omega, C([0,\tau\wedge T], H^{\gamma-1}_p)). \tag{2.4}$$

In addition, by Sobolev's embedding theorems and by properties of mollifiers, for any $v \in H^{\gamma-1}_p$ and multi-index α with $|\alpha| = n$,

$$|D^\alpha v^{(\varepsilon)}| \leq N||v^{(\varepsilon)}||_{d+n,p} \leq N\varepsilon^{-\kappa}||v||_{\gamma-1,p},$$

where N and κ are independent of v (and ε). This and (2.4) show that

$$u^{(\varepsilon)} \in L_p(\Omega, C([0,\tau\wedge T], C^n_0(\mathbb{R}^d_+))).$$

The theorem is proved.

By repeating the proof of Theorem 2.9 with obvious changes we obtain one more useful result.

THEOREM 2.10. *The statement of Theorem* 2.9 *remains true if we replace $\mathfrak{H}^n_{p,\theta}(\tau)$ and $\mathfrak{H}^\gamma_{p,\theta}(\tau)$ with $\mathbb{H}^n_{p,\theta}(\tau)$ and $\mathbb{H}^\gamma_{p,\theta}(\tau)$, respectively, or with $\mathbb{H}^n_{p,\theta}(\tau, l_2)$ and $\mathbb{H}^\gamma_{p,\theta}(\tau, l_2)$, respectively.*

As in the one-dimensional case (cf. [6]), the following embedding theorem presents certain interest.

THEOREM 2.11. *Let $T \in (0,\infty)$ be a constant and let $\tau \leq T$. Then for any function $u \in \mathfrak{H}^\gamma_{p,\theta,0}(\tau)$, we have*

$$E\sup_{t\leq\tau} ||u(t,\cdot)||^p_{\gamma-1,p,\theta} \leq N(p,d,\theta,\gamma)T^{(p-2)/2}||u||^p_{\mathfrak{H}^\gamma_{p,\theta}(\tau)}. \tag{2.5}$$

To prove this theorem we use the following fact which is similar to Remark 2.2 of [3] or Remark 4.11 of [4]. Its proof can be obtained just by repeating the proof of Lemma 2.12 of [6] and is omitted.

LEMMA 2.12. *Let $T \in (0,\infty)$ be a constant and let $\tau \le T$. Let $u \in \mathcal{H}^{\gamma}_{p,0}(\tau)$ and $du = f\,dt + g^k\,dw^k_t$. Then for any constant $c > 0$,*

$$E\sup_{t\le\tau}||u_x(t,\cdot)||^p_{\gamma-2,p} \le N(p,d)T^{(p-2)/2}(c||u_{xx}||^p_{\mathbb{H}^{\gamma-2}_p(\tau)} + c^{-1}||f||^p_{\mathbb{H}^{\gamma-2}_p(\tau)} + ||g_x||^p_{\mathbb{H}^{\gamma-2}_p(\tau,l_2)}).$$

Proof of Theorem 2.11. We proceed as in the proof of Theorem 2.11 of [6]. We have

$$E\sup_{t\le\tau}||u(t,\cdot)||^p_{\gamma-1,p,\theta} \le \sum_{n=-\infty}^{\infty} e^{n\theta}E\sup_{t\le\tau}||u(t,e^n\cdot)\zeta||^p_{\gamma-1,p}. \tag{2.6}$$

Define $u_n(t,x) := \zeta(x)u(t,e^nx)$ and notice that, since the support of $\zeta(x)u(t,e^nx)$ is not larger than the one of $\zeta(x)$, we have (see, for instance, Remark 1.12 of [5])

$$||u_n(t,\cdot)||_{\gamma-1,p} \le N||u_{nx}(t,\cdot)||_{\gamma-2,p}. \tag{2.7}$$

To estimate the right-hand side of (2.7), assume that $du = M^{-1}f\,dt + g^k\,dw^k_t$. Then

$$du_n(t,x) = f_n(t,x)\,dt + g_n(t,x)\,dw^k_t,$$

where $f_n(t,x) = (M^{-1}\zeta)(x)e^{-n}f(t,e^nx)$, $g_n(t,x) = \zeta(x)g(t,e^nx)$. By Lemma 2.12 with $c = e^{-np}$,

$$E\sup_{t\le\tau}||u_{nx}(t,\cdot)||^p_{\gamma-2,p} \le NT^{(p-2)/2}(e^{-np}||u_{nxx}||^p_{\mathbb{H}^{\gamma-2}_p(\tau)}$$

$$+e^{np}||f_n||^p_{\mathbb{H}^{\gamma-2}_p(\tau)} + ||g_{nx}||^p_{\mathbb{H}^{\gamma-2}_p(\tau,l_2)}). \tag{2.8}$$

Furthermore, $||g_{nx}||_{H^{\gamma-2}_p(l_2)} \le ||g_n||_{H^{\gamma-1}_p(l_2)}$ and

$$\sum_{n=-\infty}^{\infty} e^{n\theta}||g_n||^p_{\mathbb{H}^{\gamma-1}_p(\tau,l_2)} = ||g||^p_{\mathbb{H}^{\gamma-1}_{p,\theta}(\tau,l_2)} \le N||u||^p_{\mathfrak{H}^{\gamma}_{p,\theta}(\tau)}.$$

Also,

$$\sum_{n=-\infty}^{\infty} e^{n(\theta+p)}||f_n||^p_{\mathbb{H}^{\gamma-2}_p(\tau)} = \sum_{n=-\infty}^{\infty} e^{n\theta}||f(\cdot,e^n\cdot)M^{-1}\zeta||^p_{\mathbb{H}^{\gamma-2}_p(\tau)}$$

$$\le N||f||^p_{\mathbb{H}^{\gamma-2}_{p,\theta}(\tau)} \le N||u||^p_{\mathfrak{H}^{\gamma}_{p,\theta}(\tau)},$$

$$\sum_{n=-\infty}^{\infty} e^{n(\theta-p)}||u_{nxx}||^p_{\mathbb{H}^{\gamma-2}_p(\tau)} \le \sum_{n=-\infty}^{\infty} e^{n(\theta-p)}||u_n||^p_{\mathbb{H}^{\gamma}_p(\tau)}$$

$$= \sum_{n=-\infty}^{\infty} e^{n(\theta-p)} ||(M^{-1}u)(\cdot, e^n \cdot)M\zeta||^p_{\mathbb{H}^\gamma_p(\tau)}$$

$$\leq N||M^{-1}u||^p_{\mathbb{H}^\gamma_{p,\theta}(\tau)} \leq N||u||^p_{\mathfrak{H}^\gamma_{p,\theta}(\tau)}.$$

By combining this with (2.8) and (2.6) we get (2.5). The theorem is proved.

As always the main role is played by the spaces $\mathfrak{H}^\gamma_{p,\theta,0}(\tau)$ of functions with zero as an initial condition. In connection with this it is worth noting that while constructing our theory we could replace

$$||u(0,\cdot)||^p_{U^\gamma_{p,\theta}} := E||M^{2/p-1}u(0,\cdot)||^p_{H^{\gamma-2/p}_{p,\theta}} \tag{2.9}$$

with

$$\inf\{||v_x||_{\mathbb{H}^{\gamma-1}_{p,\theta}} + ||\tilde{\mathbb{D}}v||_{\mathbb{H}^{\gamma-2}_{p,\theta}} + ||\tilde{\mathbb{S}}v||_{\mathbb{H}^{\gamma-1}_{p,\theta}} : u - v \in \mathfrak{H}^\gamma_{p,\theta,0}\}.$$

Such an axiomatic approach to defining a norm of $u(0,\cdot)$ yields, of course, the solvability results for the widest possible class of initial data, namely, for those which are extendible at least in some way for $t > 0$. However, in applications we often want to know how to describe "admissible" initial data by knowing only their analytic properties.

A partial answer to this question is given in the following theorem, which also shows why we use the norm given by (2.9). For the only case, which we need, $\gamma = 2$, the proof of this theorem can be obtained in the same way as Theorem 2.13 of [6]. For arbitrary γ and parabolic operators with coefficients depending only on time instead of Δ this theorem is proved in [5].

THEOREM 2.13. *If $\gamma \in \mathbb{R}$, $d-1 < \theta < p+d-1$, and $1 < p < \infty$, then, for every u_0 satisfying $u_0 \in U^\gamma_{p,\theta}$, in the space $\mathfrak{H}^\gamma_{p,\theta}$ there exists a unique solution of the heat equation $du = \Delta u\, dt$ with initial data $u(0,\cdot) = u_0$. Moreover,*

$$||u||_{\mathfrak{H}^\gamma_{p,\theta}} \leq N(d,\gamma,p,\gamma)||u_0||_{U^\gamma_{p,\theta}}.$$

3. SPDEs with constant coefficients in $\mathbb{R}^d_+$. Take a stopping time τ. On $[(0,\tau]] \cap (0,\infty)] \times \mathbb{R}^d_+$ we will be dealing with the following equation:

$$du = (a^{ij}u_{x^ix^j} + M^{-1}f)\, dt + (\sigma^{ik}u_{x^i} + g^k)\, dw^k_t \tag{3.1}$$

with initial condition $u|_{t=0} = u_0$, where u_0 is a $\mathcal{D}(\mathbb{R}^d_+)$-valued, $\mathcal{F}_0$-measurable random variable, f and g^k are $\mathcal{D}(\mathbb{R}^d_+)$-valued $\mathcal{P}$-measurable functions, a^{ij} and σ^{ik} are real-valued $\mathcal{P}$-measurable functions, u is an unknown $\mathcal{D}(\mathbb{R}^d_+)$-valued function, and the equation is understood in the sense of distributions as follows. We say that u is a solution of (3.1) with initial data u_0 if for any test function $\phi \in C^\infty_0(\mathbb{R}^d_+)$ we have

$$\begin{aligned}(u(t\wedge\tau,\cdot),\phi) &= (u_0,\phi)\\ &+ \int_0^{t\wedge\tau} \big[\sum_{i,j=1}^d a^{ij}(s)(u(s,\cdot),\phi_{x^ix^j}) + (f(s,\cdot),M^{-1}\phi)\big]\, ds\\ &+ \sum_{k=1}^\infty \int_0^{t\wedge\tau} \big[-\sum_{i=1}^d \sigma^{ik}(s)(u(s,\cdot),\phi_{x^i}) + (g^k(s,\cdot),\phi)\big]\, dw^k_s\end{aligned} \tag{3.2}$$

for all $t > 0$ with probability one, where all integrals are assumed to have sense and the last series is assumed to converge uniformly on each interval of time $[0, T]$ in probability, where T is any finite constant.

Remark 3.1. If a function u belongs to $\mathfrak{H}^{\gamma}_{p,\theta}(\tau)$, then it satisfies (3.1) with

$$f = M\left(\tilde{\mathbb{D}}u - a^{ij}D_iD_ju\right), \tag{3.3}$$
$$g^k = \tilde{\mathbb{S}}^k u - \sigma^{ik}D_iu.$$

In addition (see Lemma 1.3), we have $f \in \mathbb{H}^{\gamma-2}_{p,\theta}(\tau)$ and $g \in \mathbb{H}^{\gamma-1}_{p,\theta}(\tau, l_2)$. Below we show that under additional assumptions on θ, a, and σ the mapping $u \to (f, g)$ is onto.

Assumption 3.2. There exist constants $\delta_0, \delta_1 \in (0, 1]$ such that, for every (ω, t) and every $\xi \in \mathbb{R}^d$,

$$\delta_0|\xi|^2 \le \delta_1 a^{ij}(t)\xi^i\xi^j \le \bar{a}^{ij}\xi^i\xi^j \le a^{ij}(t)\xi^i\xi^j \le \delta_0^{-1}|\xi|^2,$$

where

$$\bar{a}^{ij} := a^{ij}(t) - \alpha^{ij}(t), \quad \alpha^{ij}(t) = \tfrac{1}{2}\sigma^{ik}(t)\sigma^{jk}(t).$$

Here is our main result.

THEOREM 3.3. *Let $d - 1 < \theta < p + d - 1$, $2 \le p < \infty$, $\gamma \in \mathbb{R}$, $f \in \mathbb{H}^{\gamma-2}_{p,\theta}(\tau)$, $g \in \mathbb{H}^{\gamma-1}_{p,\theta}(\tau, l_2)$, and $u_0 \in U^{\gamma}_{p,\theta}$. Assume that*

$$d - 1 + p\left[1 - \frac{1}{p(1 - \delta_1) + \delta_1}\right] < \theta < d - 1 + p. \tag{3.4}$$

Then (3.1) *or equivalently* (3.3) *with initial data u_0 has a unique solution in $\mathfrak{H}^{\gamma}_{p,\theta}(\tau)$. In addition, for this solution it holds that*

$$||u||^p_{\mathfrak{H}^{\gamma}_{p,\theta}(\tau)} \le N\left(||f||^p_{\mathbb{H}^{\gamma-2}_{p,\theta}(\tau)} + ||g||^p_{\mathbb{H}^{\gamma-1}_{p,\theta}(\tau,l_2)} + ||u_0||^p_{U^{\gamma}_{p,\theta}}\right), \tag{3.5}$$

where $N = N(\gamma, \theta, p, d, \delta_0, \delta_1)$.

Remark 3.4. By Remark 2.7, one gets a statement equivalent to Theorem 3.3 if one replaces $M^{-1}f$ in (3.1) with D_if^i for certain $f^i \in \mathbb{H}^{\gamma-1}_{p,\theta}(\tau)$ and replaces $||f||^p_{\mathbb{H}^{\gamma-2}_{p,\theta}(\tau)}$ in (3.5) with $\sum_i ||f^i||^p_{\mathbb{H}^{\gamma-1}_{p,\theta}(\tau)}$.

Remark 3.5. If $\sigma \equiv 0$, then one can take $\delta_1 = 1$ and (3.4) becomes $d - 1 < \theta < d - 1 + p$. Furthermore, it is easy to see that, for any σ, condition (3.4) is satisfied if $d - 2 + p \le \theta < d - 1 + p$.

Remark 3.6. It is worth noting that if $\theta \ge p + d - 1$ or $\theta \le d - 1$, then the statement of Theorem 3.3 is false even in the case of the heat equation. This can be shown by simple examples.

The proof of this theorem is based on two lemmas, the first of which we prove in section 4.

LEMMA 3.7. *Theorem* 3.3 *holds if $\gamma = 2$.*

LEMMA 3.8. *Let the assumptions of Theorem* 3.3 *be satisfied and let $\mu \le \gamma$. Let $\theta_1 \in \mathbb{R}$ and let $u \in \mathfrak{H}^{\mu}_{p,\theta_1}(\tau)$ be a solution of* (3.1) *with initial condition u_0. Assume that $M^{-1}u \in \mathbb{H}^{\mu}_{p,\theta}(\tau)$. Then $u \in \mathfrak{H}^{\gamma}_{p,\theta}(\tau)$ and*

$$||u||^p_{\mathfrak{H}^{\gamma}_{p,\theta}(\tau)} \le N\left(||f||^p_{\mathbb{H}^{\gamma-2}_{p,\theta}(\tau)} + ||g||^p_{\mathbb{H}^{\gamma-1}_{p,\theta}(\tau,l_2)} + ||u_x||^p_{\mathbb{H}^{\mu-1}_{p,\theta}(\tau)} + ||u_0||^p_{U^{\gamma}_{p,\theta}}\right),$$

where $N = N(d, \gamma, \mu, \theta, p)$.

One can prove this lemma by repeating almost word for word the proof of Lemma 3.5 of [6]. The only noticeable difference is that the equations in [6] are written in the form

$$du = (au_{xx} + f_x)\,dt + (\sigma^k u_x + g^k)\,dw_t^k,$$

where we have f_x instead of $M^{-1}f$. But by Remark 3.4 we also can rewrite (3.1) with $D_i f^i$ in place of $M^{-1}f$.

Proof of Theorem 3.3. As in the proof of Theorem 3.2 of [6] we may assume that $\tau = \infty$. In the case $\gamma \geq 2$ the proof is achieved on the basis of Lemma 3.8 by repeating the proof of Theorem 3.2 of [6]. In the case $\gamma < 2$ we need only some minor adjustments which we present for completeness.

Denote by $\mathcal{R}$ the operator which maps (f, g, u_0) with $f \in \mathbb{H}^{\gamma-2}_{p,\theta}$, $g \in \mathbb{H}^{\gamma-1}_{p,\theta}(l_2)$, and $u_0 \in U^{\gamma}_{p,\theta}$ into the solution $u \in \mathfrak{H}^{\gamma}_{p,\theta}$ of (3.1) with initial data u_0. Thus far we know that $\mathcal{R}$ is well defined in spaces $\mathbb{H}^{\gamma-2}_{p,\theta} \times \mathbb{H}^{\gamma-1}_{p,\theta}(l_2) \times U^{\gamma}_{p,\theta}$ for $\gamma \geq 2$. We want to show that one can also define $\mathcal{R}$ for $\gamma < 0$.

First, let $2 > \gamma \geq 1$. Observe that by Lemma 1.3, part (iii),

$$(\mathcal{L}^{-1}f, \mathcal{L}^{-1}g, M^{1-2/p}\mathcal{L}^{-1}M^{2/p-1}u_0) \in \mathbb{H}^{\gamma}_{p,\theta} \times \mathbb{H}^{\gamma+1}_{p,\theta}(l_2) \times U^{\gamma+2}_{p,\theta}.$$

Since $\gamma > 0$, by what we know in the case $\gamma \geq 2$, the function

$$v = \mathcal{R}(\mathcal{L}^{-1}f, \mathcal{L}^{-1}g, M^{1-2/p}\mathcal{L}^{-1}M^{2/p-1}u_0)$$

is well defined and belongs to $\mathfrak{H}^{\gamma+2}_{p,\theta}$.

Define

$$\tilde{u} = \mathcal{L}v.$$

By Remark 2.8, we have $\tilde{u} \in \mathfrak{H}^{\gamma}_{p,\theta}$. Furthermore, by definition v satisfies

$$dv = (a^{ij}v_{x^ix^j} + M^{-1}\mathcal{L}^{-1}f)\,dt + (\sigma^{ik}v_{x^i} + \mathcal{L}^{-1}g^k)\,dw_t^k.$$

We apply $\mathcal{L}$ to both parts of this equality, or in other words, we substitute $\mathcal{L}^*\phi$ in place of ϕ in (3.2), where $\mathcal{L}^*$ is the formal adjoint to $\mathcal{L}$. Then we get

$$d\tilde{u} = (a^{ij}\tilde{u}_{x^ix^j} + M^{-1}f + M^{-1}\bar{f})\,dt + (\sigma^{ik}\tilde{u}_{x^i} + g^k + \bar{g}^k)\,dw_t^k,$$

$$\tilde{u}(0, \cdot) = u_0 + \bar{u}_0,$$

where

$$\bar{f} = M\mathcal{L}a^{ij}v_{x^ix^j} - Ma^{ij}(\mathcal{L}v)_{x^ix^j} + M\mathcal{L}M^{-1}\mathcal{L}^{-1}f - f,$$

$$\bar{g}^k = \mathcal{L}\sigma^{ik}v_{x^i} - \sigma^{ik}(\mathcal{L}v)_{x^i}, \quad \bar{u}_0 = \mathcal{L}M^{1-2/p}\mathcal{L}^{-1}M^{2/p-1}u_0 - u_0.$$

Next, we use

$$\mathcal{L}D_i\phi = D_i\mathcal{L}\phi - 2\delta^{i1}M^{-1}(\mathcal{L} - MD_1)\phi,$$

$$\mathcal{L}M^{-1}\phi = M^{-1}\mathcal{L}\phi - 2D_1\phi.$$

Then we find that

$$\bar{f} = -2a^{i1}MD_iM^{-1}(\mathcal{L} - MD_1)v - 2a^{1j}(\mathcal{L} - MD_1)D_jv - 2MD_1\mathcal{L}^{-1}f,$$

$$\bar{g}^k = -2\sigma^{1k}M^{-1}(\mathcal{L} - MD_1)v,$$

$$M^{2/p-1}\bar{u}_0 = (2 - 4/p)MD_1\mathcal{L}^{-1}M^{2/p-1}u_0 + c\mathcal{L}^{-1}M^{2/p-1}u_0,$$

where c is a constant. As above

$$(\mathcal{L} - MD_1)v \in \mathfrak{H}^\gamma_{p,\theta}, \quad M^{-1}(\mathcal{L} - MD_1)v \in \mathbb{H}^\gamma_{p,\theta}, \quad Dv \in \mathbb{H}^{\gamma+1}_{p,\theta},$$

$$M^{2/p-1}\bar{u}_0 \in L_p(\Omega, \mathcal{F}_0, H^{\gamma+1-2/p}_{p,\theta}).$$

It follows that

$$(\bar{f}, \bar{g}, \bar{u}_0) \in \mathbb{H}^{\gamma-1}_{p,\theta} \times \mathbb{H}^\gamma_{p,\theta}(l_2) \times U^{\gamma+1}_{p,\theta}. \tag{3.6}$$

Since $\gamma \geq 1$, it follows from (3.6) that the function $\bar{u} := \mathcal{R}(\bar{f}, \bar{g}, \bar{u}_0)$ is well defined, belongs to $\mathfrak{H}^{\gamma+1}_{p,\theta}$, and the function $u = \tilde{u} - \bar{u}$ is of class $\mathfrak{H}^\gamma_{p,\theta}$ and solves (3.1). For thus constructed u, estimate (3.5) follows from the explicit representation and known estimates for $\mathcal{R}$, $\mathcal{L}$, MD.

By repeating the above argument, we consider the case $1 > \gamma \geq 0$, this time using the fact that $\gamma + 1 \geq 1$ and relying upon the result for $\gamma \geq 1$. One can continue in the same way and it remains to prove only the uniqueness of solutions in $\mathfrak{H}^\gamma_{p,\theta}$.

It suffices to consider the case $f = 0, g = 0, u_0 = 0$ (and $\gamma < 2$). In this case any solution $u \in \mathfrak{H}^\gamma_{p,\theta,0}$ also belongs to $\mathfrak{H}^2_{p,\theta,0}$ by Lemma 3.8 and its uniqueness follows from Lemma 3.7.

The theorem is thus proved.

Remark 3.9. From the above derivation of Theorem 3.3 from Lemma 3.7 it is seen that for *any* fixed $\gamma, p, \theta, a, \sigma$ satisfying the conditions of Theorem 3.3, if the assertion of Theorem 3.3 holds for these $\gamma, p, \theta, a, \sigma$, then it holds for any $\gamma \in \mathbb{R}$ with the same p, θ, a, σ.

4. Proof of Lemma 3.7. By Remarks 2.7, we may concentrate on the following form of (3.1):

$$\begin{aligned} du(t,x) = (a^{ij}(t)u_{x^ix^j}(t,x) + D_if^i(t,x))dt \\ + (\sigma^{ik}(t)u_{x^i}(t,x) + g^k(t,x))dw^k(t). \end{aligned} \tag{4.1}$$

Next, notice that by Theorem 2.13 there is a function $\bar{u} \in \mathfrak{H}^2_{p,\theta}$ such that, $\bar{u}|_{t=0} = u_0$, $\partial\bar{u}/\partial t = D_i\bar{f}^i$ with $\bar{f} \in \mathbb{H}^1_{p,\theta}$, and appropriate estimates of $||\bar{u}_x||_{\mathbb{H}^1_{p,\theta}}$ and $||\bar{f}||_{\mathbb{H}^1_{p,\theta}}$ through $||u_0||_{U^2_{p,\theta}}$ hold. This implies that in the equation

$$du = (a^{ij}u_{x^ix^j} + (a^{ij}\bar{u}_{x^j} + f^i - \bar{f}^i)_{x^i})\,dt + (\sigma^{ik}u_{x^i} + (\sigma^{ik}\bar{u}_{x^i} + g^k))\,dw^k_t$$

we have $a^{ij}\bar{u}_{x^j} + f^i - \bar{f}^i \in \mathbb{H}^1_{p,\theta}(\tau)$ and $\sigma^{i\cdot}\bar{u}_{x^i} + g \in \mathbb{H}^1_{p,\theta}(\tau, l_2)$. Also, obviously if we can solve the above equation in $\mathfrak{H}^2_{p,\theta,0}(\tau)$, then by adding to the solution the function

$\bar{u}$ we get a solution of (4.1) with initial data u_0. Therefore, in the proof of Lemma 3.7 without loss of generality, we may and will confine ourselves only to the case $u_0 \equiv 0$.

Finally, obviously we may assume that $\tau \le T$, where the constant $T < \infty$, and we start by proving the following a priori estimate.

LEMMA 4.1. *Assume that there exists a constant $\delta_2 > 0$ such that*

$$(p-1)(d+p-1-\theta)\bar{a}^{11} - p(d+p-2-\theta)a^{11} \ge \delta_2 \tag{4.2}$$

for all ω and t. Then for any $u \in \mathfrak{H}^2_{p,\theta,0}(\tau)$,

$$||M^{-1}u||_{\mathbb{L}_{p,\theta}(\tau)} \le N(||M(\tilde{\mathbb{D}} - a^{ij}D_iD_j)u||_{\mathbb{L}_{p,\theta}(\tau)} + ||(\tilde{\mathbb{S}} - \sigma^{i\cdot}D_i)u||_{\mathbb{L}_{p,\theta}(\tau,l_2)}), \tag{4.3}$$

where N depends only on $\delta_0, \delta_2, d,$ and p.

Proof. For any γ the operators $M\tilde{\mathbb{D}}$ and $\tilde{\mathbb{S}}$ are obviously continuous on $\mathfrak{H}^\gamma_{p,\theta}(\tau)$ with values in $\mathbb{H}^{\gamma-2}_{p,\theta}(\tau)$ and $\mathbb{H}^{\gamma-1}_{p,\theta}(\tau, l_2)$, respectively. By Remark 2.5 the same is true for $M^{-1} : \mathfrak{H}^\gamma_{p,\theta}(\tau) \to \mathbb{H}^\gamma_{p,\theta}(\tau)$. By Definition 2.2 and Lemma 1.3 the operators

$$MD_iD_j : \mathfrak{H}^\gamma_{p,\theta}(\tau) \to \mathbb{H}^{\gamma-2}_{p,\theta}(\tau), \quad \sigma^{ik}D_i : \mathfrak{H}^\gamma_{p,\theta}(\tau) \to \mathbb{H}^{\gamma-1}_{p,\theta}(\tau, l_2)$$

are bounded. By Theorem 2.9, it follows that we need to prove only (4.3) for functions u belonging to set (2.3) with sufficiently large n.

Take such a function u and define f and g according to (3.3). By Sobolev's embedding theorem, if n is large, then f and g are continuous in x, have compact supports in x, and

$$E\int_0^\tau \sup_x |f(t,x)|^p\,dt < \infty, \quad E\int_0^\tau \sup_x |g(t,x)|^p_{l_2}\,dt < \infty.$$

It follows easily that u satisfies (3.1) pointwise, that is, for almost any ω for all $x \in \mathbb{R}^d_+$ and $t \in [0,\tau]$.

Next we define $c = 2 + \theta - d - p$, apply Itô's formula to $(x^1)^c|u(t,x)|^p$, and find almost surely for all $x \in \mathbb{R}^d_+$

$$(x^1)^c|u(\tau,x)|^p = \int_0^\tau \Bigg[p(x^1)^c|u|^{p-2}ua^{ij}u_{x^ix^j}$$

$$+p(x^1)^{c-1}|u|^{p-2}uf + \tfrac{1}{2}p(p-1)(x^1)^c|u|^{p-2}\sum_k(\sigma^{ik}u_{x^i} + g^k)^2\Bigg]\,ds$$

$$+\int_0^\tau p(x^1)^c|u|^{p-2}u(\sigma^{ik}u_{x^i} + g^k)\,dw^k_s. \tag{4.4}$$

We take expectations of both parts of this equality, noticing that

$$E\left[\int_0^\tau |u|^{2p-2}\sum_k|\sigma^{ik}u_{x^i} + g^k|^2\,ds\right]^{1/2} \tag{4.5}$$

$$\le NTE\sup_{s\le\tau}|u|^{p-1}|u_x| + NE\sup_{s\le\tau}|u|^{p-1}\left[\int_0^\tau |g|^2_{l_2}\right]^{1/2}.$$

Here, for instance, by Hölder's inequality the last expectation is less than

$$\big(E\sup_{s\le\tau}|u|^p\big)^{(p-1)/p}\left(T^{(p-2)/2}E\int_0^\tau|g|^p_{l_2}\,ds\right)^{1/p}<\infty.$$

Therefore, the left-hand side of (4.5) is finite and the stochastic integral will disappear after taking expectations in (4.4). After this we integrate with respect to x over $\mathbb{R}^d_+$. By the way, owing to the fact that x-supports of all functions u, f, and g belong to some G_k and the fact that even the pth power of sup's over x of these functions are integrable over $(0,\tau]\!]$, we see that all integrals converge absolutely. Hence, by using Fubini's theorem and integrating by parts, we get from (4.4) that

$$0\le E\int_0^\tau\int_{\mathbb{R}^d_+}\big[-p(p-1)(x^1)^c|u|^{p-2}\bar a^{ij}u_{x^i}u_{x^j}$$

$$-c(x^1)^{c-1}a^{i1}(|u|^p)_{x^i}+p(p-1)(x^1)^c|u|^{p-2}g^k\sigma^{ik}u_{x^i}$$

$$+p(x^1)^{c-1}|u|^{p-1}|f|+\tfrac{1}{2}p(p-1)(x^1)^c|u|^{p-2}|g|^2_{l_2}\big]\,dxdt.$$

Next, we use Young's inequality to get relations like

$$(x^1)^{c-1}|u|^{p-1}|f|\le\varepsilon(x^1)^{\theta-d}|u/x^1|^p+N(x^1)^{\theta-d}|f|^p,$$

$$g^k\sigma^{ik}u_{x^i}\le N|g|_{l_2}|u_x|\le\varepsilon\bar a^{ij}u_{x^i}u_{x^j}+N|g|^2_{l_2},$$

where $\varepsilon>0$ is arbitrary and N depends only on ε, δ_0, and p. Then we get

$$0\le E\int_0^\tau\int_{\mathbb{R}^d_+}\big[(\varepsilon-p(p-1))(x^1)^c|u|^{p-2}\bar a^{ij}u_{x^i}u_{x^j}$$

$$+(\varepsilon+c(c-1))a^{11}(x^1)^{\theta-d}|u/x^1|^p+N(x^1)^{\theta-d}|f|^p+N(x^1)^{\theta-d}|g|^p_{l_2}\big]\,dxdt.$$

By Corollary 6.2 of [5] for any t

$$\int_{\mathbb{R}^d_+}(x^1)^c|u|^{p-2}\bar a^{ij}u_{x^i}u_{x^j}\ge\bar a^{11}(1-c)^2p^{-2}\int_{\mathbb{R}^d_+}(x^1)^{\theta-d}|u/x^1|^p\,dx.$$

Hence,

$$E\int_0^\tau\{\bar a^{11}[p(p-1)-\varepsilon](1-c)^2p^{-2}+a^{11}[c(1-c)-\varepsilon]\}||M^{-1}u||^p_{0,p,\theta}\,dt$$

$$\le N(||f||^p_{\mathbb{L}_{p,\theta}(\tau)}+||g||^p_{\mathbb{L}_{p,\theta}(\tau,l_2)}).$$

It remains only to observe that for ε small enough from (4.2) we get that

$$\bar a^{11}[p(p-1)-\varepsilon](1-c)^2p^{-2}+a^{11}[c(1-c)-\varepsilon]$$

$$\geq -(1-c)p^{-1}\delta_2/2 + \bar{a}^{11}(p-1)(1-c)^2p^{-1} + a^{11}c(1-c)$$

$$= -(1-c)p^{-1}\delta_2/2 + (1-c)p^{-1}[(p-1)(d+p-1-\theta)\bar{a}^{11} - p(d+p-2-\theta)a^{11}]$$

$$\geq (1-c)p^{-1}\delta_2/2.$$

The lemma is proved.

We divide the remaining part of the proof of Lemma 3.7 into the following subcases:

(1) $\sigma \equiv 0$;

(2) general case.

4.1. Case $\boldsymbol{\sigma \equiv 0}$. Observe that in this case $\bar{a} = a$ and (4.2) becomes

$$a^{11}(\theta - d + 1) \geq \delta_2,$$

which is satisfied for δ_2 sufficiently small because we always assume that $\theta > d-1$ (and, for that matter, $\theta < p+d-1$). Therefore, estimate (4.3) holds. Of course, this estimate implies uniqueness.

To prove existence again use (4.3) and proceed as in the proof of Lemma 4.2 of [6] or Lemma 5.7 of [5]. Since this can be done in quite a straightforward way, we give only a sketch.

First, bearing in mind the a priori estimate and the method of continuity, we see that it suffices to consider the case $a^{ij} = \delta^{ij}$. Furthermore, owing to Theorem 2.10 and Lemma 4.1 we may and will additionally assume that

$$f \in L_p(\Omega, C((0,\tau], C_0^n(G_k))), \quad g \in L_p(\Omega, C((0,\tau], C_0^n(G_k))).$$

Continue f and g across $x^1 = 0$ so that f becomes an even smooth function and g an odd smooth function of x^1. By Theorem 3.2 of [3] or Theorem 5.1 of [4] there exists a unique solution $u \in \mathcal{H}_p^n(\tau)$ of (3.1) considered in the whole $\mathbb{R}^d$ with zero initial condition. If n is large enough, u is smooth with respect to x and satisfies (3.1) pointwise. From the uniqueness, it follows that $u(t,x) = 0$ for $x^1 = 0$. Next use the fact that the functions f and g have compact support and that outside this support u satisfies the deterministic equation $du = \Delta u\, dt$. Then as in the proof of Lemma 4.2 of [6] we derive that $u \in \mathfrak{H}^2_{p,\theta,0}(\tau)$. Using Lemma 3.8 with $\gamma = 2$ and $\mu = 0$ and Lemma 4.1 we conclude that u belongs to $\mathfrak{H}^2_{p,\theta,0}(\tau)$, satisfies (3.1), and estimate (3.5) holds for $\gamma = 2$ and $u_0 = 0$. This proves Lemma 3.7 in our first particular case.

4.2. General case. The left inequality in (3.4) means that

$$\delta_1(p-1)(d+p-1-\theta) > p(d+p-2-\theta),$$

which by virtue of Assumption 3.2 implies (4.2) with

$$\delta_2 = \delta_0[\delta_1(p-1)(d+p-1-\theta) - p(d+p-2-\theta)] > 0.$$

Therefore, a priori estimate (4.3) holds. Using Lemma 3.8 with $\gamma = 2$ and $\mu = 0$, we get that estimate (3.5) holds for $\gamma = 2$ and $u_0 = 0$. In particular, we get the uniqueness.

Furthermore, the same estimate with the same constant N holds if we take $\lambda\sigma^{ik}$ instead of σ^{ik} if $|\lambda| \leq 1$. Now to get the result in our present case from the case $\sigma \equiv 0$ it remains only to use the method of continuity (cf., for instance, the end of the proof of Theorem 5.1 of [4]).

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