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No Class: Russell on Contextual Definition and the Elimination of Sets

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Michael Kremer complains that my first-order presentation of Russell's logicist reduction "misrepresents the technical achievement of Russell's theory of classes" – which he identifies with the "ontological economy" of eliminating commitment to classes (sets). I disagree. Russell did attempt to eliminate classes, and he connected this attempt to other interesting projects. These projects aside, however, the purported elimination was not, in my opinion, a genuine achievement, and the relevant issues have little to do with first-order vs. second-order symbolizations.

Kremer's discussion focuses on what he calls Russell's "contextual definition of classes," the purpose of which is "to eliminate the notion of classes."

1. $F(\ \{x: Gx\}\ ) =_{df} \exists H(\forall y(Hy \leftrightarrow Gy) \,\&\, F(H))$

This characterization of the definition is tendentious. What (1) defines is a notation containing complex singular terms for classes. By identifying contents of sentences containing such terms with those of sentences that don't, Russell eliminates the terms from the primitive vocabulary of his theory. Whether or not an ontological reduction is achieved depends on how predicate quantifiers are understood. On standard second-order interpretations, there is no reduction, since they range over sets. By contrast, Russell took the quantifiers to range over "propositional functions." Kremer tells us little about what these are -- for good reason. Russell's discussion vacillates between taking them to be formulas, properties, or functions from nonlinguistic arguments to nonlinguistic propositions. Mostly, he seems to have inclined to the former.[1] So, if strict historical fidelity is the issue, then Russell should be taken to have seen himself as reducing classes to expressions. But this is problematic.

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[1] To take two, albeit retrospective, indicators: "In the language of the second order, variables denote symbols, not what is symbolized," Russell (1940: 192), and "Whitehead and I thought of a propositional function as an expression," Russell (1959: 124). See Sainsbury (1979: 278-295) for a good discussion of Russell's view of quantification into predicate position, plus Sainsbury's take on how he could have improved it.

For one thing, the ontology of formulas isn't innocent. Since there are infinitely many that will never be produced, they must be types – abstract sequences of existing simple expressions. And what are sequences, if not the set-theoretic constructions they are standardly taken to be? On this account, the ontological " achievement" of the "no-class theory" is to have eliminated classes in favor of classes. Nor would matters be helped by contending that sequences are *sui generous*, since their formal and ontological features parallel those of classes. Then there is the logical problem. To treat predicate variables as ranging over formulas is to risk artificially limiting their range. If, as is natural, the formulas are those of a single language, then, on the standard assumption that no formula is infinitely long, the range of the variables will be restricted to a denumerable infinity -- with crippling results on set theory, and the expressive power of the logic.[2] Far from being a "technical achievement," this would be a technical slip. One could, of course, relax the usual assumptions. But then the ontology of formulas would become even more contentious, and the resulting logic more complex – with no gain in historical accuracy, since Russell didn't foresee any of this. Finally, there is the semantic problem. Arithmetical sentences aren't about language, which, on Russell's account, they would be if his predicate quantifiers ranged over formulas. For all these reasons, something else is needed as the value of a predicate variable.

Since his semantics identifies the meanings of subsentential expressions – including predicates – with nonlinguistic entities they denote, such entities are a natural choice. A basis for implementing this idea is given in Volume 1, chapter 5 of my history. We let 'x is green' denote the propositional function g, which assigns to any object o the proposition expressed by 'x is green' relative to an assignment of o to 'x'. This proposition is a structured complex in which the

[2] The expressive distinctiveness of second-order logic – e.g. the completeness of second-order arithmetic and the impossibility of complete second-order proof procedures – depends on the ability of second-order quantifiers to range over the nondenumerable infinity of subsets of a denumerable domain.

property of being green is predicated of o. '∃x x is green' says that g "is sometimes true" – i.e. that g assigns a true proposition to at least one argument. '∃F (F grass)' says the same thing about the propositional function denoted by '(F grass)' -- which assigns to any propositional function f the proposition that f assigns to grass. How one thinks of the metaphysical nature of such functions – set theoretically, or in some other way – is not crucial. Nor, contrary to Kremer, is there any semantic reason not to identify functions that agree on the values of all arguments. On this conception, g is *the* function that assigns to any o the proposition that o is green. Knowing that it assigns a truth to something is a pretty good approximation to knowing that something is green.

With propositional functions as values of predicate variables, (1) stipulates that to say that the class of Gs is F is to say that there is a propositional function h which is both F and equivalent to g (where g/h are denoted by 'Gy'/'Hy', and the functions are equivalent iff they assign true propositions to the same things). Talk of propositional functions can, in this way, be substituted for talk of classes. However, this is not a genuine reduction. If one recognizes functions of this sort from objects to propositions, one can hardly refuse to recognize similar functions from objects to other entities, including truth values. But the distinction between the class of Gs and the function that assigns truth to all and only Gs is a distinction without a difference. So, on this interpretation, the "no-class theory" achieves no real "ontological economy".

It appears that Kremer doesn't accept this conception of propositional functions. Although he agrees that Russellian propositional functions should, ultimately, be understood as nonlinguistic, he regards them as "intensional." What does that mean? In one sense, *extensional functions* are correlations of arguments and values that are identical whenever they map the same arguments onto the same values. Since Kremer denies that propositional functions are extensional in this sense, it is tempting to interpret him as taking *intensional functions* to be correlations that can differ in such cases, when the same values are reached by different procedures. But on this interpretation, his position is untenable. Since *intensional functions* in this sense are just *extensional functions*, plus the murky notion of a procedure (when are two of them the same?), eliminating the latter (which are interdefinable with sets) in terms of the former makes little sense.

The problem, at bottom, is that Kremer is confused about the intensional / extensional distinction. The notion, "intensional function," is obscure, and Kremer's, "one thing that is certain is that Russell's propositional functions are *intensional* entities" is worse. It is clear what it is for a linguistic construction to be intensional or extensional, but unclear what it is for an arbitrary *entity* to be one of the other. Sometimes, as with properties vs. classes, it is easy figure out what the contrast is supposed to be. The pair <property, instantiation-relation> differs from the pair <class, membership-relation> in that properties with the same instantiators can differ, while classes with the same members can't. But this is fortuitous. Since arbitrary entities don't come with obvious analogues of the membership relation, the general notion of an intensional (or extensional) *entity* has no definite sense.

Russell does, of course, speak of intensional vs. extensional functions. But this talk is obscured by his habitual conflation of formulas with the nonlinguistic functions they denote. Typically, he has in mind a linguistic construction that is intensional/extensional, while leaving the relation between the function and the construction in his usual muddle. For example, in *Principia* *20, p. 187, he says, " 'I believe (x) Φx' is an *intensional* function, because even if (x) (Φx ↔Ψx), it by no means follows that I believe (x) Ψx provided I believe (x) Φx." Here a *sentence* is said to be both intensional, which it is, and a function, which it isn't. To make sense of this, one must tread carefully. If one wishes to distinguish formulas from the functions they denote, while continuing to apply the labels "intensional" and "extensional" to the latter, one must explicitly state what the distinction comes to.

Kremer's failure to do this leads him to make a critical mistake. He assumes that Russell's distinction between intensional and extensional functions *of functions* requires propositional functions that are not mere correlations of arguments and values. It doesn't. Let *extensional*$_2$ and *intensional*$_2$ stand the for the notions relevant to Russell's distinction. When 'F' is a predicate variable, the functions denoted by '~(Fa)' and 'I believe Fa' are functions from functions to propositions. The former is *extensional*$_2$ and the latter is *intensional*$_2$, because the former always maps *equivalent propositional functions* onto *equivalent propositions*, while the latter doesn't. (Propositions are equivalent iff they agree in truth value; propositional functions are equivalent iff the same arguments are always mapped to equivalent propositions.) Nevertheless, *both* functions are mere pairings of arguments and values – and so, extensional *in the sense defined earlier.* Thus, when Kremer wrongly concludes that since (some) Russellian propositional functions are *intensional*$_2$, they can't be *extensional*$_1$, he conflates two fundamentally different understandings of *extensional* vs. *intensional*. Since even *intensional*$_2$ *functions* are extensional in the sense that characteristic functions of sets are, Kremer's admission that trading sets for their characteristic functions yields no ontological economy undermines his central claim. If, (i) the propositional functions that Russell's predicate quantifiers range over are *extensional*$_1$, and (ii), one who is committed to these functions can't plausibly avoid commitment to other functions that are *extensional*$_1$, including characteristic functions (which Kremer admits are on a par with sets), then Russell can't have achieved an ontological economy by adopting his contextual definition.[3]

[3] There are, of course, advantages to second-order quantification over Russellian propositional functions rather than sets – especially if one wants to make room for belief, or other intensional, operators. Such systems need not explicitly invoke sets, and may even, as Russell shows, use (1) to simulate certain kinds of talk of sets. However, one who adopts such a system does not achieve a genuine ontological reduction, because the propositional functions one is committed to make it impossible to deny the existence of characteristic functions, which are acceptable iff sets are.

Can one salvage such an economy by abandoning Russellian propositional functions, and identifying values of predicate variables with *properties* that are the meanings of predicates, and open sentences? Though this seems to be more of a revision than an interpretation of Russell, such a system makes sense. According to it '∃x x is green' says that being green has the property *combining with some (appropriate) argument to form a truth*, while '∃F (F grass)' says that the property *being instantiated by grass* – denoted by '(F grass)' – also has that property. The idea works fine. However, to use it in Russellian semantics, one needs a way of specifying the compound properties expressed by complex formulas. That, of course, was just what propositional functions were made for. Given an open formula $\ldots v_1 \ldots v_n \ldots$, we identify the compound property it denotes with the function that maps n-tuples of arguments onto the structured propositions expressed by the formula relative to assignments of those n-tuples to the variables. Nothing could be easier. If, in the name of salvaging the "no-class theory," such functions are forbidden, then something else must take their place.

Here's a thought. Since the meanings of complete sentences are structured propositions, the meanings of open sentences – the compound properties they express – can be identified with structured propositional matrices – propositions with gaps in them. To say that such a matrix "is sometimes true" is to say that there is a way of filling the gaps that results in a truth. This works as an account of quantification in essentially the way that propositional functions do. However, the no-class theory is still in trouble. For what is a propositional matrix? It would seem, at the very least, to be a kind of sequence (with gaps) the elements of which are either individual meanings or sequences of meanings. But now we are back to an earlier objection. For what are sequences, if not set-theoretic constructions, or something so similar as to be ontologically on a par with sets? No matter how we twist and turn, the requirements of Russellian semantics seem to preclude using the language of *Principia* to eliminate sets.[4] Hence, even on the most favorable interpretation, the "no-class theory" remains dubious.

This is one main reason I didn't discuss Russell's ideas about eliminating classes in my history. Too many problems for too little result. The entities to which classes are supposed to be reduced – propositional functions – are more taken for granted by Russell than seriously investigated. As already indicated, he speaks confusingly and inconsistently about them, and the view that seems to be uppermost in his mind – that they are expressions – is obviously inadequate. Although other choices – *extensional*$_1$ functions and gappy propositions – make more sense, he doesn't systematically explore them, and, as we have seen, they aren't promising candidates for achieving ontological economies anyway. In addition, the "no-class theory" has been more or less historically inert – central figures in logic, mathematics, and formal semantics haven't adopted it. In light of this, it should be clear why there was no room for Russell's ideas about the eliminability of sets in the history I aimed to write. It should be equally clear why I presented the reduction in a simple first-order system of naïve set theory. Since sets weren't going to be eliminated anyway, there was little point in putting readers through the extra complexity of a second-order system.[5] Russell's most important achievement was in showing how numbers can be identified with sets,

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[4] If, as might be maintained, our mastery of Russell's language – including our recognition of (1) as defining the contents of one set of sentences in terms of another (as opposed to simply being a true biconditional) -- requires being given a systematic semantics, then the attempt to use that language to eliminate sets requires a set- and sequence-free ontology not only of compound properties, but also of sentences.

[5] The important expressive differences between first- and second-order quantification were unknown (and unimportant) to Russell. If the values of the predicate variables are taken to be properties or propositional functions, then his second-order sentences can be rewritten using many sorted, first-order quantification. E.g. '∃F(Fx)' can be taken to say that for some property p, x instantiates p. (See Sainsbury, pp. 283, 290-92, for a defense of this procedure as true to Russell.) If we, further, give up the fruitless attempt to eliminate sets, then '∃F(Fx)' can (for purposes of the reduction) be taken to say that for some set s, x is a member of s – and, so, can be symbolized '∃s (x ε s)' -- as it is in my volumes.

and how Russell's Paradox can be avoided. Since these are things that all students of the analytic tradition need to know the rudiments of, they are what I tried to explain.

Evidently, this evaluative perspective irritates Kremer, and those on whose behalf he purports to write. They are deeply suspicious of my attempt to evaluate which ideas of past philosophers marked the most significant advances, and therefore merit the most attention. That's fine, so long as they identify fruitful philosophical truths I missed, or misleading falsehoods to which I succumbed. In the case of the "no-class theory," they haven't done so. On the contrary, Kremer's assumption that it was a "technical achievement" resulting in an "ontological economy" is a serious mistake. That, of course, doesn't mean that Russell's ideas on the subject aren't interesting, or that they aren't connected to other ideas that are more significant. In fact, they are intertwined with his dauntingly complex and far-reaching ideas about predicativity, the semantic paradoxes, and the ramified vs. the simple theory of types – all of which are fascinating, and some of which involve real technical achievements.[6] Kremer and others may feel that at least some hint of this should have found its way into my history. If so, I agree. I also agree with Kremer that Russell's "no-class theory" can be used to shed light on the thinking behind (what I believe to be) his disastrous attempt, in *Our Knowledge of the External World*, to reduce talk of physical objects to talk of sense-data. I will try to correct these shortcomings in an appendix to the section on Russell, and an enlarged set of suggestions for further reading, in the expanded edition of my history.[7]

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[6] See chapter 1 of Burgess (2005) for an explanation of Russell's leading technical result about the relationship between higher-order logic with an axiom of extensionality and higher-order logic without such an axiom.

[7] Thanks to John Burgess for comments on an earlier draft.

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