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A SOBOLEV SPACE THEORY OF SPDEs WITH CONSTANT COEFFICIENTS ON A HALF LINE*

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Abstract. Equations of the form $du = (au_{xx} + f_x)\,dt + \sum_k (\sigma^k u_x + g^k)\,dw_t^k$ are considered for $t > 0$ and $x > 0$. The unique solvability of these equations is proved in weighted Sobolev spaces with fractional positive or negative derivatives, summable to the power $p \in [2, \infty)$.

Key words. stochastic partial differential equations, Sobolev spaces with weights

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Introduction. We are considering the equation

$$du = (au_{xx} + f_x)\,dt + \sum_{k=1}^{\infty} (\sigma^k u_x + g^k)\,dw_t^k$$

in one space dimension for $x > 0$ and $t > 0$ with some initial condition at $t = 0$ and zero boundary condition at $x = 0$. Here w_t^k are independent one-dimensional Wiener processes and f and g^k are some given functions of (ω, t, x). The functions a and σ^k are assumed to depend only on ω and t. Such equations with a finite number of the processes w_t^k appear, for instance, in nonlinear filtering problems for partially observable diffusions (see [11]). Considering infinitely many w_t^k turns out to be instrumental in treating equations for measure valued processes, for instance, driven by space–time white noise (see [8] or [6]).

Our main goal is to prove solvability of such equations in spaces similar to Sobolev spaces, in which derivatives are understood as generalized functions, the number of derivatives may be fractional or negative, and underlying power of summability is $p \in [2, \infty)$.

The motivation for this goal is explained in detail in [5] or [8], where an L_p-theory is developed for the equations in the whole space. We only mention that if $p = 2$, the theory was developed long ago and an account of it can be found, for instance, in [11]. The case of equations in domains is also treated in [11]. However, the solvability is only proved in spaces W_2^1 of functions having one generalized derivative in x square summable in (ω, t, x). It turns out that going to better smoothness of solutions is not possible in spaces W_2^n and one needs to consider Sobolev spaces with weights, allowing derivatives to blow up near the boundary. The theory of solvability in Hilbert spaces like W_2^n with weights is developed in [1] and [10], where n is an integer. Here we show what happens if one takes a fractional or negative number of derivatives and replaces 2 with any $p \geq 2$. By the way, according to [2], it is not possible to take $p < 2$ when stochastic terms are present in the equation.

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Unlike the above mentioned works, we only concentrate on the one-dimensional case. There are several reasons for that, the main being that even in the case of Hilbert spaces in [1] the central estimates are first proved in the one-dimensional case and after this there is still a rather long way to go to get to multidimensional domains. Our treatment of the one-dimensional case is long itself.

One of main difficulties in developing the theory presented below was finding right spaces. The idea was to find a scale of spaces like in [11], [5], or [8] generated by fractional powers of a certain operator, which is $1-\Delta$ in [11], [5], and [8]. From the results of [1] and [10] one can guess that $xD = x\partial/\partial x$ should be such an operator in our case. Elliptic second-order operators are more appropriate if one wants to define fractional powers and expects them to have nice properties. Therefore, our first attempt was to try the operator $L = xD(xD) + xD - c$, which is formally self-adjoint for any constant c. However, after having constructed the theory we noticed that the same spaces can be defined as images of spaces from [5] or [8] under certain linear mapping. This made using the results from [5] and [8] easier and allowed us to avoid developing solvability theory for L and investigating the semigroup and the resolvent associated with this operator.

In [11], [5], and [8] the solution is sought for in the same scale of spaces (at least as far as the space variables are concerned) as the one to which the free terms f and g belong. Surprisingly enough this is not the case in our situation, and this causes many difficulties practically at each step. The origin of all unusual features of our theory lies in the fact that there are no operators commuting with $\partial/\partial x$ and generating our scale of spaces. To give one more example of what is unusual we state the following theorem, which can be obtained from Theorem 3.2 after changing variables $v(t,x) = e^{x(\alpha-1)}u(t,e^x)$, where $\alpha = \theta/p$.

THEOREM 0.1. *Let $\alpha \in (0,1)$, $p \in (1,\infty)$, $T \in (0,\infty]$, and $f \in L_p([0,\infty)\times\mathbb{R})$. Then in the class of functions $v(t,x)$, $t \in [0,T]$, $x \in \mathbb{R}$ such that*

$$\int_0^T \int_{\mathbb{R}} [|v_x|^p + |v|^p]\,dxdt < \infty,$$

the equation

$$e^{2x}v_t = v_{xx} + (1-2\alpha)v_x - (1-\alpha)\alpha v + f_x \tag{0.1}$$

on $(0,T)\times\mathbb{R}$ with zero initial condition has a unique solution. In addition, this solution satisfies

$$\int_0^T \int_{\mathbb{R}} [|v_x|^p + |v|^p]\,dxdt \le N(\alpha,p)\int_0^T \int_{\mathbb{R}} |f|^p\,dxdt.$$

Surprising in this theorem is that if we replace e^{2x} with 1 in (0.1), then the result becomes well known and is true for any finite T (now with N depending on T too). The presence of e^{2x} makes (0.1) degenerate, and usually results for degenerate equations differ very much from those for nondegenerate cases. Actually, we do not know much about (0.1). In particular, it would be interesting to know whether Theorem 0.1 remains true if we replace the term $(1-2\alpha)v_x$ in (0.1) with bv_x where b is an arbitrary constant.

The article is organized as follows. In section 1 we introduce and investigate basic spaces with weights of functions of $x \in (0,\infty)$. Section 2 is devoted to stochastic Banach spaces of functions of (ω,t,x) satisfying zero boundary condition at $x=0$.

This condition is expressed by means of requirement (2.1). In section 3 we prove our main Theorem 3.2 about unique solvability of our equations. The reader will see the very core of our technique in the proof of Lemma 3.6. Rather long section 4 contains the proof of the main particular case of Theorem 3.2, which is stated as Lemma 3.5.

1. Sobolev spaces with weights. For $\gamma \in \mathbb{R}$ and $p \in (1,\infty)$ let $H_p^\gamma = H_p^\gamma(\mathbb{R})$ be the spaces of Bessel potentials (see, for instance, [13]) which are formally given by $H_p^\gamma = \Lambda^{-\gamma} L_p(\mathbb{R})$, where $\Lambda := (1-D^2)^{1/2}$ and $D = d/dx$. One knows that the elements of H_p^γ are distributions and $C_0^\infty = C_0^\infty(\mathbb{R})$ is dense in H_p^γ. Let $\mathcal{D}(\mathbb{R})$ and $\mathcal{D}(\mathbb{R}_+)$ be the sets of all distributions on $C_0^\infty(\mathbb{R})$ and $C_0^\infty(\mathbb{R}_+)$, respectively, where $\mathbb{R}_+ = (0,\infty)$. If $f \in \mathcal{D}(\mathbb{R}_+)$ and $\theta \in \mathbb{R}$, then the expression $h(x) := f(e^x)e^{x\theta/p}$ is well defined and is a distribution on $\mathbb{R}$. Indeed, the action of h on a test function $\phi \in C_0^\infty(\mathbb{R})$ is defined as $(h,\phi) = (f,\psi)$, where $\psi(x) := \phi(\log x)x^{\theta/p-1}$. We denote $h = Q_{p,\theta} f$ in this way defining a one-to-one operator

$$Q_{p,\theta} : f(x) \to f(e^x)e^{x\theta/p}.$$

DEFINITION 1.1. *We write* $f \in H_{p,\theta}^\gamma$ $(= H_{p,\theta}^\gamma(\mathbb{R}_+))$ *if and only if* $Q_{p,\theta} f = h \in H_p^\gamma$. *We write* $L_{p,\theta} = H_{p,\theta}^0$. *For* $f \in H_{p,\theta}^\gamma$ *we define*

$$||f||_{H_{p,\theta}^\gamma} = ||Q_{p,\theta} f||_{H_p^\gamma}.$$

Remark 1.2. Since H_p^γ is a Banach space, so is $H_{p,\theta}^\gamma$ with the norm introduced above. Also since $C_0^\infty(\mathbb{R})$ is dense in H_p^γ, the set $C_0^\infty(\mathbb{R}_+)$ is dense in $H_{p,\theta}^\gamma$.

Remark 1.3. Define $\Lambda_{p,\theta}^\gamma = Q_{p,\theta}^{-1}\Lambda^\gamma Q_{p,\theta}$. Then for any $\gamma,\mu,\theta \in \mathbb{R}$ the operator $\Lambda_{p,\theta}^\gamma$ is an isometric operator from $H_{p,\theta}^\mu$ onto $H_{p,\theta}^{\mu-\gamma}$.

Indeed, by definition,

$$||\Lambda_{p,\theta}^\gamma u||_{H_{p,\theta}^{\mu-\gamma}} = ||Q_{p,\theta}\Lambda_{p,\theta}^\gamma u||_{H_p^{\mu-\gamma}} = ||\Lambda^\gamma Q_{p,\theta} u||_{H_p^{\mu-\gamma}}$$

$$= ||Q_{p,\theta} u||_{H_p^\mu} = ||u||_{H_{p,\theta}^\mu}.$$

Remark 1.4. The norm in $H_{p,\theta}^\gamma$ contains norms of, so to speak, γ derivatives of u. However, it scales in the same way for any γ. We mean that, due to translation invariance of norms in H_p^γ, for any constant $a > 0$ and $u \in H_{p,\theta}^\gamma$,

$$||u(a\,\cdot)||_{H_{p,\theta}^\gamma}^p = a^{-\theta}||u||_{H_{p,\theta}^\gamma}^p.$$

Remark 1.5. Define M as the operator of multiplying by x, $M : u(x) \to xu(x)$. It turns out that for any $\gamma \in \mathbb{R}$ the operator MD is a bounded operator from $H_{p,\theta}^\gamma$ into $H_{p,\theta}^{\gamma-1}$ and if, in addition, $\theta \neq 0$, then MD maps $H_{p,\theta}^\gamma$ onto $H_{p,\theta}^{\gamma-1}$ and its inverse is also bounded.

Indeed, an easy computation shows that

$$Q_{p,\theta} MDu = LQ_{p,\theta} u, \quad MDu = Q_{p,\theta}^{-1} LQ_{p,\theta} u,$$

where $Lv = Dv - v\theta/p$. One knows (see, for instance, p. 263 in [12]) that for any constant ν the operator $v \to Dv + \nu v$ is a bounded operator from H_p^γ into $H_p^{\gamma-1}$ and if ν is real and $\nu \neq 0$, then it maps H_p^γ onto $H_p^{\gamma-1}$ and its inverse is bounded. This and the definition of $H_{p,\theta}^\gamma$ obviously imply our assertion.

Remark 1.6. Functions in $H^{\gamma}_{p,\theta}$ are different from those in H^{γ}_{p} only in what concerns their behavior near zero and infinity. More precisely, if $[a,b]\subset\mathbb{R}_+$ and $f=0$ outside $[a,b]$, then by the results on changing variables and pointwise multipliers (see Theorem 4.3.2 and Corollary 4.2.2 of [13]) $||f||_{H^{\gamma}_{p,\theta}}\le N||f||_{H^{\gamma}_{p}}\le N||f||_{H^{\gamma}_{p,\theta}}$, where N is independent of f.

It is convenient here also to notice that for the same f we have

$$||f||_{H^{\gamma}_{p}}\le N||Df||_{H^{\gamma-1}_{p}}\le N||f||_{H^{\gamma}_{p}},$$

with N independent of f.

Indeed, the inequality on the right is known to be true even for any $f\in H^{\gamma}_{p}$. As far as the left inequality is concerned, by Remark 1.5 we have

$$||f||_{H^{\gamma}_{p}}\le N||f||_{H^{\gamma}_{p,1}}\le N||MDf||_{H^{\gamma-1}_{p,1}}\le N||\eta Df||_{H^{\gamma-1}_{p}},$$

where $\eta\in C^{\infty}_0(\mathbb{R})$ and $\eta(x)=x$ on $[a,b]$. It only remains to remember (see [13]) that such η is a pointwise multiplier in any space $H^{\gamma-1}_{p}$.

Remark 1.7. Upon noticing that $DMu=MDu+u$, as in Remark 1.5 we conclude that for any $\gamma\in\mathbb{R}$ the operator DM is a bounded operator from $H^{\gamma}_{p,\theta}$ into $H^{\gamma-1}_{p,\theta}$ and if, in addition, $\theta\ne p$, then DM maps $H^{\gamma}_{p,\theta}$ onto $H^{\gamma-1}_{p,\theta}$ and its inverse is also bounded.

Remark 1.8. Let $\theta\ne 0$, $u\in\bigcup_{\mu}H^{\mu}_{p,\theta}$, and $MDu\in H^{\gamma}_{p,\theta}$. Then $u\in H^{\gamma+1}_{p,\theta}$ and $||u||_{H^{\gamma+1}_{p,\theta}}\le N||MDu||_{H^{\gamma}_{p,\theta}}$.

Indeed, by Remark 1.5 there is $v\in H^{\gamma+1}_{p,\theta}$ such that $MDv=MDu$ and $||v||_{H^{\gamma+1}_{p,\theta}}\le N||MDu||_{H^{\gamma}_{p,\theta}}$. Then $v'=u'$ and $v-u=c$, where c is a constant. Since $v,u\in H^{\mu}_{p,\theta}$ for some μ, we have $c\in H^{\mu}_{p,\theta}$, which is only possible if $c=0$. Therefore, $u=v\in H^{\gamma+1}_{p,\theta}$.

Remark 1.9. Let $\theta\ne p$, $u\in\bigcup_{\mu}H^{\mu}_{p,\theta}$, and $DMu\in H^{\gamma}_{p,\theta}$. Then $u\in H^{\gamma+1}_{p,\theta}$ and $||u||_{H^{\gamma+1}_{p,\theta}}\le N||DMu||_{H^{\gamma}_{p,\theta}}$.

Indeed, one can repeat the argument in Remark 1.8 relying on Remark 1.7 instead of Remark 1.5 and noticing that from the equality $DMv=DMu$ it follows that $v-u=c/x$, where c is a constant.

Remark 1.5 and the observation that $H^{0}_{p,\theta}=L_{p,\theta}$ is just an L_p-space of functions on $\mathbb{R}_+$ with measure $m_\theta(dx)=x^{\theta-1}\,dx$ yield inequalities (1.1) in the following useful result, which can also be restated in a natural way on the basis of Remark 1.7.

THEOREM 1.10. *If γ is an integer satisfying $\gamma\ge 1$ and $\theta\ne 0$, then for any $u\in H^{\gamma}_{p,\theta}$ we have*

$$||(MD)^{\gamma}u||_{L_p(\mathbb{R}_+,m_\theta)}\le N||u||_{H^{\gamma}_{p,\theta}}\le N||(MD)^{\gamma}u||_{L_p(\mathbb{R}_+,m_\theta)},\tag{1.1}$$

$$\sum_{n=1}^{\gamma}||M^nD^nu||_{L_p(\mathbb{R}_+,m_\theta)}\le N||u||_{H^{\gamma}_{p,\theta}}\le N\sum_{n=1}^{\gamma}||M^nD^nu||_{L_p(\mathbb{R}_+,m_\theta)},\tag{1.2}$$

where N is independent of u. Thus, the space $H^{\gamma}_{p,\theta}$ can also be defined as a closure of the set $C^{\infty}_0(\mathbb{R}_+)$ with respect to either of the norms

$$||(MD)^{\gamma}\ \cdot||_{L_p(\mathbb{R}_+,m_\theta)},\qquad \sum_{n=1}^{\gamma}||M^nD^n\ \cdot||_{L_p(\mathbb{R}_+,m_\theta)}.$$

To prove (1.2) observe that for any integer $k \geq 1$,

$$(MD)^k = \sum_{n=1}^{k} c^{k,n} M^n D^n, \tag{1.3}$$

where $c^{k,n}$ are some constants and $c^{k,k} = 1$. This and the inequality on the right in (1.1) give us the inequality on the right in (1.2). On the other hand, one can solve the triangular system (1.3) with respect to $M^n D^n$. Then from the inequality on the left in (1.1) we get

$$\sum_{n=1}^{\gamma} ||M^n D^n u||_{L_p(\mathbb{R}_+, m_\theta)} \leq N \sum_{n=1}^{\gamma} ||(MD)^n u||_{L_p(\mathbb{R}_+, m_\theta)}$$
$$\leq N \sum_{n=1}^{\gamma} ||u||_{H^n_{p,\theta}} \leq N ||u||_{H^\gamma_{p,\theta}},$$

which proves the inequality on the left in (1.2).

The following theorem will play the most important role in obtaining results for equations on $\mathbb{R}_+$ from those on $\mathbb{R}$.

THEOREM 1.11. *Let $\zeta \in C_0^\infty(\mathbb{R}_+)$, $\gamma, \theta \in \mathbb{R}$, and $p \in (1, \infty)$. Then there exists a constant N depending only on ζ, γ, p, and θ such that, for any $u \in H^\gamma_{p,\theta}$,*

$$\sum_{n=-\infty}^{\infty} e^{n\theta} ||\zeta u(e^n \cdot)||^p_{H^\gamma_p} \leq N ||u||^p_{H^\gamma_{p,\theta}}.$$

In addition, if there is a $\delta > 0$ such that

$$\sum_{n=-\infty}^{\infty} e^{(n-x)\theta} |\zeta(e^{x-n})|^p \geq \delta \tag{1.4}$$

for all $x \in [0, 1]$, then

$$||u||^p_{H^\gamma_{p,\theta}} \leq N \sum_{n=-\infty}^{\infty} e^{n\theta} ||\zeta u(e^n \cdot)||^p_{H^\gamma_p},$$

where N depends on δ as well.

Proof. Since the functions $\zeta(x)u(e^n x)$ vanish outside the support of ζ, by the change of variables (see Theorem 4.3.2 in [13])

$$e^{n\theta} ||\zeta u(e^n \cdot)||^p_{H^\gamma_p} \leq N e^{n\theta} ||\zeta(e^\cdot) u(e^{\cdot + n})||^p_{H^\gamma_p}$$

with N independent of n, u. By translation invariance of the norm in H^γ_p the last expression equals

$$e^{n\theta} ||\zeta(e^{\cdot - n}) u(e^\cdot)||^p_{H^\gamma_p} = ||\eta(e^{\cdot - n}) Q_{p,\theta} u||^p_{H^\gamma_p},$$

where $\eta(e^{x-n}) = \zeta(e^{x-n}) e^{(n-x)\theta/p}$. Next it is easy to find a finite m such that for $x \in [0, 1]$,

$$I(x) := \sum_{n=-\infty}^{\infty} |\eta(e^{x-n})|^p = \sum_{n=-\infty}^{\infty} |\zeta(e^{x-n})|^p e^{(x-n)\theta}$$
$$= \sum_{|n| \leq m} |\zeta(e^{x-n})|^p e^{(x-n)\theta}.$$

It follows that $I(x)$ is bounded on $[0,1]$. On the other hand $I(x)$ is obviously periodic with period 1. Thus $I(x)$ is bounded on $\mathbb{R}$. The same is true for

$$\sum_{n=-\infty}^{\infty} |(\eta(e^{x-n}))'|^p, \quad \sum_{n=-\infty}^{\infty} |(\eta(e^{x-n}))''|^p,$$

and so on. By Theorem 2.2 and Remark 2.1 of [2]

$$\sum_{n=-\infty}^{\infty} ||\eta(e^{\cdot-n})Q_{p,\theta}u||^p_{H^\gamma_p} \le N||Q_{p,\theta}u||^p_{H^\gamma_p},$$

which yields our first assertion.

To prove the second one we use the same resources as above and get

$$||Q_{p,\theta}u||^p_{H^\gamma_p} \le N \sum_{n=-\infty}^{\infty} ||\eta(e^{\cdot-n})Q_{p,\theta}u||^p_{H^\gamma_p} = N \sum_{n=-\infty}^{\infty} e^{n\theta}||\zeta(e^{\cdot-n})u(e^{\cdot})||^p_{H^\gamma_p}$$

$$= N \sum_{n=-\infty}^{\infty} e^{n\theta}||\zeta(e^{\cdot})u(e^{\cdot+n})||^p_{H^\gamma_p} \le N \sum_{n=-\infty}^{\infty} e^{n\theta}||\zeta u(e^n \cdot)||^p_{H^\gamma_p}.$$

The theorem is proved.

Remark 1.12. Similar to properties of $I(x)$ in the above proof, we find that if $\zeta \in C_0^\infty(\mathbb{R}_+)$ and $\beta \in \mathbb{R}$, then $\sum_n e^{(n+x)\beta}\zeta(e^{n+x})$ is bounded on $\mathbb{R}$, which after substituting $\log x$ in place of x implies that $\sum_n e^{n\beta}\zeta(e^n x) \le Nx^{-\beta}$ on $\mathbb{R}_+$.

The following theorem is used in establishing some properties of our stochastic Banach spaces.

THEOREM 1.13. *Recall that the operator M is defined by $Mu(x) = xu(x)$ and let $\theta, \gamma \in \mathbb{R}$, $\theta \ne p$. Then*

$$M^{-1}u \in H^{\gamma+1}_{p,\theta} \iff Du \in H^\gamma_{p,\theta} \text{ and } M^{-1}u \in \bigcup_\mu H^\mu_{p,\theta}. \tag{1.5}$$

In addition, under either one of the above conditions

$$||M^{-1}u||_{H^{\gamma+1}_{p,\theta}} \le N||Du||_{H^\gamma_{p,\theta}} \le N||M^{-1}u||_{H^{\gamma+1}_{p,\theta}}. \tag{1.6}$$

Proof. If $M^{-1}u \in H^{\gamma+1}_{p,\theta}$, then by Remark 1.7 we have

$$Du = DM(M^{-1}u) \in H^\gamma_{p,\theta}$$

and the right inequality in (1.6) holds. On the other hand, under the condition on the right in (1.5) we have

$$DM(M^{-1}u) \in H^\gamma_{p,\theta} \quad \text{and} \quad M^{-1}u \in \bigcup_\mu H^\mu_{p,\theta},$$

which by Remark 1.9 yields $M^{-1}u \in H^{\gamma+1}_{p,\theta}$ and the inequality on the left in (1.6). The theorem is proved.

The following result will also be used in the future.

LEMMA 1.14. *For any constants p, θ, α we have*

$$Q_{p,\theta}^{-1} D Q_{p,\theta} = bI + MD, \quad Q_{p,\theta}^{-1} D^2 Q_{p,\theta} = (bI + MD)^2, \quad DM = MD + I,$$
$$M^\alpha \Lambda_{p,\theta}^2 M^{-\alpha} = \Lambda_{p,\theta}^2 + c_1 I + c_2 MD, \quad M\Lambda_{p,\theta}^2 - \Lambda_{p,\theta}^2 M = MP_1,$$
$$\Lambda_{p,\theta}^2 D - D\Lambda_{p,\theta}^2 = P_1 D, \tag{1.7}$$
$$\Lambda_{p,\theta}^2 DM - D\Lambda_{p,\theta}^2 M = P_1 DM, \quad \Lambda_{p,\theta}^2 D^2 M - D^2 \Lambda_{p,\theta}^2 M = 4DP_2,$$

where $b = \theta/p$, I is the identity operator, c_i are certain constants, and

$$P_1 := (2b+1)I + 2MD, \quad P_2 := bDM + (MD)(DM).$$

Furthermore, for any $\theta, \gamma \in \mathbb{R}$ there exists a constant $N = N(\gamma, \theta, p)$ such that for any $u \in H_{p,\theta}^{\gamma+2}$,

$$||P_1 u||_{H_{p,\theta}^{\gamma+1}} + ||P_2 u||_{H_{p,\theta}^{\gamma}} \le N||u||_{H_{p,\theta}^{\gamma+2}}. \tag{1.8}$$

Indeed, equalities (1.7) are checked out by straightforward computations and (1.8) follows immediately from Remarks 1.5 and 1.7.

2. Stochastic Banach spaces on $\mathbb{R}_+$. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, $(\mathcal{F}_t, t \ge 0)$ be an increasing filtration of σ-fields $\mathcal{F}_t \subset \mathcal{F}$ containing all P-null subsets of Ω, and $\mathcal{P}$ be the predictable σ-field generated by $(\mathcal{F}_t, t \ge 0)$. Let $\{w_t^k; k = 1, 2, ...\}$ be a family of independent one-dimensional $\mathcal{F}_t$-adapted Wiener processes defined on $(\Omega, \mathcal{F}, P)$. We are going to use the Banach spaces $\mathbb{H}_p^\gamma(\tau)$, $\mathbb{H}_p^\gamma(\tau, l_2)$, and $\mathcal{H}_p^\gamma(\tau)$ introduced in [5] or [8], where we take $d = 1$. Also throughout the remaining part of the paper $\theta \ne 0$, $\theta \ne p$, and $p \ge 2$ unless another range of p is specified explicitly.

DEFINITION 2.1. *Let τ be a stopping time, f and g^k, $k = 1, 2...$, be $\mathcal{D}(\mathbb{R}_+)$-valued $\mathcal{P}$-measurable functions defined on $(0, \tau]\!]$. We write $f \in \mathbb{H}_{p,\theta}^\gamma(\tau)$ and $g \in \mathbb{H}_{p,\theta}^\gamma(\tau, l_2)$ if and only if $Q_{p,\theta} f \in \mathbb{H}_p^\gamma(\tau)$ and $Q_{p,\theta} g \in \mathbb{H}_p^\gamma(\tau, l_2)$, respectively. We also denote*

$$||f||_{\mathbb{H}_{p,\theta}^\gamma(\tau)} = ||Q_{p,\theta} f||_{\mathbb{H}_p^\gamma(\tau)}, \quad ||g||_{\mathbb{H}_{p,\theta}^\gamma(\tau, l_2)} = ||Q_{p,\theta} g||_{\mathbb{H}_p^\gamma(\tau, l_2)},$$

$$\mathbb{H}_{p,\theta}^\gamma = \mathbb{H}_{p,\theta}^\gamma(\infty), \quad \mathbb{H}_{p,\theta}^\gamma(l_2) = \mathbb{H}_{p,\theta}^\gamma(\infty, l_2), \quad \mathbb{L}_{...}... = \mathbb{H}_{...}^0... .$$

In the case $f \in \mathbb{H}_{p,\theta}^\gamma(\tau)$, $g \in \mathbb{H}_{p,\theta}^{\gamma+1}(\tau, l_2)$ we write $(f, g) \in \mathcal{F}_{p,\theta}^\gamma(\tau)$ and

$$||(f, g)||_{\mathcal{F}_{p,\theta}^\gamma(\tau)} = ||f||_{\mathbb{H}_{p,\theta}^\gamma(\tau)} + ||g||_{\mathbb{H}_{p,\theta}^{\gamma+1}(\tau, l_2)}.$$

Finally, we introduce spaces of initial data. We write $u_0 \in U_{p,\theta}^\gamma$ if and only if $M^{2/p-1} u_0 \in L_p(\Omega, \mathcal{F}_0, H_{p,\theta}^{\gamma-2/p})$ and denote

$$||u_0||_{U_{p,\theta}^\gamma}^p = E||M^{2/p-1} u_0||_{H_{p,\theta}^{\gamma-2/p}}^p.$$

DEFINITION 2.2. *For a $\mathcal{D}(\mathbb{R}_+)$-valued function u defined on $\Omega \times [0, \infty)$ with $u(0, \cdot) \in U_{p,\theta}^{\gamma+1}$ and*

$$M^{-1} u \in \bigcup_\mu \bigcap_{T > 0} \mathbb{H}_{p,\theta}^\mu(\tau \wedge T), \tag{2.1}$$

we write $u \in \mathfrak{H}^{\gamma+1}_{p,\theta}(\tau)$ *if and only if* $u_x \in \mathbb{H}^{\gamma}_{p,\theta}(\tau)$ *and there exists* $(f,g) \in \mathcal{F}^{\gamma-1}_{p,\theta}(\tau)$ *such that for any* $\phi \in C_0^\infty(\mathbb{R}_+)$ *we have*

$$(u(t,\cdot),\phi) = (u(0,\cdot),\phi) + \int_0^t (M^{-1}f(s,\cdot),\phi)\,ds + \sum_{k=1}^{\infty}\int_0^t (g^k(s,\cdot),\phi)\,dw^k_s \tag{2.2}$$

for all $t \le \tau$ *at once with probability one. In this situation we also write* $M^{-1}f = \tilde{\mathbb{D}}u$, $g = \tilde{\mathbb{S}}u$,

$$du = M^{-1}f\,dt + g^k\,dw^k_t$$

and define $\mathfrak{H}^{\gamma+1}_{p,\theta,0}(\tau) = \mathfrak{H}^{\gamma+1}_{p,\theta}(\tau) \cap \{u:\ u(0,\cdot)=0\}$,

$$||u||^p_{\mathfrak{H}^{\gamma+1}_{p,\theta}(\tau)} = ||u_x||^p_{\mathbb{H}^{\gamma}_{p,\theta}(\tau)} + ||(f,g)||^p_{\mathcal{F}^{\gamma-1}_{p,\theta}(\tau)} + ||u(0,\cdot)||^p_{U^{\gamma+1}_{p,\theta}}. \tag{2.3}$$

As always, we drop τ *in* $\mathfrak{H}^{\gamma}_{p,\theta}(\tau)$ *and* $\mathcal{F}^{\gamma}_{p,\theta}(\tau)$ *if* $\tau = \infty$.

Remark 2.3 (cf. Remark 3.3 in [8]). Given $u \in \mathfrak{H}^{\gamma}_{p,\theta}(\tau)$, there exists only one pair of functions f and g in Definition 2.2. Therefore, the notation $M^{-1}f = \tilde{\mathbb{D}}u$, $g = \tilde{\mathbb{S}}u$, and (2.3) make sense.

It is also worth noting that the last series in (2.2) converges uniformly in t on each interval $[0,\tau\wedge T]$, $T \in (0,\infty)$, in probability.

Remark 2.4. It follows from Theorem 1.13 that, in Definition 2.2, the two requirements (2.1) and $u_x \in \mathbb{H}^{\gamma}_{p,\theta}(\tau)$ can be replaced with only one: $M^{-1}u \in \mathbb{H}^{\gamma+1}_{p,\theta}(\tau)$. In addition,

$$||M^{-1}u||_{\mathbb{H}^{\gamma+1}_{p,\theta}(\tau)} \le N||u_x||_{\mathbb{H}^{\gamma}_{p,\theta}(\tau)} \le N||M^{-1}u||_{\mathbb{H}^{\gamma+1}_{p,\theta}(\tau)},$$

where $N = N(\gamma,\theta,p)$.

Remark 2.5. The space $\mathfrak{H}^{\gamma}_{p,\theta}(\tau)$ is *not* $Q^{-1}_{p,\theta}\mathcal{H}^{\gamma}_{p}(\tau)$. However, obviously ϕu lies in $Q^{-1}_{p,\theta}\mathcal{H}^{\gamma}_{p}(\tau)$ for any $\phi \in C_0^\infty(\mathbb{R}_+)$ if $u \in \mathfrak{H}^{\gamma}_{p,\theta}(\tau)$. By Theorem 3.7 of [8] this easily implies that if $u \in \mathfrak{H}^{\gamma}_{p,\theta}(\tau)$ and $||u||_{\mathfrak{H}^{\gamma}_{p,\theta}(\tau)} = 0$, then u is indistinguishable from zero.

Of course, we identify elements of $\mathfrak{H}^{\gamma}_{p,\theta}(\tau)$ which are indistinguishable.

Remark 2.6. The spaces $\mathfrak{H}^{\gamma}_{p,\theta}(\tau)$ and $\mathfrak{H}^{\gamma}_{p,\theta,0}(\tau)$ are Banach spaces.

Indeed, their completeness is obtained as follows. If u_n is a Cauchy sequence in $\mathfrak{H}^{\gamma}_{p,\theta}(\tau)$, then $M^{-1}u_n$ is a Cauchy sequence in $\mathbb{H}^{\gamma}_{p,\theta}(\tau)$ by Remark 2.4 and hence it converges to some $M^{-1}u \in \mathbb{H}^{\gamma}_{p,\theta}(\tau)$. Also, $M\tilde{\mathbb{D}}u_n \to f$ and $\tilde{\mathbb{S}}u_n \to g$ for some $(f,g) \in \mathcal{F}^{\gamma-2}_{p,\theta}(\tau)$.

Next, for any $\phi \in C_0^\infty(\mathbb{R}_+)$ the sequence ϕu_n is a Cauchy sequence in $\mathcal{H}^{\gamma}_{p}(\tau)$, which is a Banach space by Theorem 3.7 of [8]. This easily implies that u has a modification $\bar{u}$ such that $\phi\bar{u}$ belongs to $\mathcal{H}^{\gamma}_{p}(\tau)$ for any $\phi \in C_0^\infty(\mathbb{R}_+)$, and $\bar{u}$ satisfies (2.2), so that $\bar{u} \in \mathfrak{H}^{\gamma}_{p,\theta}(\tau)$. One treats $\mathfrak{H}^{\gamma}_{p,\theta,0}(\tau)$ similarly.

Remark 2.7. By Remark 1.5 it follows that $f \in \mathbb{H}^{\gamma-1}_{p,\theta}(\tau)$ if and only if there exists a unique $h \in \mathbb{H}^{\gamma}_{p,\theta}(\tau)$ such that $M^{-1}f = Dh$. In addition, the norms of f and h are equivalent. Hence, one obtains the same space $\mathfrak{H}^{\gamma+1}_{p,\theta}(\tau)$ if in Definition 2.2 one replaces $M^{-1}f$ with f_x and instead of the condition $(f,g) \in \mathcal{F}^{\gamma-1}(\tau)$ requires $f \in \mathbb{H}^{\gamma}_{p,\theta}(\tau)$, $g \in \mathbb{H}^{\gamma}_{p,\theta}(\tau,l_2)$. In this case one obtains an equivalent norm by replacing $||(f,g)||^p_{\mathcal{F}^{\gamma-1}_{p,\theta}(\tau)}$ in (2.3) with

$$||f||^p_{\mathbb{H}^{\gamma}_{p,\theta}(\tau)} + ||g||^p_{\mathbb{H}^{\gamma}_{p,\theta}(\tau,l_2)}.$$

Remark 2.8. If $u \in \mathfrak{H}^{\gamma}_{p,\theta}(\tau)$, then $v := MDu \in \mathfrak{H}^{\gamma-1}_{p,\theta}(\tau)$ and

$$||MDu||_{\mathfrak{H}^{\gamma-1}_{p,\theta}(\tau)} \leq N(\gamma,\theta,p)||u||_{\mathfrak{H}^{\gamma}_{p,\theta}(\tau)}.$$

Indeed, we have $M^{-1}v = Du \in \mathbb{H}^{\gamma-1}_{p,\theta}(\tau)$, which by Remark 2.4 gives us a part of the needed properties of v. Also by Remark 2.7, $du = f_x\,dt + g^k\,dw^k_t$ with $f \in \mathbb{H}^{\gamma-1}_{p,\theta}(\tau)$ and $g \in \mathbb{H}^{\gamma-1}_{p,\theta}(\tau, l_2)$, so that $dv = (MDf - f)_x\,dt + MDg^k\,dw^k_t$, where by Remark 1.5

$$||MDf - f||_{\mathbb{H}^{\gamma-2}_{p,\theta}(\tau)} \leq N||f||_{\mathbb{H}^{\gamma-1}_{p,\theta}(\tau)}, \quad ||MDg||_{\mathbb{H}^{\gamma-2}_{p,\theta}(\tau,l_2)} \leq N||g||_{\mathbb{H}^{\gamma-1}_{p,\theta}(\tau,l_2)},$$

$$\begin{aligned}||M^{2/p-1}v(0,\cdot)||_{H^{\gamma-1-2/p}_{p,\theta}} &= ||MD(M^{2/p-1}u(0,\cdot)) - (2/p-1)M^{2/p-1}u(0,\cdot)||_{H^{\gamma-1-2/p}_{p,\theta}} \\ &\leq N||M^{2/p-1}u(0,\cdot)||_{H^{\gamma-2/p}_{p,\theta}}.\end{aligned}$$

Remark 2.9. From Remark 1.3 we have

$$||\Lambda^{\gamma}_{p,\theta}u||_{\mathbb{H}^{\mu}_{p,\theta}(\tau)} = ||u||_{\mathbb{H}^{\mu+\gamma}_{p,\theta}(\tau)}.$$

The assertions of the following theorem are straightforward corollaries of Remark 2.4 and of two Sobolev theorems. One says that $H^{\gamma}_p \subset \mathcal{C}^{\delta}$ if $\delta := \gamma - 1/p > 0$, where $\mathcal{C}^{\delta} = \mathcal{C}^{\delta}(\mathbb{R})$ is the Zygmund space (which differs from the usual Hölder space $C^{\delta} = C^{\delta}(\mathbb{R})$ only if δ is an integer; see [13]). The second one says that $H^{\gamma}_p \subset H^{\mu}_q$ if $\mu < \gamma$ and $\gamma - 1/p = \mu - 1/q$. These theorems are easily rewritten in terms of our spaces $H^{\gamma}_{p,\theta} = Q^{-1}_{p,\theta}H^{\gamma}_p$.

Theorem 2.10. (i) *If $\alpha := \gamma - 1/p > 0$ and $u \in \mathfrak{H}^{\gamma}_{p,\theta}(\tau)$, then $Q_{p,\theta}M^{-1}u \in L_p((0,\tau]], \mathcal{C}^{\alpha})$, where $\mathcal{C}^{\alpha}$ is the Zygmund space. In addition,*

$$E\int_0^{\tau} ||Q_{p,\theta}M^{-1}u(t,\cdot)||^p_{\mathcal{C}^{\alpha}}\,dt \leq N(d,\gamma,p)||u||^p_{\mathfrak{H}^{\gamma}_{p,\theta}(\tau)}.$$

(ii) *If $\mu < \gamma$, $\gamma - 1/p = \mu - 1/q$, and $u \in \mathfrak{H}^{\gamma}_{p,\theta}(\tau)$, then*

$$E\int_0^{\tau} ||M^{-1}u(t,\cdot)||^p_{H^{\mu}_{q,\theta q/p}}\,dt \leq N(d,\gamma,\mu,p)||u||^p_{\mathfrak{H}^{\gamma}_{p,\theta}(\tau)}.$$

In order to prove the solvability even of the simplest equations we need the following embedding theorem. However, the way in which the right-hand side of (2.4) depends on T will not be used.

Theorem 2.11. *Let $T \in (0,\infty)$ be a constant and let $\tau \leq T$. Then for any function $u \in \mathfrak{H}^{\gamma}_{p,\theta,0}(\tau)$, we have*

$$E\sup_{t\leq\tau}||u(t,\cdot)||^p_{H^{\gamma-1}_{p,\theta}} \leq N(p,\theta,\gamma)T^{(p-2)/p}||u||^p_{\mathfrak{H}^{\gamma}_{p,\theta}(\tau)}. \tag{2.4}$$

To prove this theorem we use the following fact, which is similar to Remark 2.2 of [5] or Remark 4.11 of [8].

Lemma 2.12. *Let $T \in (0,\infty)$ be a constant and let $\tau \leq T$. Let $u \in \mathcal{H}^{\gamma}_{p,0}(\tau)$ and $du = f\,dt + g^k\,dw^k_t$. Then for any constant $c > 0$,*

$$\begin{aligned}E\sup_{t\leq\tau}||u_x(t,\cdot)||^p_{H^{\gamma-2}_p} \leq N(p)T^{(p-2)/2}(c||u_{xx}||^p_{\mathbb{H}^{\gamma-2}_p(\tau)} \\ +c^{-1}||f||^p_{\mathbb{H}^{\gamma-2}_p(\tau)} + ||g_x||^p_{\mathbb{H}^{\gamma-2}_p(\tau,l_2)}).\end{aligned} \tag{2.5}$$

Proof. As always, it suffices to prove (2.5) for any particular γ and $\tau = T$ (regarding τ see, for instance, the proof of Theorem 7.1 in [8]). We take $\gamma = 2$. Then (2.5) becomes

$$E\sup_{t\le T}||u_x(t,\cdot)||^p_{L_p} \le N(p)T^{(p-2)/2}(c||u_{xx}||^p_{\mathbb{L}_p(T)}$$

$$+c^{-1}||f||^p_{\mathbb{L}_p(T)} + ||g_x||^p_{\mathbb{L}_p(T,l_2)}). \tag{2.6}$$

It suffices to prove this inequality for $c = 1$. Indeed, for any constant $a > 0$ we have $du(t, ax) = f(t, ax)\,dt + g^k(t, ax)\,dw^k_t$ and if (2.6) holds with $c = 1$, then

$$a^{p-1}E\sup_{t\le T}||u_x(t,\cdot)||^p_{L_p} = E\sup_{t\le T}||(u(t,a\,\cdot))_x||^p_{L_p}$$

$$\le NT^{(p-2)/2}(||(u(\cdot,a\,\cdot))_{xx}||^p_{\mathbb{L}_p(T)} + ||f(\cdot,a\,\cdot)||^p_{\mathbb{L}_p(T)} + ||(g(\cdot,a\,\cdot))_x||^p_{\mathbb{L}_p(T,l_2)})$$

$$= NT^{(p-2)/2}(a^{2p-1}||u_{xx}||^p_{\mathbb{L}_p(T)} + a^{-1}||f||^p_{\mathbb{L}_p(T)} + a^{p-1}||g_x||^p_{\mathbb{L}_p(T,l_2)}).$$

This proves (2.6) with a^p in place of c.

We further transform (2.6) with $c = 1$ by denoting $v = u_x$ and $h^k = g^k_x$, so that $dv = f_x\,dt + h^k\,dw^k_t$ and $v \in \mathcal{H}^1_{p,0}(T)$. We see that we only need to prove that

$$E\sup_{t\le T}||v(t,\cdot)||^p_{L_p} \le N(p)T^{(p-2)/2}(||v_x||^p_{\mathbb{L}_p(T)}$$

$$+||f||^p_{\mathbb{L}_p(T)} + ||h||^p_{\mathbb{L}_p(T,l_2)}). \tag{2.7}$$

By Theorem 2.1 of [5] or Theorem 4.10 of [8] and by the observation that $dv = (v_{xx} + (f - v_x)_x)\,dt + h^k\,dw^k_t$, for any $\lambda, T > 0$ we have

$$E\sup_{t\le T}(e^{-p\lambda t}||v(t,\cdot)||^p_{L_p}) \le N(||e^{-\lambda t}\bar{f}||^p_{\mathbb{L}_p(T)} + ||e^{-\lambda t}h||^p_{\mathbb{L}_p(T,l_2)}),$$

where $N = N(p, \lambda)$ and $\bar{f} = f - v_x$. For $\lambda = 1/p$ this yields

$$E\sup_{t\le T}||v(t,\cdot)||^p_{L_p} \le Ne^T(||\bar{f}||^p_{\mathbb{L}_p(T)} + ||h||^p_{\mathbb{L}_p(T,l_2)}).$$

By using the self-similarity of the equation $dv = (v_{xx} + \bar{f}_x)\,dt + h^k\,dw^k_t$ (that is, by considering equations like (3.6)), for any constant $c > 0$ we get

$$E\sup_{t\le T}||v(c^2t,c\,\cdot)||^p_{L_p} \le Ne^T(||c\bar{f}(c^2t,c\,\cdot)||^p_{\mathbb{L}_p(T)} + ||ch(c^2t,c\,\cdot)||^p_{\mathbb{L}_p(T,l_2)})$$

with $N = N(p)$. Changing variables we obtain

$$E\sup_{t\le T}||v(t,\cdot)||^p_{L_p} \le Ne^{T/c^2}c^{p-2}(||\bar{f}||^p_{\mathbb{L}_p(T)} + ||h||^p_{\mathbb{L}_p(T,l_2)}).$$

For $c^2 = T$ this is even a little bit stronger than (2.7) and the lemma is proved.

Proof of Theorem 2.11. For an appropriate $\zeta \in C_0^\infty(\mathbb{R}_+)$ we have

$$E\sup_{t\le\tau}||u(t,\cdot)||^p_{H^{\gamma-1}_{p,\theta}} \le N\sum_{n=-\infty}^{\infty} e^{n\theta} E\sup_{t\le\tau}||\zeta u(t,e^n\cdot)||^p_{H^{\gamma-1}_p}. \tag{2.8}$$

Let $du = f_x\,dt + g^k\,dw^k_t$. Then

$$d(\zeta(x)u(t,e^nx)) = \zeta(x)(f_x)(t,e^nx)\,dt + \zeta(x)g^k(t,e^nx)\,dw^k_t.$$

By Lemma 2.12 for $u_n(t,x) := \zeta(x)u(t,e^nx)$, $f_n(t,x) := \zeta(x)(f_x)(t,e^nx)$, $g_n(t,x) := \zeta(x)g(t,e^nx)$, and $c = e^{-np}$ we have

$$E\sup_{t\le\tau}||u_{nx}(t,\cdot)||^p_{H^{\gamma-2}_p} \le NT^{(p-2)/2}(e^{-np}||u_{nxx}||^p_{\mathbb{H}^{\gamma-2}_p(\tau)}$$

$$+e^{np}||f_n||^p_{\mathbb{H}^{\gamma-2}_p(\tau)} + ||g_{nx}||^p_{\mathbb{H}^{\gamma-2}_p(\tau,l_2)}). \tag{2.9}$$

To transform this inequality notice that all the functions $u_n(t,x)$ as functions of x have supports inside the support of ζ which is bounded. Therefore (see Remark 1.6),

$$||\zeta u(t,e^n\cdot)||_{H^{\gamma-1}_p} = ||u_n(t,\cdot)||_{H^{\gamma-1}_p} \le N||u_{nx}(t,\cdot)||_{H^{\gamma-2}_p}.$$

Furthermore, $||g_{nx}||_{H^{\gamma-2}_p(l_2)} \le ||g_n||_{H^{\gamma-1}_p(l_2)}$ and

$$\sum_{n=-\infty}^{\infty} e^{n\theta}||g_n||^p_{\mathbb{H}^{\gamma-1}_p(\tau,l_2)} \le N||g||^p_{\mathbb{H}^{\gamma-1}_{p,\theta}(\tau,l_2)} \le N||u||^p_{\mathfrak{H}^{\gamma}_{p,\theta}(\tau)}.$$

Also,

$$\sum_{n=-\infty}^{\infty} e^{n(\theta+p)}||f_n||^p_{\mathbb{H}^{\gamma-2}_p(\tau)} = \sum_{n=-\infty}^{\infty} e^{n\theta}||(M^{-1}\zeta)(MDf)(\cdot,e^n\cdot)||^p_{\mathbb{H}^{\gamma-2}_p(\tau)}$$

$$\le N||MDf||^p_{\mathbb{H}^{\gamma-2}_{p,\theta}(\tau)} \le N||f||^p_{\mathbb{H}^{\gamma-1}_{p,\theta}(\tau)} \le N||u||^p_{\mathfrak{H}^{\gamma}_{p,\theta}(\tau)},$$

$$\sum_{n=-\infty}^{\infty} e^{n(\theta-p)}||u_{nxx}||^p_{\mathbb{H}^{\gamma-2}_p(\tau)} \le \sum_{n=-\infty}^{\infty} e^{n(\theta-p)}||u_n||^p_{\mathbb{H}^{\gamma}_p(\tau)}$$

$$= \sum_{n=-\infty}^{\infty} e^{n\theta}||(M\zeta)(M^{-1}u)(\cdot,e^n\cdot)||^p_{\mathbb{H}^{\gamma}_p(\tau)}$$

$$\le N||M^{-1}u||^p_{\mathbb{H}^{\gamma}_{p,\theta}(\tau)} \le N||u||^p_{\mathfrak{H}^{\gamma}_{p,\theta}(\tau)}.$$

By combining this with (2.9) and (2.8) we get (2.4). The theorem is proved.

As always the main role is played by the spaces $\mathfrak{H}^{\gamma}_{p,\theta,0}(\tau)$ of functions with zero initial conditions. In connection with this it is worth noting that while constructing our theory we could replace

$$||u(0,\cdot)||^p_{U^{\gamma+1}_{p,\theta}} := E||M^{2/p-1}u(0,\cdot)||^p_{H^{\gamma+1-2/p}_{p,\theta}} \tag{2.10}$$

with

$$\inf\{||v_x||^p_{\mathbb{H}^{\gamma+1}_{p,\theta}} + ||M\tilde{\mathbb{D}}v||^p_{\mathbb{H}^{\gamma-1}_{p,\theta}} + ||\tilde{\mathbb{S}}v||^p_{\mathbb{H}^{\gamma}_{p,\theta}} : u - v \in \mathfrak{H}^{\gamma+1}_{p,\theta,0}\}.$$

Such an axiomatic approach to defining a norm of $u(0,\cdot)$ yields, of course, the solvability results for the widest possible class of initial data, namely, for those which are extendible at least in some way for $t > 0$. However, in applications we often want to know how to describe "admissible" initial data by knowing only their analytic properties. A partial answer to this question is given in the following theorem, which also shows why we use the norm given by (2.10).

THEOREM 2.13. *If $0 < \theta < p$ and $\gamma = 2$ and $1 < p < \infty$, then for every u_0 satisfying $M^{2/p-1}u_0 \in H^{\gamma-2/p}_{p,\theta}$ there exists a deterministic $u \in \mathfrak{H}^{\gamma}_{p,\theta}$ such that $du = D^2 u\,dt$, $u|_{t=0} = u_0$, and*

$$||u||^p_{\mathfrak{H}^{\gamma}_{p,\theta}} \le N(p,\gamma,\theta)||M^{2/p-1}u_0||^p_{H^{\gamma-2/p}_{p,\theta}}. \tag{2.11}$$

Proof. If $u_0 \in C_0^\infty(\mathbb{R}_+)$, then there is a unique function $u(t,x)$ which is bounded in $\mathbb{R}^2_+$ together with all its derivatives and which is a unique bounded solution of the heat equation $\partial u/\partial t = D^2 u$, $t > 0$ in $\mathbb{R}^2_+$ with initial condition $u(0,x) = u_0(x)$ and boundary condition $u(t,0) = 0$. Observe that u is given by $u(t,\cdot) = p_t * \bar{u}_0$, where $p_t(x) = (4\pi t)^{-1/2}\exp(-|x|^2/(4t))$ and $\bar{u}_0$ is an odd extension of u_0 on $\mathbb{R}$. By the way, from this representation it follows that $u(t,x) \to 0$ exponentially fast as $x \to \infty$ and the same is true for any derivative of u.

Next, we observe that $\partial|u(t,x)|^p/\partial t = p|u|^{p-2}uD^2u$, multiply this equality by x^c, with $c := \theta + 1 - p \in (1-p,1)$, and integrate by parts, and also use $|u(t,x)| \le N|x|$, $|u(t,x)|^{p-1}x^c \le Nx^\theta$, $|u(t,x)|^p x^{c-1} \le Nx^\theta$ for x close to zero. Finally we fix $T \in (0,\infty)$ and find that

$$\begin{aligned}\int_{\mathbb{R}_+} x^c|u(T,x)|^p\,dx - \int_{\mathbb{R}_+} x^c|u_0(x)|^p\,dx &= \int_0^T\int_{\mathbb{R}_+} px^c|u|^{p-2}uD^2u\,dxdt \\ &= -c\int_0^T\int_{\mathbb{R}_+} x^{c-1}D(|u|^p)\,dxdt - p(p-1)I = c(c-1)J - p(p-1)I,\end{aligned} \tag{2.12}$$

where

$$I := \int_0^T\int_{\mathbb{R}_+} x^c|u|^{p-2}(Du)^2\,dxdt, \quad J := \int_0^T\int_{\mathbb{R}_+} x^{c-2}|u|^p\,dxdt.$$

To estimate I from below through J, denote $v := |u|^{p/2}$ and observe that we have $|u|^{p-2}(Du)^2 = (2/p)^2(Dv)^2$ and by Minkowski's inequality

$$\int_0^\infty x^{c-2}|u|^p\,dx = \int_0^\infty x^{c-2}v^2\,dx = \int_0^\infty x^c\left(\int_0^1 v'(yx)\,dy\right)^2 dx$$

$$\le \left(\int_0^1 dy\left(\int_0^\infty x^c(v'(yx))^2\,dx\right)^{1/2}\right)^2 = \int_0^\infty x^c(v'(x))^2\,dx\left(\int_0^1 y^b\,dy\right)^2,$$

where $b = -1/2 - c/2 > -1$. By evaluating the last integral we get

$$\int_0^\infty x^{c-2}|u|^p\,dx \le p^2(1-c)^{-2}\int_0^\infty x^c|u|^{p-2}(Du)^2\,dx. \tag{2.13}$$

Hence $p(p-1)I \ge q^{-1}(1-c)^2J$, where $1/q = 1 - 1/p$, and from (2.12) we get

$$[q^{-1}(1-c)^2 - c(c-1)]J \le \int_{\mathbb{R}_+} x^c|u_0(x)|^p\,dx = ||M^{2/p-1}u_0||^p_{L_{p,\theta}}. \tag{2.14}$$

Here $L_{p,\theta} \supset H^{2-2/p}_{p,\theta}$ with the corresponding inequality for the norms since $2-2/p > 0$. Also, one can easily check that $q^{-1}(1-c)^2 - c(c-1) > 0$ for $0 < \theta < p$ and therefore, after passing to the limit as $T \to \infty$, we obtain the following intermediate estimate:

$$\int_0^\infty ||M^{-1}u(t,\cdot)||^p_{L_{p,\theta}}\,dt \le N||M^{2/p-1}u_0||^p_{H^{2-2/p}_{p,\theta}}. \tag{2.15}$$

An attentive reader might have noticed that the above derivation of (2.13) and (2.15) falls into some trouble if $1 < p < 2$. Indeed, then we get terms containing $|u|$ to a negative power and also the absolute continuity of v is not clear. However, the following fact is true even if $1 < p < 2$:

(i) the functions $|u|^{p/2}$ and $|u|^{p-2}uu_x$ are absolutely continuous on $\mathbb{R}$;

(ii) almost everywhere on $\mathbb{R}$ ($\infty \cdot 0 := 0$)

$$(|u|^{p/2})_x = \tfrac{p}{2}|u|^{p/2-2}uu_x,$$

$$(|u|^{p-2}uu_x)_x = |u|^{p-2}uu_{xx} + (p-1)|u|^{p-2}(u_x)^2.$$

Above we have only used this fact. However, we do not prove (i) and (ii). Instead, we show how to get (2.15) for $1 < p < 2$ by using an approximation argument.

For $\varepsilon > 0$ define $G_\varepsilon(s) = (s^2+\varepsilon)^{p/2} - \varepsilon^{p/2}$. As it is easy to see, we have $|G_\varepsilon(u)| \le (1+\varepsilon^{p/2})|u|^p$ and, for $|u| \le 1$,

$$|G'_\varepsilon(u)| = p(u^2+\varepsilon)^{p/2-1}|u| \le N(\varepsilon)|u| \le N(\varepsilon)|u|^{p-1}.$$

Also $G''_\varepsilon \ge 0$. Hence, owing to $\partial G_\varepsilon(u)/\partial t = G'_\varepsilon(u)D^2u$ and introducing

$$v(t,x) := \int_0^{u(t,x)} (G''_\varepsilon(s))^{1/2}\,ds,$$

we get as above

$$\int_{\mathbb{R}_+} x^cG_\varepsilon(u(T,x))\,dx - \int_{\mathbb{R}_+} x^cG_\varepsilon(u_0(x))\,dx$$

$$= c(c-1)\int_0^T\int_{\mathbb{R}_+} x^{c-2}G_\varepsilon(u)\,dxdt - \int_0^T\int_{\mathbb{R}_+} x^c(v')^2\,dxdt$$

$$\le c(c-1)\int_0^T\int_{\mathbb{R}_+} x^{c-2}G_\varepsilon(u)\,dxdt - 4^{-1}(1-c)^2\int_0^T\int_{\mathbb{R}_+} x^{c-2}v^2\,dxdt.$$

By letting $\varepsilon \downarrow 0$, noticing that $\lim_{\varepsilon\downarrow 0} G''_\varepsilon(s) = p(p-1)|s|^{p-2}$ and $c<1$, and using Fatou's lemma, we again arrive at (2.14) and (2.15).

Next, take a function $\zeta \in C_0^\infty(\mathbb{R}_+)$ and notice that for $u_n(t,x) := u(e^{2n}t, e^n x)$ we have

$$\frac{\partial}{\partial t}(\zeta u_n) = (\zeta u_n)_{xx} - 2(\zeta_x u_n)_x + \zeta_{xx} u_n. \tag{2.16}$$

Hence by inequalities (IV.3.1) and (IV.3.2) in [9] (also see Remark 2.3.2 in [13]) for any n we obtain

$$\int_0^\infty ||(\zeta u_n)_{xx}(t,\cdot)||^p_{H_p^{-1}}\,dt \le N||\zeta u_n(0,\cdot)||^p_{H_p^{1-2/p}}$$

$$+\int_0^\infty ||((2\zeta_x u_n)_x - \zeta_{xx}u_n)(t,\cdot)||^p_{H_p^{-1}}\,dt.$$

We make the change of variable t replacing it with $e^{2n}t$; then we multiply through the inequality by $e^{2n-np+\theta n}$ and observe that by Remark 1.6

$$||(\zeta u_n)_{xx}||_{H_p^{-1}} \ge N||(\zeta u_n)_x||_{L_p} \ge N||\zeta u_{nx}||_{L_p} - N||\zeta_x u_n||_{L_p},$$

where $N = N(\zeta, p)$. Also use the fact that

$$||(2\zeta_x u_n)_x - \zeta_{xx}u_n||_{H_p^{-1}} \le 2||(\zeta_x u_n)_x||_{H_p^{-1}} + N||\zeta_{xx}u_n||_{L_p} \le N||\eta u_n||_{L_p},$$

where $N = N(\zeta, p, \eta)$ and η is a more or less arbitrary function of class $C_0^\infty(\mathbb{R}_+)$ with support covering that of ζ.

Then we get

$$\int_0^\infty \sum_n e^{\theta n}||\zeta u_x(t, e^n\cdot)||^p_{L_p}\,dt \le N\sum_n e^{\theta n}||\xi(M^{2/p-1}u_0)(e^n\cdot)||^p_{H_p^{1-2/p}}$$

$$+N\int_0^\infty \sum_n e^{\theta n}||\eta_1 M^{-1}u(t,e^n\cdot)||^p_{L_p}\,dt,$$

where $\xi = M^{1-2/p}\zeta$ and η_1 is a function of type η. For the right choice of ζ we rewrite the last inequality as

$$\int_0^\infty ||u_x(t,\cdot)||^p_{L_{p,\theta}}\,dt \le N||M^{2/p-1}u_0||^p_{H^{1-2/p}_{p,\theta}} + N\int_0^\infty ||M^{-1}u(t,\cdot)||^p_{L_{p,\theta}}\,dt. \tag{2.17}$$

Next, we use (2.16) and inequalities (IV.3.1) and (IV.3.2) in [9] to write

$$\int_0^\infty ||(\zeta u_n)_{xx}(t,\cdot)||^p_{L_{p,\theta}}\,dt \le N||\zeta u_n(0,\cdot)||^p_{H_p^{2-2/p}}$$

$$+\int_0^\infty ||((2\zeta_x u_n)_x - \zeta_{xx}u_n)(t,\cdot)||^p_{L_{p,\theta}}\,dt.$$

If η_1 and η_2 are functions of class $C_0^\infty(\mathbb{R}_+)$ with supports covering that of ζ, then, for the same reasons as before, this inequality yields

$$\int_0^\infty ||\zeta u_{nxx}(t,\cdot)||^p_{L_{p,\theta}}\,dt \le N||\zeta u_n(0,\cdot)||^p_{H_p^{2-2/p}}$$

$$+\int_0^\infty ||\eta_1 u_{nx}(t,\cdot)||^p_{L_{p,\theta}}\,dt + \int_0^\infty ||\eta_2 u_n(t,\cdot)||^p_{L_{p,\theta}}\,dt,$$

and

$$\int_0^\infty ||Mu_{xx}(t,\cdot)||^p_{L_{p,\theta}}\,dt \le N||M^{2/p-1}u_0||^p_{H^{2-2/p}_{p,\theta}}$$

$$+N\int_0^\infty ||u_x(t,\cdot)||^p_{L_{p,\theta}}\,dt + N\int_0^\infty ||M^{-1}u(t,\cdot)||^p_{L_{p,\theta}}\,dt.$$

Together with (2.15), (2.17), and the equation $\partial u/\partial t = M^{-1}(Mu_{xx})$ the last inequality implies that $u \in \mathfrak{H}^2_{p,\theta}$ and that (2.11) holds with $\gamma = 2$.

Actually, above we have constructed a mapping $u_0 \in C_0^\infty(\mathbb{R}_+) \to u \in \mathfrak{H}^\gamma_{p,\theta}$. If we introduce an operator $\Pi : u_0 \to u$, then what is proved means that (for $\gamma = 2$)

$$||\Pi u_0||_{\mathfrak{H}^\gamma_{p,\theta}} \le N(p,\theta)||M^{2/p-1}u_0||_{H^{\gamma-2/p}_{p,\theta}} \tag{2.18}$$

if $u_0 \in C_0^\infty(\mathbb{R}_+)$. Remembering that $\mathfrak{H}^\gamma_{p,\theta}$ is a Banach space and relying on the usual continuity argument based on (2.18), we see that Π can be extended on all u_0 satisfying $M^{2/p-1}u_0 \in H^{\gamma-2/p}_{p,\theta}$ in such a way that $\partial \Pi u_0/\partial t = D^2\Pi u_0$, $\Pi u_0|_{t=0} = u_0$, and (2.18) holds. The theorem is proved.

Remark 2.14. We will see from Theorem 3.2 that Theorem 2.13 holds for any $\gamma \in \mathbb{R}$ and the solution is unique in $\mathfrak{H}^\gamma_{p,\theta}$.

In connection with this it is interesting to notice that Theorem 2.13 without weights and on $\mathbb{R}$ instead of $\mathbb{R}_+$ cannot hold for all $1 < p < 2$ if $\gamma = 1$. For instance, if $1 < p < 3/2$, then, for the solution u of the equation $du = D^2u\,dt$, $t > 0$, $x \in \mathbb{R}$, with initial condition given by the delta function, we have $u(0,\cdot) \in H_p^{1-2/p}$, but the pth power of the function u_x is not integrable over $\mathbb{R}^+ \times \mathbb{R}$.

3. SPDEs with constant coefficients on $\mathbb{R}_+$. Take a stopping time τ. On $\mathbb{R}_+$ we will be dealing with the following equation:

$$du = (au_{xx} + f_x)\,dt + (\sigma^k u_x + g^k)\,dw^k_t, \quad t \in (0,\tau), \tag{3.1}$$

where f and g^k are given $\mathcal{D}(\mathbb{R}_+)$-valued $\mathcal{P}$-measurable functions, a and σ^k are given real-valued $\mathcal{P}$-measurable functions, u is an unknown $\mathcal{D}(\mathbb{R}_+)$-valued function, and the equation is understood in the sense of distributions as follows. We say that u is a solution of (3.1) with given initial condition u_0 if for any test function $\phi \in C_0^\infty(\mathbb{R}_+)$ we have

$$\begin{aligned}(u(t,\cdot),\phi) &= (u_0,\phi)\\ &\quad+\int_0^t [a(s)(u(s,\cdot),\phi_{xx}) - (f(s,\cdot),\phi_x)]\,ds\\ &\quad+\sum_{k=1}^\infty \int_0^t [-\sigma^k(s)(u,\phi_x) + (g^k,\phi)]\,dw^k_t \end{aligned} \tag{3.2}$$

for all $t \leq \tau$ with probability one, where all integrals are assumed to have sense and the last series is also assumed to converge uniformly on each interval of time $[0, T \wedge \tau]$ in probability, where T is any finite constant.

Remark 3.1. If a function u belongs to $\mathfrak{H}^{\gamma+1}_{p,\theta}(\tau)$, then it satisfies (3.1) with $f = (MD)^{-1}M\tilde{\mathbb{D}}u - aDu$ and $g^k = \tilde{\mathbb{S}}^k u - \sigma^k Du$. In addition (see Remark 1.5), we have $f \in \mathbb{H}^{\gamma}_{p,\theta}(\tau)$ and $g \in \mathbb{H}^{\gamma}_{p,\theta}(\tau, l_2)$. Below we show that under an additional assumption on a and σ the mapping $u \to (f, g)$ is onto.

We always assume that for some constants $K \geq \delta > 0$ and all ω, t we have

$$K \geq 2a \geq 2a - |\sigma|^2_{l_2} \geq \delta.$$

Here is the main result of this section.

THEOREM 3.2. (i) *Let* $0 < \theta < p$, $1 < p < \infty$, $\gamma \in \mathbb{R}$, $f \in \mathbb{H}^{\gamma}_{p,\theta}(\tau)$, $g \in \mathbb{H}^{\gamma}_{p,\theta}(\tau, l_2)$, *and* $u_0 \in U^{\gamma+1}_{p,\theta}$. (ii) *Assume that one of the following conditions is satisfied:*

(a) $p \geq 2$ *and* $\theta \in [p-1, p)$*;*

(b) $p \geq 2$ *and* $\sigma \equiv 0$*;*

(c) $\sigma \equiv 0$ *and* $g \equiv 0$.

Then (3.1) *with initial data* u_0 *has a unique solution in class* $\mathfrak{H}^{\gamma+1}_{p,\theta}(\tau)$. *In addition, for this solution it holds that*

$$||u||_{\mathfrak{H}^{\gamma+1}_{p,\theta}(\tau)} \leq N\big(||f||_{\mathbb{H}^{\gamma}_{p,\theta}(\tau)} + ||g||_{\mathbb{H}^{\gamma}_{p,\theta}(\tau,l_2)} + ||u_0||_{U^{\gamma+1}_{p,\theta}}\big), \tag{3.3}$$

where $N = N(\gamma, \theta, p, K, \delta)$. *Finally, the uniqueness holds even if we replace condition* (a) *with:* $p \geq 2$ *and* $\theta \in (0, p)$.

Remark 3.3. In a subsequent paper on equations in $\mathbb{R}^d_+$ we will show that condition (a) can be relaxed to be $p \geq 2$ and $1 \leq \theta < p$. This could be done here too if one uses interpolation with respect to θ and the result of [7], where the case $\theta = 1$ is treated. However, there is a small gap in the arguments proving (2.9) of [7], so that strictly speaking we cannot use the result of [7].

Remark 3.4. Notice that when conditions (b) or (c) are satisfied, θ may be any number in $(0, p)$.

It is also worth noting that if $\theta \geq p$ or $\theta \leq 0$, then the statement of Theorem 3.2 is false even in the case of the heat equation. This can be shown by simple examples.

The proof of this theorem is based on two lemmas, the first of which we prove in section 4.

LEMMA 3.5. *Theorem* 3.2 *holds if* $\gamma = 1$.

LEMMA 3.6. *Let assumption* (i) *of Theorem* 3.2 *be satisfied and let* $\mu \leq \gamma$. *Assume that either* $p \geq 2$ *or* $\sigma \equiv g \equiv 0$. *Let* $\theta_1 \in \mathbb{R}$ *and let* $u \in \mathfrak{H}^{\mu+1}_{p,\theta_1}(\tau)$ *be a solution of* (3.1) *with initial condition* u_0. *Assume that* $M^{-1}u \in \mathbb{H}^{\mu+1}_{p,\theta}(\tau)$. *Then* $u \in \mathfrak{H}^{\gamma+1}_{p,\theta}(\tau)$ *and*

$$||u||_{\mathfrak{H}^{\gamma+1}_{p,\theta}(\tau)} \leq N\big(||f||_{\mathbb{H}^{\gamma}_{p,\theta}(\tau)} + ||g||_{\mathbb{H}^{\gamma}_{p,\theta}(\tau,l_2)} + ||u_x||_{\mathbb{H}^{\mu}_{p,\theta}(\tau)} + ||u_0||_{U^{\gamma+1}_{p,\theta}}\big),$$

where $N = N(\gamma, \mu, \theta, p)$.

Proof. For simplicity of notation we will only consider the case $\tau \equiv \infty$. The reader can easily make the necessary changes for general τ.

By virtue of (3.2) we have (2.2) with $x(au_{xx} + f_x)$ instead of f and $\sigma^k u_x + g^k$ instead of g^k. Upon taking into account the assumptions on f and g and remembering Remark 1.5, we conclude that we only need to prove that

$$||u_x||^p_{\mathbb{H}^{\gamma}_{p,\theta}} \leq N(||f||^p_{\mathbb{H}^{\gamma}_{p,\theta}} + ||g||^p_{\mathbb{H}^{\gamma}_{p,\theta}(l_2)} + ||u_x||^p_{\mathbb{H}^{\mu}_{p,\theta}} + ||u_0||^p_{U^{\gamma+1}_{p,\theta}}). \tag{3.4}$$

Since $||u_x||_{\mathbb{H}^{\nu}_{p,\theta}} \leq ||u_x||_{\mathbb{H}^{\mu}_{p,\theta}}$ for $\nu \leq \mu$, it suffices to prove (3.4) with some $\nu \leq \mu$ in place of μ. This shows that we may assume that $\gamma - \mu$ is an integer. Also we can go from μ up to γ in several steps each time getting an increase by one. Therefore, without loss of generality we may and will assume that $\gamma = \mu + 1$, so that (3.4) becomes

$$||u_x||^p_{\mathbb{H}^{\gamma}_{p,\theta}} \leq N(||f||^p_{\mathbb{H}^{\gamma}_{p,\theta}} + ||g||^p_{\mathbb{H}^{\gamma}_{p,\theta}(l_2)} + ||u_x||^p_{\mathbb{H}^{\gamma-1}_{p,\theta}} + ||u_0||^p_{U^{\gamma+1}_{p,\theta}}). \tag{3.5}$$

Take a function $\zeta \in C_0^{\infty}(\mathbb{R}_+)$ with $M\zeta$ satisfying condition (1.4). One can easily check that the functions $u_n(t,x) := u(e^{2n}t, e^n x)$ satisfy the equation

$$du_n = (a_n u_{nxx} + f_n)\, dt + (\sigma^k_n u_{nx} + g^k_n)\, dw^k_t(n), \tag{3.6}$$

where

$$a_n(t) = a(e^{2n}t), \quad \sigma^k_n(t) = \sigma^k(e^{2n}t), \quad w^k_t(n) = e^{-n} w_{e^{2n}t},$$

$$f_n(t,x) = e^{2n}(f_x)(e^{2n}t, e^n x), \quad g^k_n(t,x) = e^n g^k(e^{2n}t, e^n x).$$

Observe that for any n, the processes $w^k_t(n)$ are independent Wiener processes. From (3.6) we get

$$d(\zeta u_n) = (a_n(\zeta u_n)_{xx} + \bar{f}_n)\, dt + (\sigma^k_n(\zeta u_n)_x + \bar{g}^k_n)\, dw^k_t(n), \tag{3.7}$$

where

$$\bar{f}_n = \zeta f_n - 2a_n \zeta_x u_{nx} - a_n \zeta_{xx} u_n, \quad \bar{g}^k_n = \zeta g^k_n - \sigma^k_n \zeta_x u_n.$$

Since $M^{-1}u \in \mathbb{H}^{\gamma}_{p,\theta}$, it is easy to see that for any $\eta \in C_0^{\infty}(\mathbb{R}_+)$ we have $\eta u_n \in \mathbb{H}^{\gamma}_p$ and $\eta u_{nx} \in \mathbb{H}^{\gamma-1}_p$, so that $\bar{f}_n \in \mathbb{H}^{\gamma-1}_p$ and $\bar{g}_n \in \mathbb{H}^{\gamma}_p(l_2)$. By Theorem 2.1 of [5] or Theorem 4.10 of [8] for $p \geq 2$ (with uniqueness in $\mathcal{H}^{\gamma}_p(\tau)$ and existence in $\mathcal{H}^{\gamma+1}_p(\tau)$, here we use $\zeta u_n \in \mathcal{H}^{\gamma}_p(\tau)$), (3.7) implies that

$$||(\zeta u_n)_{xx}||^p_{\mathbb{H}^{\gamma-1}_p} \leq N(||\bar{f}_n||^p_{\mathbb{H}^{\gamma-1}_p} + ||\bar{g}_n||^p_{\mathbb{H}^{\gamma}_p(l_2)} + E||\zeta u_0(e^n\, \cdot)||^p_{H^{\gamma+1-2/p}_p}), \tag{3.8}$$

where $u_{0n}(x) = u_0(e^n x)$. Actually, Theorem 2.1 of [5] or Theorem 4.10 of [8] treats the case $u_0 = 0$. One deals with arbitrary u_0 as in the beginning of the proof of Theorem 5.1 of [8] by just subtracting the solution of the heat equation $\partial v/\partial t = v_{xx}$ with initial condition u_0. Owing to the fact that supports of all functions ζu_n coincide with that of ζ, from (3.8) by Remark 1.6, we get

$$||\zeta u_n||^p_{\mathbb{H}^{\gamma+1}_p} \leq N(||\bar{f}_n||^p_{\mathbb{H}^{\gamma-1}_p} + ||\bar{g}_n||^p_{\mathbb{H}^{\gamma}_p(l_2)} + E||\zeta u_0(e^n\, \cdot)||^p_{H^{\gamma+1-2/p}_p}). \tag{3.9}$$

The same conclusions are true if $1 < p < 2$ and $\sigma \equiv g \equiv 0$, which can be seen from section 9, Chapter IV of [9] or from the proof of Theorem 2.1 of [5] or Theorem 4.10 of [8], where one can take any $p \in (1, \infty)$ if $\sigma \equiv g \equiv 0$. In particular, in all cases $\zeta u_n \in \mathbb{H}^{\gamma+1}_p$ and (3.9) holds.

Now we multiply (3.9) through by $e^{(2-p+\theta)n}$ and sum up over all n. We also use

$$\sum_{n=-\infty}^{\infty} e^{(2-p+\theta)n}||\zeta f_n||^p_{\mathbb{H}^{\gamma-1}_p} = \sum_{n=-\infty}^{\infty} e^{(2-p+\theta)n}||e^{2n}(f_x)(e^{2n}\cdot, e^n\cdot)\zeta||^p_{\mathbb{H}^{\gamma-1}_p}$$

$$= \sum_{n=-\infty}^{\infty} e^{\theta n}||e^n(f_x)(\cdot, e^n\cdot)\zeta||^p_{\mathbb{H}^{\gamma-1}_p} = \sum_{n=-\infty}^{\infty} e^{\theta n}||(Mf_x)(\cdot, e^n\cdot)M^{-1}\zeta||^p_{\mathbb{H}^{\gamma-1}_p}$$

$$\le N||Mf_x||^p_{\mathbb{H}^{\gamma-1}_{p,\theta}} \le N||f||^p_{\mathbb{H}^{\gamma}_{p,\theta}},$$

$$\sum_{n=-\infty}^{\infty} e^{(2-p+\theta)n}||\zeta g_n||^p_{\mathbb{H}^{\gamma}_p(l_2)} = \sum_{n=-\infty}^{\infty} e^{\theta n}||g(\cdot, e^n\cdot)\zeta||^p_{\mathbb{H}^{\gamma}_p(l_2)} \le N||g||^p_{\mathbb{H}^{\gamma}_{p,\theta}(l_2)},$$

$$\sum_{n=-\infty}^{\infty} e^{(2-p+\theta)n}||\zeta_x u_{nx}||^p_{\mathbb{H}^{\gamma-1}_p} = \sum_{n=-\infty}^{\infty} e^{(2-p+\theta)n}||e^n(u_x)(e^{2n}\cdot, e^n\cdot)\zeta_x||^p_{\mathbb{H}^{\gamma-1}_p}$$

$$= \sum_{n=-\infty}^{\infty} e^{\theta n}||(u_x)(\cdot, e^n\cdot)\zeta_x||^p_{\mathbb{H}^{\gamma-1}_p} \le N||u_x||^p_{\mathbb{H}^{\gamma-1}_{p,\theta}},$$

$$\sum_{n=-\infty}^{\infty} e^{(2-p+\theta)n}||\zeta_{xx} u_n||^p_{\mathbb{H}^{\gamma-1}_p} = \sum_{n=-\infty}^{\infty} e^{\theta n}||(M^{-1}u)(\cdot, e^n\cdot)M\zeta_{xx}||^p_{\mathbb{H}^{\gamma-1}_p}$$

$$\le N||M^{-1}u||^p_{\mathbb{H}^{\gamma-1}_{p,\theta}} \le N||u_x||^p_{\mathbb{H}^{\gamma-1}_{p,\theta}}.$$

Similarly, we estimate $\zeta_x u_n$, we notice that

$$\begin{aligned}
&\sum_n e^{(2-p+\theta)n}E||\zeta u_0(e^n\cdot)||^p_{H^{\gamma+1-2/p}_p} \\
&\quad = \sum_n e^{\theta n}E||(M^{2/p-1}u_0)(e^n\cdot)M^{1-2/p}\zeta||^p_{H^{\gamma+1-2/p}_p} \\
&\quad \le NE||M^{2/p-1}u_0||^p_{H^{\gamma+1-2/p}_{p,\theta}} = N||u_0||^p_{U^{\gamma+1}_{p,\theta}},
\end{aligned}$$

and we get

$$\sum_{n=-\infty}^{\infty} e^{(2-p+\theta)n}||\zeta u_n||^p_{\mathbb{H}^{\gamma+1}_p} \le I,$$

where I is the right-hand side of (3.5). Here the left-hand side equals

$$\sum_{n=-\infty}^{\infty} e^{\theta n}||(M^{-1}u)(\cdot, e^n\cdot)M\zeta||^p_{\mathbb{H}^{\gamma+1}_p} \ge N^{-1}||M^{-1}u||^p_{\mathbb{H}^{\gamma+1}_{p,\theta}} \ge N^{-1}||u_x||^p_{\mathbb{H}^{\gamma}_{p,\theta}}$$

and the lemma is proved.

Proof of Theorem 3.2. For simplicity of notation we only consider the case $\tau \equiv \infty$. Actually, as it is easy to see, the statement of existence for $\tau \equiv \infty$ implies the statement of existence for other τ, and the proof of uniqueness for general τ can be done in the same way as in the case $\tau \equiv \infty$.

Case $\gamma \geq 1$. The uniqueness follows from Lemma 3.5 and the fact that $\mathfrak{H}^2_{p,\theta,0} \supset \mathfrak{H}^{\gamma+1}_{p,\theta,0}$, which implies that the difference of two solutions belongs to $\mathfrak{H}^2_{p,\theta,0}$. The existence and estimate (3.3) follow from Lemmas 3.5 and 3.6 (applied with $\mu = 1$) and the observation that by Lemma 3.5

$$||u_x||^p_{\mathbb{H}^1_{p,\theta}} \leq N(||f||^p_{\mathbb{H}^1_{p,\theta}} + ||g||^p_{\mathbb{H}^1_{p,\theta}(l_2)} + ||u_0||^p_{U^2_{p,\theta}})$$

$$\leq N(||f||^p_{\mathbb{H}^\gamma_{p,\theta}} + ||g||^p_{\mathbb{H}^\gamma_{p,\theta}(l_2)} + ||u_0||^p_{U^{\gamma+1}_{p,\theta}}).$$

Case $\gamma < 1$. Denote by $\mathcal{R}$ the operator which maps (f, g, u_0) with $f \in \mathbb{H}^\gamma_{p,\theta}$, $g \in \mathbb{H}^\gamma_{p,\theta}(l_2)$, and $u_0 \in U^{\gamma+1}_{p,\theta}$ into the solution $u \in \mathfrak{H}^{\gamma+1}_{p,\theta}$ of (3.1) with initial data u_0. So far we know that $\mathcal{R}$ is well defined in spaces $\mathbb{H}^\gamma_{p,\theta} \times \mathbb{H}^\gamma_{p,\theta}(l_2) \times U^{\gamma+1}_{p,\theta}$ for $\gamma \geq 1$. If $\gamma < 1$, as a candidate for the solution of (3.1) we try

$$\tilde{u} = \Lambda^n_{p,\theta}\mathcal{R}(\Lambda^{-n}_{p,\theta}f, \Lambda^{-n}_{p,\theta}g, M^{1-2/p}\Lambda^{-n}_{p,\theta}M^{2/p-1}u_0),$$

where $n + \gamma \geq 1$ and (see Remark 1.3)

$$(\Lambda^{-n}_{p,\theta}f, \Lambda^{-n}_{p,\theta}g, M^{1-2/p}\Lambda^{-n}_{p,\theta}M^{2/p-1}u_0) \in \mathbb{H}^{n+\gamma}_{p,\theta} \times \mathbb{H}^{n+\gamma}_{p,\theta}(l_2) \times U^{n+\gamma+1}_{p,\theta}.$$

If the operators $\Lambda_{p,\theta}$, $M^{2/p-1}$, and D were commuting, then our candidate would be an exact solution of (3.1). Since this is not the case, we need an additional argument based on Lemma 1.14.

Take $n = 2$ and first let $1 > \gamma \geq 0$. Then by what we know in the case $\gamma \geq 1$, we have

$$v := \mathcal{R}(\Lambda^{-2}_{p,\theta}f, \Lambda^{-2}_{p,\theta}g, M^{1-2/p}\Lambda^{-2}_{p,\theta}M^{2/p-1}u_0) \in \mathfrak{H}^{\gamma+3}_{p,\theta},$$

$$dv = (av_{xx} + (\Lambda^{-2}_{p,\theta}f)_x)\,dt + (\sigma^k v_x + \Lambda^{-2}_{p,\theta}g^k)\,dw^k_t.$$

We apply $\Lambda^2_{p,\theta}$ to both parts of this equality, or in other words we substitute $(\Lambda^2_{p,\theta})^*\phi$, where $(\Lambda^2_{p,\theta})^*$ is the formal adjoint to $\Lambda^2_{p,\theta}$, in place of ϕ in (3.2). Now our candidate becomes

$$\tilde{u} = \Lambda^2_{p,\theta}v.$$

We claim that $\tilde{u}$ belongs to $\mathfrak{H}^{\gamma+1}_{p,\theta}$ and there exists

$$(\bar{f}, \bar{g}, \bar{u}_0) \in \mathbb{H}^{\gamma+1}_{p,\theta} \times \mathbb{H}^{\gamma+1}_{p,\theta}(l_2) \times U^{\gamma+2}_{p,\theta} \tag{3.10}$$

such that

$$d\tilde{u} = (a\tilde{u}_{xx} + f_x + \bar{f}_x)\,dt + (\sigma^k\tilde{u}_x + g^k + \bar{g}^k)\,dw^k_t, \tag{3.11}$$

$$\tilde{u}(0, \cdot) = u_0 + \bar{u}_0.$$

Indeed, by Remarks 2.4 and 2.8 and Lemma 1.14 we easily get that

$$\tilde{u} \in \mathfrak{H}^{\gamma+1}_{p,\theta}, \quad M^{-1}v \in \mathbb{H}^{\gamma+3}_{p,\theta}, \quad M^{-1}\tilde{u} = \Lambda^2_{p,\theta} M^{-1}v + P_1 M^{-1}v \in \mathbb{H}^{\gamma+1}_{p,\theta},$$

$$D\tilde{u} = \Lambda^2_{p,\theta} Dv + P_1 Dv \in \mathbb{H}^{\gamma}_{p,\theta}$$

and that $\tilde{u}$ satisfies (3.11) with

$$\bar{f} = 4P_2 M^{-1}v + ((2b-1)I + 2MD)\Lambda^{-2}_{p,\theta} f, \quad \bar{g}^k = \sigma^k P_1 Dv.$$

Obviously, $\bar{f}$ and $\bar{g}$ are as in (3.10). Also by Lemma 1.14 at $t = 0$,

$$M^{2/p-1}\tilde{u} = M^{2/p-1}\Lambda^2_{p,\theta} M^{1-2/p}(M^{2/p-1}v)$$

$$= \Lambda^2_{p,\theta} M^{2/p-1}v + c_1 M^{2/p-1}v + c_2 MDM^{2/p-1}v =: M^{2/p-1}u_0 + M^{2/p-1}\bar{u}_0,$$

where

$$M^{2/p-1}\bar{u}_0 \in L_p(\Omega, \mathcal{F}_0, H^{\gamma+2-2/p}_{p,\theta}).$$

This finishes the proofs of (3.10) and our claim.

Since $\gamma + 1 \geq 1$, it follows from (3.10) that the function $\bar{u} := \mathcal{R}(\bar{f}, \bar{g}, \bar{u}_0)$ is well defined, belongs to $\mathfrak{H}^{\gamma+2}_{p,\theta}$, and the function $u = \tilde{u} - \bar{u}$ is of class $\mathfrak{H}^{\gamma+1}_{p,\theta}$ and solves (3.1). For thus constructed u estimate (3.3) follows from the explicit representation and known estimates for $\mathcal{R}$, P_i, MD.

By repeating the above argument, we consider the case $0 > \gamma \geq -1$, this time using the fact that $\gamma + 1 \geq 0$ and relying upon the result for $\gamma \geq 0$. One can continue in the same way, and it only remains to prove the uniqueness of solutions in $\mathfrak{H}^{\gamma+1}_{p,\theta}$.

It suffices to consider the case $f = 0, g = 0, u_0 = 0$ (and $\gamma < 1$). In this case any solution $u \in \mathfrak{H}^{\gamma+1}_{p,\theta,0}$ also belongs to $\mathfrak{H}^2_{p,\theta,0}$ by Lemma 3.6 and its uniqueness follows from Lemma 3.5.

The theorem is thus proved.

Remark 3.7. In the above argument one can use $(MD)^2$ instead of $\Lambda^2_{p,\theta}$, which would make the argument shorter. We prefer $\Lambda^2_{p,\theta}$ bearing in mind a generalization to a multidimensional case.

Remark 3.8. From the above derivation of Theorem 3.2 from Lemma 3.5 it is seen that, if the assertions of Theorem 3.2 hold for some particular γ, p, θ, a, and σ satisfying the conditions of Theorem 3.2, then they hold for any $\gamma \in \mathbb{R}$ with the same p, θ, a, σ.

4. Proof of Lemma 3.5. First notice that by Theorem 2.13 for almost every ω the function $\bar{u} := \Pi u_0$ is well defined, $\bar{u} \in \mathfrak{H}^2_{p,\theta}$, $\bar{u}|_{t=0} = u_0$, $\partial\bar{u}/\partial t = \bar{f}_x$ with $\bar{f} \in \mathbb{H}^1_{p,\theta}$, and an appropriate estimate of $||\bar{u}_x||_{\mathbb{H}^1_{p,\theta}}$ and $||\bar{f}||_{\mathbb{H}^1_{p,\theta}}$ through $||u_0||_{U^2_{p,\theta}}$ holds. This implies that in the equation

$$du = (au_{xx} + (a\bar{u}_x + f - \bar{f})_x)\,dt + (\sigma^k u_x + (\sigma^k \bar{u}_x + g^k))\,dw^k_t$$

we have $a\bar{u}_x + f - \bar{f} \in \mathbb{H}^1_{p,\theta}$ and $\sigma\bar{u}_x + g \in \mathbb{H}^1_{p,\theta}(l_2)$. Also, obviously if we can solve the above equation in $\mathfrak{H}^2_{p,\theta,0}$, then by adding to the solution the function $\bar{u}$ we get a solution of (3.1) with initial data u_0. Therefore, in the proof of Lemma 3.5 without loss of generality, we may and will confine ourselves only to the case $u_0 \equiv 0$.

Furthermore, we may assume that $a \equiv 1$. Indeed, to get the result for the general case one only needs to use a random time change. Namely, let us define

$$\psi(t) = \int_0^t a(s)ds, \quad \tau(t) = \inf\{s \geq 0 : \psi(s) \geq t\},$$

$$\tilde{w}_k(t) = \int_0^{\tau(t)} \sqrt{a(s)}dw_k(s), \quad \tilde{f}(t,x) = f(\tau(t),x)/a(\tau(t)),$$

$$\tilde{\sigma}(t) = \sigma(\tau(t))/\sqrt{a(\tau(t))}, \quad \tilde{g}(t,x) = g(\tau(t),x)/\sqrt{a(\tau(t))},$$

$$\tilde{u}(t,x) = u(\tau(t),x).$$

Direct computations (see, for instance, Lemma IV.2.2 and Theorem IV.2.3 in [3]) show that $\tilde{w}_k(t)$ are independent Wiener processes and also that u is a solution of (3.1) if and only if $\tilde{u}$ is a solution of

$$d\tilde{u} = (\tilde{u}_{xx} + \tilde{f})\,dt + (\tilde{\sigma}_k \tilde{u}_x + \tilde{g}_k)\,d\tilde{w}_k(t).$$

Therefore, we easily get the desired result for general a from the result for $a \equiv 1$. Finally, obviously we may assume that $\tau \leq T$ where the constant $T < \infty$. Thus, we may and will assume that $u_0 = 0$, $a \equiv 1$, and unless stated explicitly otherwise $\tau \leq T$.

We divide the proof of the lemma in this case into the following subcases:

1. $p \geq 2$ and $\theta \in [p-1, p)$, existence;
2. $p \geq 2$ and $\theta \in (0, p)$, uniqueness;
3. $p \geq 2$ and $\sigma \equiv 0$;
4. $\sigma \equiv 0$ and $g \equiv 0$.

4.1. Case $p \geq 2$ and $\theta \in [p-1, p)$. Existence. We use the following simple lemma.

LEMMA 4.1. *Let functions f, h be defined on $\mathbb{R}_+$, be locally absolutely continuous, and satisfy*

$$\int_0^\infty |f(x)g(x)|\,dx < \infty. \tag{4.1}$$

Then

$$\int_0^\infty x f(x) g'(x)\,dx = -\int_0^\infty x f'(x) g(x)\,dx - \int_0^\infty f(x)g(x)\,dx$$

if at least one of the sides of this equality makes sense.

This fact easily follows if one integrates by parts between a, b with $0 < a < b < \infty$ and then lets $a \downarrow 0$ and $b \to \infty$ after noticing that (4.1) implies that

$$\liminf_{a \downarrow 0} |af(a)g(a)| = \liminf_{b \to \infty} |bf(b)g(b)| = 0.$$

Denote by $\mathcal{E}$ the collection of functions of the form

$$f(t,x) = \sum_{i=1}^m I_{(\!(\tau_{i-1},\tau_i]\!]}(t) f_i(x),$$

where $f_i \in C_0^\infty(\mathbb{R}_+)$ and τ_i are stopping times, $\tau_i \leq \tau_{i+1} \leq \tau$. The set $\mathcal{E}$ is dense in $\mathbb{H}^1_{p,\theta}(\tau)$, which follows from a similar fact for spaces $\mathbb{H}^\gamma_p$ (see [5] or [8]) and the definition of $\mathbb{H}^\gamma_{p,\theta}(\tau)$. Also, the collection of sequences $g = (g_k)$, such that each g_k belongs to $\mathcal{E}$ and only finitely many of g_k are different from 0, is dense in $\mathbb{H}^1_{p,\theta}(\tau, l_2)$. It follows that in the proof of existence and estimate (3.3) we may assume that f and g are of this type.

Next, we use an argument from [7]. We continue $f(t,x)$ to be an even function and $g(t,x)$ to be an odd function of $x \in \mathbb{R}$. Also take an infinitely differentiable odd function $\alpha(x)$ such that $\alpha(x) = 1$ for large x, $\alpha(x) = 0$ for $|x| \leq 2$ and on $\mathbb{R}$ consider the equation

$$du = (u_{xx} + f_x)\,dt + (\alpha\sigma^k u_x + g^k)\,dw^k_t. \tag{4.2}$$

The following lemma is proved in the end of this subsection.

LEMMA 4.2. *In $\mathcal{H}^0_p(\tau)$ there exists a unique solution u of* (4.2) *with zero initial condition. Moreover, $u \in \mathfrak{H}^0_{p,\theta}(\tau)$ and*

$$||u||_{\mathfrak{H}^0_{p,\theta}(\tau)} \leq N||f||_{\mathbb{H}^1_{p,\theta}(\tau)} + N||g||_{\mathbb{L}_{p,\theta}(\tau,l_2)}, \tag{4.3}$$

where N is independent of τ, f, and g.

Now notice that the equation

$$du = (u_{xx} + f_x)\,dt + (\alpha_n\sigma^k u_x + g^k)\,dw^k_t, \tag{4.4}$$

where $\alpha_n(x) = \alpha(e^n x)$, also has a solution $u \in \mathfrak{H}^0_{p,\theta}(\tau)$ for which (4.3) holds with the same N. To prove this, it suffices to use scaling properties of the norms in $H^\gamma_{p,\theta}$ (see Remark 1.4) and to observe that if u is a solution of (4.2), then the function $u_n(t,x) = u(e^{2n}t, e^n x)$ satisfies (3.6) with the same f_n, g_n, and $w_t(n)$ and with $a_n = 1$ and $\sigma_n(t) = \alpha(e^n x)\sigma(e^{2n}t)$.

Denote u_n the solution of (4.4). Then u_n satisfies (4.3) and, in particular, $M^{-1}u_n$ form a bounded sequence in $\mathbb{L}_{p,\theta}(\tau)$. Denote u a weak limit of a subsequence of u_n. As in the proof of Theorem 3.11 of [8] we get that $u \in \mathfrak{H}^0_{p,\theta}(\tau)$. Then passing to the limit in (4.4) and observing that $\alpha(e^n x) \to 1$ for $x > 0$, we get that u satisfies (3.1) and estimate (4.3). It follows from Lemma 3.6 that $u \in \mathfrak{H}^2_{p,\theta}(\tau)$ and (3.3) holds with $\gamma = 1$ and $u_0 = 0$. This finishes the proof of existence.

Proof of Lemma 4.2. The existence and uniqueness of solution $u \in \mathcal{H}^1_p(\tau)$ of (4.2) is asserted in Theorem 3.2 of [5] or Theorem 5.1 of [8]. Therefore, we only need to prove that $u \in \mathfrak{H}^0_{p,\theta}(\tau)$ and that (4.3) holds.

By the definition of the norm in $\mathfrak{H}^\gamma_{p,\theta}(\tau)$ and by Remarks 2.4 and 2.7, it is sufficient to show that $M^{-1}u \in \mathbb{L}_{p,\theta}(\tau)$ and

$$||M^{-1}u||^p_{\mathbb{L}_{p,\theta}(\tau)} \leq N||f||^p_{\mathbb{H}^1_{p,\theta}(\tau)} + N||g||^p_{\mathbb{L}_{p,\theta}(\tau,l_2)}. \tag{4.5}$$

Owing to our choice of f and g, from [5] or [8] we know that $u \in \mathcal{H}^\gamma_p(\tau)$ for any γ and, in particular, for almost any ω, the function $u(t,x)$ is infinitely differentiable with respect to x and all its derivatives are continuous in t. This implies that (4.2) holds pointwise (a.s.). In addition, by uniqueness the function $u(t,x)$ is odd with respect to x, so that, in particular, $u(t,0) = 0$.

Again by choice of f and g, the function u satisfies the heat equation $u_t = u_{xx}$ for $0 < x < 2$ with zero initial and zero boundary value for $x = 0$. If we set $u(t,x) = 0$

for $t < 0$, then it satisfies the heat equation for all $t \le T$ and $0 < x < 2$. For such functions it is well known (see, for instance, the maximum principle and Theorem 8.4.4 in [4]) that for any integer $n \ge 0$,

$$\sup_{0<x<1, t\le T} |D^n u(t,x)| \le N(n) \sup_{t\le T} |u(t,2)|.$$

Therefore, for $\theta > 0$,

$$E\int_0^\tau \int_0^1 |u/x|^p x^{\theta-1}\,dxdt \le NE \sup_{0<x<1,t\le T} |u_x|^p \le NE\sup_{t\le\tau} |u(t,2)|^p.$$

In addition, as has been mentioned above, we have $u \in \mathcal{H}_p^\gamma(\tau)$ for any γ. By embedding theorems (see [5] or [8])

$$E \sup_{[0,\tau]\times\mathbb{R}_+} |u|^p < \infty,$$

which proves that, for any $\theta > 0$, we have $M^{-1}u\eta \in \mathbb{L}_{p,\theta}(\tau)$ if $\eta = \eta(x)$ is smooth and vanishes for $x \ge 1$. In the same way it is proved that for any integer $n \ge 0$ and $\theta > 0$,

$$E\int_0^\tau \int_0^1 |D^n u|^p x^{\theta-1}\,dxdt < \infty. \tag{4.6}$$

On the other hand, $|M^{-1}u|^p x^{\theta-1} \le |u|^p$ if $x \ge 1$ and $\theta \le p+1$. Hence, $M^{-1}u \in \mathbb{L}_{p,\theta}(\tau)$ not only for $\theta \in [p-1,p)$ but for all $\theta \in (0, p+1]$.

Next, we claim that, actually, for any $\theta \in (0,p+1]$ and $\gamma \ge 0$, we have

$$u \in \mathfrak{H}_{p,\theta}^\gamma(\tau). \tag{4.7}$$

To prove this claim, let $\zeta \in C^\infty(\mathbb{R})$ be such that $\zeta(x) = 1$ for $x \le 1/2$ and $\zeta(x) = 0$ for $x \ge 1$. We want to apply Theorem 1.10 to prove that $\zeta u \in \mathfrak{H}_{p,\theta}^\gamma(\tau)$. Notice that we already know that $M^{-1}\zeta u \in \mathbb{L}_{p,\theta}(\tau)$. Also from (4.6) it follows that $M^n D^n(\zeta u)_x \in \mathbb{L}_{p,\theta}$ for any integer n. Hence the inclusion $\zeta u \in \mathfrak{H}_{p,\theta}^\gamma(\tau)$ follows indeed from Theorem 1.10.

To prove the claim it only remains to prove that $v := (1-\zeta)u \in \mathfrak{H}_{p,\theta}^\gamma(\tau)$. Observe that $u \in \mathcal{H}_p^\gamma(\tau)$ and $v \in \mathcal{H}_p^\gamma(\tau)$ for any γ. Also, v satisfies

$$dv = (v_{xx} + \bar f)\,dt + (\alpha\sigma^k v_x + \bar g^k)\,dw_t^k, \tag{4.8}$$

where

$$\bar f = (1-\zeta)f_x + 2\zeta_x u_x + \zeta_{xx}u, \quad \bar g^k = (1-\zeta)g^k + \alpha\sigma^k\zeta_x u.$$

Now, consider the following equation on $\mathbb{R}$:

$$d\tilde u = (\tilde u_{xx} - 2\tilde u_x \tanh x + (2\tanh^2 x - 1)\tilde u + \bar f\cosh x)\,dt$$

$$+(\alpha\sigma^k\tilde u_x - \alpha\sigma^k\tilde u\tanh x + \bar g^k\cosh x)\,dw_t^k,$$

with zero initial condition. Because of compactness of supports of $\bar f$ and $\bar g$, by already cited results from [5] or [8] there is a unique solution $\tilde u$ in class $\mathcal{H}_p^\gamma(\tau)$ for any γ. Of course, $\tilde u/\cosh x \in \mathcal{H}_p^\gamma(\tau)$ for any γ. In addition, one can easily check that $\tilde u/\cosh x$ satisfies (4.8). By the uniqueness of solutions of (4.8) in class $\mathcal{H}_p^\gamma(\tau)$, we conclude that $v = \tilde u/\cosh x$ and, in particular, $v\cosh x \in \mathcal{H}_p^\gamma(\tau)$ for any γ. Now the fact that $v \in \mathfrak{H}_{p,\theta}^\gamma(\tau)$ for any γ follows easily from the observation that $v = 0$ if $x \le 1$ and $x^n/\cosh x$ is bounded.

Next we remember that (4.2) holds pointwise and we apply Itô's formula to $|u(t,x)|^p x^c$, where $c = \theta + 1 - p$. We get that, for any $x \in \mathbb{R}_+$ and $t \le \tau$, a.s.

$$\int_0^t I(s,x)\,ds + \sum_k \int_0^t px^c|u|^{p-2}u(\alpha\sigma^k u_x - g^k)\,dw_s^k = |u(t,x)|^p x^c \ge 0, \tag{4.9}$$

where

$$I := px^{\theta-1}G(v)(xu_{xx}) + px^{\theta-1}G(v)(xf_x) + bx^{\theta-1}\sum_k |v|^{p-2}(\alpha\sigma^k u_x - g^k)^2,$$

$$b := p(p-1)/2, \quad v := u/x, \quad G(r) := |r|^{p-2}r.$$

It follows that for any $x \in \mathbb{R}_+$ there is a sequence of stopping times $\tau(n) \uparrow \tau$ localizing the stochastic integral in (4.9) so that

$$E\int_0^{\tau(n)} I(s,x)\,ds \ge 0. \tag{4.10}$$

It turns out that for almost any $x \in \mathbb{R}_+$, here one can replace $\tau(n)$ with τ and integrate with respect to x over $\mathbb{R}_+$. To prove this it suffices to prove that

$$E\int_0^\tau \int_{\mathbb{R}_+} |I(s,x)|\,dxds < \infty. \tag{4.11}$$

Observe that (4.7) for $\gamma = 2$ means that

$$M^{-1}u,\ u_x,\ Mu_{xx} \in \mathbb{L}_{p,\theta}(\tau), \tag{4.12}$$

which implies (4.11) since by Hölder's inequality

$$\begin{aligned} &E\int_0^\tau \int_0^\infty |G(v)|\,|xu_{xx}|x^{\theta-1}\,dxdt \\ &\quad = E\int_0^\tau \int_0^\infty |u(t,x)/x|^{p-1}|xu_{xx}(t,x)|x^{\theta-1}\,dxdt \\ &\quad \le ||M^{-1}u||^{p-1}_{\mathbb{L}_{p,\theta}(\tau)}||Mu_{xx}||_{\mathbb{L}_{p,\theta}(\tau)}, \end{aligned}$$

$$E\int_0^\tau \int_0^\infty |G(v)|\,|xf_x|x^{\theta-1}\,dxdt \le ||M^{-1}u||^{p-1}_{\mathbb{L}_{p,\theta}(\tau)}||Mf_x||_{\mathbb{L}_{p,\theta}(\tau)},$$

$$E\int_0^\tau \int_0^\infty |v|^{p-2}|u_x|^2x^{\theta-1}\,dxdt \le ||M^{-1}u||^{p-2}_{\mathbb{L}_{p,\theta}(T)}||u_x||^2_{\mathbb{L}_{p,\theta}(\tau)},$$

$$E\int_0^\tau \int_0^\infty |v|^{p-2}|g|^2_{l_2}x^{\theta-1}\,dxdt \le ||M^{-1}u||^{p-2}_{\mathbb{L}_{p,\theta}(\tau)}||g||^2_{\mathbb{L}_{p,\theta}(\tau,l_2)}.$$

Having thus proved (4.11), from (4.10) we conclude

$$E\int_0^\tau \int_{\mathbb{R}_+} I(s,x)\,dxds \geq 0. \tag{4.13}$$

While estimating the integral with respect to x in (4.13) we integrate by parts after noticing that (4.12) also implies that

$$E\int_0^\tau \int_0^\infty |G(u)||u_x|x^{\theta-1}\,dxdt < \infty.$$

By Lemma 4.1 for almost all $(\omega,t) \in (0,\tau]\!]$ we get

$$p\int_0^\infty x^{\theta-1}G(v)(xu_{xx})\,dx = p\int_0^\infty x^c G(u)u_{xx}\,dx$$

$$= -p(p-1)\int_0^\infty |u|^{p-2}|u_x|^2 x^c\,dx - c\int_0^\infty x^{c-1}(|u|^p)_x\,dx$$

$$= -p(p-1)\int_0^\infty |u|^{p-2}|u_x|^2 x^c\,dx + c(c-1)\int_0^\infty |M^{-1}u|^p x^{\theta-1}\,dx.$$

Furthermore,

$$\begin{aligned}\left|\int_0^\infty x^{\theta-1}G(v)(xf_x)\,dx\right| &\leq ||v||^{p-1}_{L_{p,\theta}}||Mf_x||_{L_{p,\theta}} \\ &\leq \varepsilon||M^{-1}u||^p_{L_{p,\theta}} + N(\varepsilon,p)||f||^p_{H^1_{p,\theta}},\end{aligned}$$

where $\varepsilon > 0$ is arbitrary. Finally, while estimating the terms in (4.13) which came from stochastic integrals we also use

$$(\alpha\sigma^k u_x - g^k)^2 \leq (1+\varepsilon)|\sigma^k|^2|u_x|^2 + (1+\varepsilon^{-1})|g^k|^2.$$

Then from (4.13) we conclude that for any $\varepsilon > 0$,

$$p(p-1)E\int_0^\tau \int_0^\infty [(1+\varepsilon)|\sigma|^2_{l_2}/2 - 1]|u|^{p-2}|u_x|^2 x^c\,dxdt$$

$$+[(\theta+1-p)(\theta-p)+\varepsilon]E\int_0^\tau \int_0^\infty |M^{-1}u|^p x^{\theta-1}\,dxdt$$

$$+N(\varepsilon,p)(||f||^p_{\mathbb{H}^1_{p,\theta}(\tau)} + ||g||^p_{\mathbb{L}_{p,\theta}(\tau,l_2)})\,dxdt \geq 0. \tag{4.14}$$

Now comes the only place where we need $\theta \in [p-1,p)$. This condition implies that $(\theta+1-p)(\theta-p) \leq 0$. Also $|\sigma|^2_{l_2} \leq 2-\delta$. By using (2.13) we conclude that the first term in (4.14) is strong enough if ε is small and (4.14) implies (4.5). This brings the proof of Lemma 4.2 to an end.

4.2. Case $p \geq 2$ and $\theta \in (0,p)$. Uniqueness. Suppose that $u \in \mathfrak{H}^{\gamma}_{p,\theta}(\tau)$ is a solution of

$$du = u_{xx}\,dt + \sigma^k u_x\,dw^k_t \tag{4.15}$$

with zero initial condition. By Lemma 3.6 it follows that $u \in \mathfrak{H}^{\gamma}_{p,\theta}(\tau)$ for all γ and also $u\zeta \in \mathcal{H}^{\gamma}_{p}(\tau)$ for all $\zeta \in C^{\infty}_{0}(\mathbb{R}_+)$. Hence we again have (4.12) and the equation is satisfied pointwise. For $\theta \in [p-1,p)$, this makes it possible to estimate the norm $||u||_{\mathfrak{H}^{0}_{p,\theta}(\tau)}$ using the same computations as in Lemma 4.2. Since now $f = g_k = 0$, the result is $||u||_{\mathfrak{H}^{1}_{p,\theta}(\tau)} = 0$.

Next notice that, for any $\theta \in (0,p)$, there exists $\theta_1 \in (p-1,p)$ such that $\theta < \theta_1 < \theta + p$. Also as above, for any γ any solution of (4.15) in $\mathfrak{H}^{\gamma}_{p,\theta}(\tau)$ with zero initial condition also belongs to $\mathfrak{H}^{1}_{p,\theta}(\tau)$. Hence, the following result implies the uniqueness for general $\theta \in (0,p)$.

LEMMA 4.3. *Let γ, θ_1, and p be such that the first two assertions of Theorem* 3.2 *hold for $u_0 \equiv 0$, any stopping time τ, and these γ, θ_1, and p (for instance, $\gamma = 1$, $\theta_1 \in [p-1,p)$, and $p \geq 2$). Let $q \geq p$, $\theta \neq 0$, and $\theta \neq q$ satisfy $\theta/q < \theta_1/p \leq \theta/q + 1$. Let τ be a stopping time and $u \in \mathfrak{H}^{1}_{q,\theta,0}(\tau)$ satisfy* (3.1) *with some $f \in \mathbb{L}_{p,\theta_1}(\tau)$ and $g \in \mathbb{L}_{p,\theta_1}(\tau,l_2)$. Then $u \in \mathfrak{H}^{1}_{p,\theta_1,0}(\tau)$.*

Proof. By Remark 3.8 we may assume that $\gamma = 0$. Let v be the unique solution of (3.1) in $\mathfrak{H}^{1}_{p,\theta,0}(\tau)$ with given f and g. To prove the lemma we prove that $u = v$.

Let κ be an infinitely differentiable function such that $\kappa(x) = 1$ for $|x| \leq 1$ and $\kappa(x) = 0$ for $|x| \geq 2$. Define $\kappa_n = \kappa(x/n)$.

First we prove that for any n,

$$u\kappa_n \in \mathfrak{H}^{1}_{p,\theta_1,0}(\tau). \tag{4.16}$$

To this end observe that

$$E\int_0^{\tau}\int_0^{\infty} |(u\kappa_n)_x|^p x^{\theta_1 - 1}\,dxdt \leq 2^{p-1}E\int_0^{\tau}\int_0^{\infty} |u_x\kappa_n|^p x^{\theta_1-1}\,dxdt$$

$$+2^{p-1}E\int_0^{\tau}\int_0^{\infty} |u\kappa_{nx}|^p x^{\theta_1-1}\,dxdt, \tag{4.17}$$

where by Hölder's inequality the first term on the right is less than a constant times

$$E\int_0^{\tau}\int_0^{2n} |u_x x^{(\theta-1)/q}|^p x^{\theta_1 - 1 - (\theta-1)p/q}\,dxdt$$

$$\leq \left(E\int_0^{\tau}\int_0^{2n} |u_x|^q x^{\theta-1}\,dxdt\right)^{p/q} T^{1-p/q}\left(\int_0^{2n} x^c\,dx\right)^{1-p/q},$$

with

$$c = [\theta_1 - 1 - (\theta-1)p/q]q/(q-p) = \frac{qp}{(q-p)}\left(\frac{\theta_1}{p} - \frac{\theta}{q}\right) - 1.$$

Since $c > -1$, the first term on the right in (4.17) is finite. One can similarly treat the second term after noticing that $|u\kappa_{nx}| \leq N|u/x|$ and $u/x \in \mathbb{L}_{q,\theta}(\tau)$. The same argument yields $u\kappa_n/x \in \mathbb{L}_{p,\theta_1}(\tau)$ and this proves (4.16).

Now, let $\bar{u} = u - v$. By what we have just proved, $\bar{u}\kappa_n$ belongs to $\mathfrak{H}^1_{p,\theta_1,0}(\tau)$. Also $\bar{u}\kappa_n$ satisfies the following equation similar to (3.7)

$$d(\bar{u}\kappa_n) = (a(\bar{u}\kappa_n)_{xx} + \bar{f}_{nx})\,dt + (\sigma^k(\bar{u}\kappa_n)_x + \bar{g}^k_n)\,dw^k_t,$$

where

$$\bar{f}_n(t,x) = a(t)\int_x^\infty [2\kappa_{nx}(y)\bar{u}_x(t,y) + \kappa_{nxx}(y)\bar{u}(t,y)]\,dy$$

$$= -2a\kappa_{nx}\bar{u} + (MD)^{-1}(Ma\kappa_{nxx}\bar{u}), \quad \bar{g}^k = -\sigma^k\kappa_{nx}\bar{u}.$$

Hence, by our assumptions and Remark 1.5

$$||\bar{u}\kappa_n||_{\mathfrak{H}^1_{p,\theta_1}(\tau)} \leq N||\kappa_{nx}\bar{u}||_{\mathbb{L}_{p,\theta_1}(\tau)} + N||M\kappa_{nxx}\bar{u}||_{\mathbb{L}_{p,\theta_1}(\tau)}. \tag{4.18}$$

Here, for instance, ($\kappa_{nx} \leq N/n$)

$$||\kappa_{nx}\bar{u}||^p_{\mathbb{L}_{p,\theta_1}(\tau)} \leq Nn^{-p}E\int_0^\tau\int_n^{2n} |\bar{u}|^p x^{\theta_1-1}\,dxdt$$

$$\leq NE\int_0^\tau\int_n^{2n} |v/x|^p x^{\theta_1-1}\,dxdt + Nn^{\theta_1-p-1}E\int_0^\tau\int_n^{2n} |u|^p\,dxdt.$$

The first term on the right tends to zero as $n \to \infty$ since $v/x \in \mathbb{H}^0_{p,\theta_1}(\tau)$. To prove the same for the second term use Hölder's inequality to get that it is less than

$$NT^{1-p/q}n^{\theta_1-p-p/q}\left(E\int_0^\tau\int_n^{2n} |u|^q\,dxdt\right)^{p/q}$$

$$\leq Nn^c\left(E\int_0^\tau\int_n^{2n} |u|^q x^{\theta-1}\,dxdt\right)^{p/q}, \tag{4.19}$$

where $c = \theta_1 - p - p/q - (\theta-1)p/q \leq 0$ by virtue of $\theta_1/p \leq 1 + \theta/q$. Theorem 2.11 implies that the right-hand side of (4.19) tends to zero as $n \to \infty$.

In the same way using the fact that $|M\kappa_{nxx}| \leq N/n$ we get that the second term on the right in (4.18) tends to zero as well. Thus (use Theorem 2.11)

$$E\sup_{t\leq\tau}\int_0^\infty |\bar{u}(t,x)|^p x^{\theta_1-1}\,dxdt \leq \liminf_{n\to\infty} E\sup_{t\leq\tau}||\bar{u}(t,\cdot)\kappa_n||^p_{H^0_{p,\theta_1}}$$

$$\leq N\liminf_{n\to\infty}||\bar{u}\kappa_n||^p_{\mathfrak{H}^1_{p,\theta_1}(\tau)} = 0.$$

The lemma is proved.

4.3. Case $\sigma \equiv 0$ and $p \geq 2$. Uniqueness follows directly from section 4.2. To prove existence notice that as has been emphasized in section 4.1 the only place where we used $\theta \in [p-1, p)$ is right after (4.14). But in our present situation $\sigma \equiv 0$ and from (4.14) and (2.13) we conclude that

$$[p^{-1}(p-1)(p-\theta)^2 + (\theta+1-p)(p-\theta) + \varepsilon]||M^{-1}u||^p_{\mathbb{L}_{p,\theta}(\tau)}$$

$$\leq N(\varepsilon,p)(||f||^p_{\mathbb{H}^1_{p,\theta}} + ||g||^p_{\mathbb{L}_{p,\theta}(\tau,l_2)}). \tag{4.20}$$

Observe that the condition $0 < \theta < p$ is equivalent to $p^{-1}(p-1)(p-\theta)^2 + (\theta+1-p)(p-\theta) > 0$. Therefore, for ε small enough we again get (4.5). This takes care of the existence.

4.4. Case $\sigma \equiv 0$ and $g \equiv 0$. Actually, this is the case of the heat equation without any stochastic terms. In this case Lemma 3.6 is available for any $p > 1$ and as in section 4.1, to prove existence, it suffices to prove (4.5) for f as in section 4.1. This time we get (4.20) with $\varepsilon = 0$ even for $1 < p < 2$, which is proved by the same approximating argument as in the proof of Theorem 2.13 right after (2.15). Hence, we have existence.

The uniqueness is proved as in the beginning of section 4.2 observing that this time we do not need condition $\theta \in [p-1, p)$ to be satisfied and yet have (4.20).

This finishes the proof of Lemma 3.5.

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