An Insertion Algorithm for Multiplying Schubert Polynomials by Schur Polynomials

Sami Assaf

joint work with

Nantel Bergeron

Online Schubert Seminar
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Outline

A Littlewood–Richardson Rule for Grassmannian permutations

Schubert structure constants $c^w_{u,v}$, triply indexed by permutations, defined by

$$\mathfrak{S}_u \mathfrak{S}_v = \sum_w c^w_{u,v} \mathfrak{S}_w,$$

enumerate flags in a suitable triple intersection of Schubert varieties, and so $c^w_{u,v} \in \mathbb{N}$.

Fundamental open problem: Give a simple positive combinatorial formula for $c^w_{u,v}$.

A permutation v is k-grassmannian if $v_i < v_{i+1}$ for all $i \neq k$; e.g. 1367245 is 4-grassmannian.

A k-partition is a weakly increasing sequence of length k; e.g. $(0,1,3,3)$ is a 4-partition.

The map $\nu \mapsto v(\nu,k)$ given by $\nu_i = v_i - i$ for $i \leq k$ is a bijection between k-grassmannian permutations and k-partitions; e.g. $1367245 = v((0,1,3,3),4)$.

My conjectured formula, proved with **Nantel Bergeron**, states for all u, ν, k, w,

$$c^w_{u,v(\nu,k)} = \# \left\{ \begin{array}{l} \text{saturated chains in } k\text{-Bruhat order} \\ \text{from } u \text{ to } w \text{ with lattice weight } \nu \end{array} \right\}$$

k-Bruhat Order — Lattice chains

Bruhat order has cover relations $u \lessdot ut_{r,s}$ if $\ell(ut_{r,s}) = \ell(u)+1$, where $\ell(u) = \#\{i<j \mid u_i > u_j\}$.

Definition (Bergeron–Sottile 1998)

$u \leq_k w$ is the transitive closure of $u \lessdot_k ut_{a,b}$ whenever $a \leq k < b$ with $\ell(ut_{a,b}) = \ell(u) + 1$

Bergeron and Sottile decorate saturated chains by labeling the cover $u \lessdot_k ut_{a,b}$ with u_b.

$$136\underline{2}|\underline{5}847 \xrightarrow{5} 13\underline{6}5|284\underline{7} \xrightarrow{7} 13\underline{7}5|2\underline{8}46 \xrightarrow{8} \underline{1}385|\underline{2}746 \xrightarrow{2} 2\underline{3}85|17\underline{4}6 \xrightarrow{4} 248\underline{5}|1\underline{7}36 \xrightarrow{7} 2\underline{4}87|1\underline{5}36 \xrightarrow{5} 25871436$$

For $d_1, \ldots, d_m$ the decorations of $\mathcal{C}$, set

$$e_i = \begin{cases} k & \text{if } i = 1 \\ e_{i-1} & \text{if } d_{i-1} < d_i \\ e_{i-1} - 1 & \text{if } d_{i-1} > d_i \end{cases}$$

The weight is $\mathrm{wt}(\mathcal{C})_i = \{j \mid e_j = i\} = (0,1,3,3)$

$$\begin{pmatrix} 5 & 7 & 8 & 2 & 4 & 7 & 5 \\ 4 & 4 & 4 & 3 & 3 & 3 & 2 \end{pmatrix} \xrightarrow{\text{sort top}} \begin{pmatrix} 2 & 4 & 5 & 5 & 7 & 7 & 8 \\ 3 & 3 & 4 & 2 & 4 & 3 & 4 \end{pmatrix}$$

Definition (A. 2021)

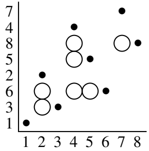
A chain $\mathcal{C}$ is lattice if for all d, for all $i \leq k$, we have $\#\{j \mid e_j = i-1,\ d_j \geq d\} \leq \#\{j \mid e_j = i,\ d_j \geq d\}$.

Theorem (A.–Bergeron 2023)

$$c^{w}_{u,v(\nu,k)} = \#\{\textit{saturated chains in } k\textit{-Bruhat order from } u \textit{ to } w \textit{ with lattice weight } \nu\}$$

Lattice Permutation Tableaux

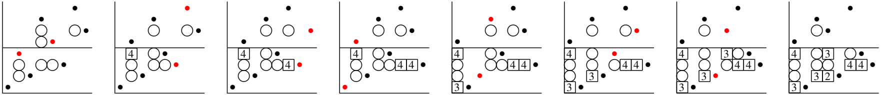
$\mathbb{D}(u) = \{(u_j, i) \mid i < j, u_i > u_j\}$



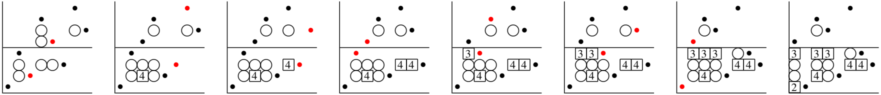
Definition (A. 2021)

For $\mathcal{C}$ from u to w, set $T_0 = \mathbb{D}(u)$, and define T_{i+1} by

- move cells in row b_i right of column $u^{(i)}_{a_i}$ down to row a_i
- place a new skew cell with entry e_i in row a_i, column $u^{(i)}_{a_i}$
- between a_i, b_i and $u^{(i)}_{a_i}, u^{(i)}_{b_i}$, move cells left to avoid dots



1362|5847 1365|2847 1375|2846 1385|2746 2385|1746 2485|1736 24|871536 2587|1436



1362|5847 1562|3847 1572|3846 1582|3746 1583|2746 1584|2736 1587|2436 2587|1436

Definition (A. 2021)

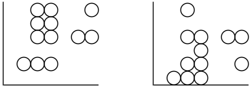
$\mathrm{LPT}_k(w/u, \nu)$ is the set of permutation tableaux from u to w in k-Bruhat with lattice weight ν.

Kohnert's rule

Definition (Kohnert 1991)

A Kohnert move on a row selects the rightmost cell c, moves c down to the first open position below.

$\mathrm{KD}(D) = \{T \mid T$ obtained by Kohnert moves on $D\}$



$x_2^3 x_4^4 x_5^2 x_6^3$ $\quad$ $x_1^3 x_2^3 x_3 x_4^4 x_6$

Theorem (Kohnert 1991)

The Schur polynomial indexed by ν is

$$s_\nu(x_1, \ldots, x_k) = \sum_{T \in \mathrm{KD}(\mathbb{D}(v(\nu,k)))} x_1^{\mathrm{wt}(T)_1} \cdots x_n^{\mathrm{wt}(T)_n}$$

Placing an r into cells in row r and floating up to row k gives a weight-preserving bijection

$$\mathrm{KD}(\mathbb{D}(v(\nu,k))) \xrightarrow{\varphi} \mathrm{SSRT}_k(\nu)$$

with semi-standard reverse tableaux.

Theorem (Winkel 1999, Winkel 2002, A. 2022, Armon–A.–Bowling–Ehrhard 2023)

The Schubert polynomial indexed by w is $\mathfrak{S}_w = \sum_{T \in \mathrm{KD}(\mathbb{D}(w))} x_1^{\mathrm{wt}(T)_1} \cdots x_n^{\mathrm{wt}(T)_n}$

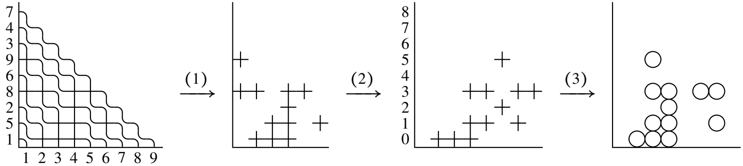
AABE 2023: Kohnert moves generate the character of any southwest flagged Schur module.

Pipe dreams — Kohnert diagrams

Theorem (Billey–Jockush–Stanley 1992, Bergeron–Billey 1993)

The Schubert polynomial $\mathfrak{S}_w = \sum_{P \in \mathrm{RPD}(w)} x_1^{\mathrm{wt}(T)_1} \cdots x_n^{\mathrm{wt}(T)_n}$ *is generated by reduced pipe dreams.*

(1) Remove all ╯╭s (2) Shift +s in row r right $r-1$ columns (3) Change + to ○ and rectify



To rectify at (c, r), pair cells $\{(c+1, s) \mid s > r\}$ to the nearest unpaired cell $\{(c, s) \mid s > r\}$ weakly above, if it exists, and then move all unpaired cells in column $c+1$ left to column c.

Theorem (A. 2022)

This is a weight-preserving bijection between reduced pipe dreams and Kohnert diagrams.

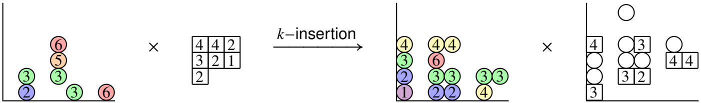
A bijective proof

Recall RSK gives a bijection $\mathrm{SSRT}_k(\mu) \times \mathrm{SSRT}_k(\nu) \xrightarrow{\sim} \bigsqcup_{\mu \subset \lambda} \mathrm{SSRT}_k(\lambda) \times \mathrm{LRT}_k(\lambda/\mu, \nu)$

Theorem (A.–Bergeron 2023)

The k-insertion algorithm on Kohnert diagrams gives a weight-preserving bijection

$$\mathrm{KD}(u) \times \mathrm{SSRT}_k(\nu) \xrightarrow{\sim} \bigsqcup_{u \prec_k w} \mathrm{KD}(w) \times \mathrm{LPT}_k(w/u, \nu).$$



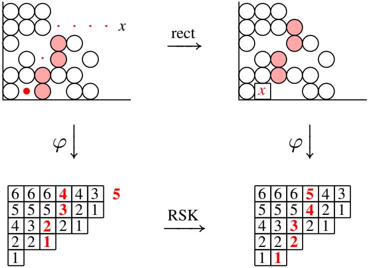
Corollary (A.–Bergeron 2023)

$$\mathfrak{S}_u s_\nu(x_1, \ldots, x_k) = \sum_{u \prec_k w} \#\mathrm{LPT}_k(w/u, \nu)\mathfrak{S}_w$$

Insertion on Kohnert diagrams

Placing an r into cells in row r and floating up to row k gives a weight-preserving bijection

$$\mathrm{KD}(\mathbb{D}(v(\nu,k))) \xrightarrow{\varphi} \mathrm{SSRT}_k(\nu)$$



Theorem (A.–Quijada 2019$^+$)

For $T \in \mathrm{KD}(v(\mu,k))$ and x in row $r \le k$,

$$\varphi\,(\textit{rectify}(T \sqcup x)) = \varphi(T) \xleftarrow{\mathrm{RSK}} r$$

Rectification is well-defined on any T and well-behaved on $T \in \mathrm{KD}(D)$ when D is southwest.

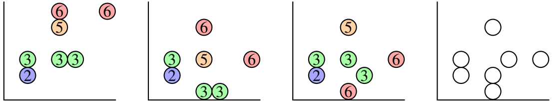
- **Question 1**: At which cells $x \notin T$ can we land?
- **Question 2**: Which cells $y \in T$ can $x \notin T$ bump?

Answer: We may k-land at and k-bump a cell if and only if it has semi-proper label at most k.

Semi-proper labelings

A.–**Searles** (JCT-A 2018) define the proper labeling of cells of a diagram $T \in \mathrm{KD}(\mathbb{D}(\alpha))$.
A. (Trans. AMS 2022) generalized this to (semi-)proper labelings of cells of $T \in \mathrm{KD}(\mathbb{D}(u))$.

Given a diagram D, the super-standard labeling puts r into cells in row r.



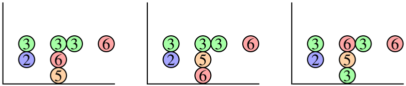
$\mathbb{D}(13625847)$ $\quad \in\mathrm{KD}(13625847)$ $\quad \in\mathrm{KD}(13625847)$ $\quad \notin\mathrm{KD}(13625847)$

Using label rectification, we can (attempt to) properly label T w.r.t. D explicitly.

Theorem (A. 2022)

For T and D, the following are equivalent:

- $T \in \mathrm{KD}(D)$
- *T can be properly labeled w.r.t. D*
- *T has a semi-proper labeling w.r.t. D*

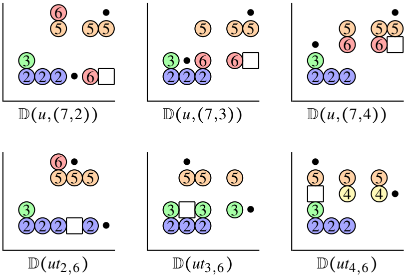


proper $\quad$ semi-proper $\quad$ not semi-proper

Landing columns

For $u \prec ut_{r,s}$, construct $\mathbb{D}_0(u,(u_s,r)) \in \mathrm{KD}(u)$ by moving cells in row s down to row r.

Set $\mathbb{D}(u,(u_s,r)) = \mathbb{D}_0(u,(u_s,r)) \sqcup (u_s,r)$.



Theorem (A.–Bergeron 2023)

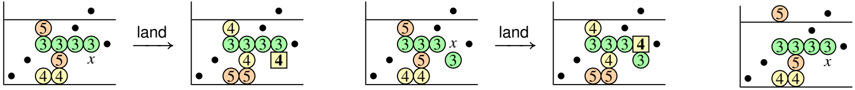
We have a weight-preserving bijection

$$\bigcup_{u \prec_k ut_{r,s}} \mathrm{KD}(\mathbb{D}(u,(u_s,r))) \xrightarrow{\sim} \bigsqcup_{u \prec_k ut_{r,s}} \mathrm{KD}(ut_{r,s})$$

Definition (A.–Bergeron 2023)

For $T \in \mathrm{KD}(u)$ and $x \notin T$ in column u_s with $s > k$, say x is a k-landing spot for T if

$$T \cup x \in \mathrm{KD}(\mathbb{D}(u,(u_s,r))) \quad \text{and} \quad \mathcal{L}(x) = r \text{ for some semi-proper } \mathcal{L}$$



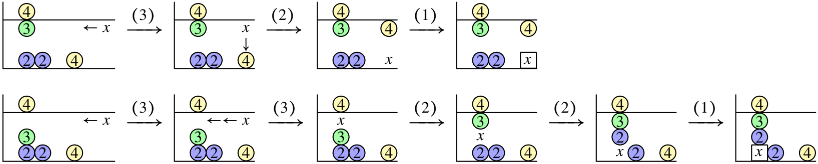
Bumpable cells

Definition (A.–Bergeron 2023)

For $T \in \mathrm{KD}(u)$ and $x \notin T$, say $y \in T$ below x is k-bumpable if there exists semi-proper $\mathcal{L}$ s.t.

$$\mathcal{L} \text{ w.r.t. } \mathbb{D}(u) \text{ and } \mathcal{L}(y) \leq k \qquad \text{or} \qquad \mathcal{L} \text{ w.r.t. } \mathbb{D}_0(u, (u_s, r)) \text{ and } \mathcal{L}(y) = r$$

and the labeling on $T \sqcup x - y$ obtained by swapping x and y is semi-proper.



Definition (A.–Bergeron 2023)

For $T \in \mathrm{KD}(\mathbb{D}(u))$ and $x \notin T$, the k-insertion of x into T, denoted by $T \xleftarrow{k} x$, is

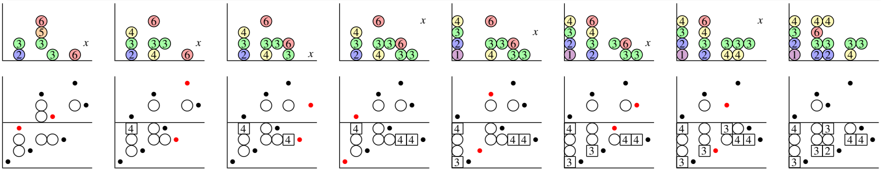
1. IF x is a k-landing spot for T, THEN RETURN $T \cup x$;
2. ELSE IF there exists a lowest k-bumpable cell y, THEN RETURN $(T \sqcup x - y) \xleftarrow{k} y$;
3. ELSE RETURN $T \xleftarrow{k} z$ for rightmost $z \notin T$ left of x.

Row-Bumping Lemma

Lemma (A.–Bergeron 2023)

For $S \in \mathrm{KD}(u)$ and $i, j \leq k$, let $T = (S \overset{k}{\leftarrow} i) \in \mathbb{D}(ut_{r,s}) = \mathbb{D}(v)$ and $U = (T \overset{k}{\leftarrow} j) \in \mathbb{D}(vt_{q,t})$. Then

if $i < j$, then $u_s > v_t$ and if $i \geq j$, then $u_s < v_t$.



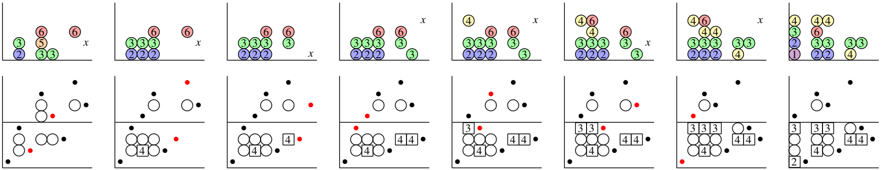
Insert $R \in \mathrm{SSRT}_k(\nu)$ into $T \in \mathrm{KD}(u)$ by k-inserting the Yamanouchi work for R.

$$\begin{pmatrix} 4 & 4 & 4 & 3 & 3 & 3 & 2 \\ 2 & 2 & 1 & 4 & 3 & 2 & 4 \end{pmatrix} \xrightarrow{\mathrm{RSK}} \begin{array}{|c|c|c|} \hline 4 & 4 & 2 \\ \hline 3 & 2 & 1 \\ \hline 2 \\ \cline{1-1} \end{array} \times \begin{array}{|c|c|c|} \hline 4 & 4 & 4 \\ \hline 3 & 3 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}$$

Lemma (A.–Bergeron 2023)

For $T \in \mathrm{KD}(u)$ and $R \in \mathrm{SSRT}_k(\nu)$, the recording tableau of $T \overset{k}{\leftarrow} R \in \mathrm{LPT}_k(w/u, \nu)$ for $u \prec_k w$.

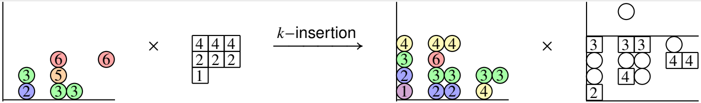
Reversing insertion



Theorem (A.–Bergeron 2023)

The k-insertion algorithm on Kohnert diagrams gives a weight-preserving bijection

$$\mathrm{KD}(u) \times \mathrm{SSRT}_k(\nu) \xrightarrow{\sim} \bigsqcup_{u \prec_k w} \mathrm{KD}(w) \times \mathrm{LPT}_k(w/u, \nu).$$



Corollary (A.–Bergeron 2023)

$$\mathfrak{S}_u s_\nu(x_1, \ldots, x_k) = \sum_{u \prec_k w} \#\mathrm{LPT}_k(w/u, \nu) \mathfrak{S}_w$$

References on arXiv

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Sami Assaf and Nantel Bergeron.
An insertion algorithm for multiplying Schubert polynomials by Schur polynomials.
arXiv:*coming soon!*

Thank You