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A biographical sketch of Robert J. Sacker

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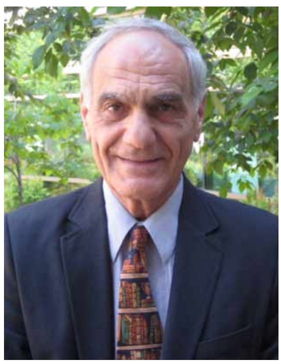
Bob Sacker and I met the first time in December 1965 in Mayaguez, Puerto Rico, at an International Conference on Differential Equations and Dynamical Systems, organized by Lefschetz, LaSalle and Hale [16,18]. We also met Henry Antosiewicz, who later played a major role in the Sacker's career.

Robert John Sacker was born in Miami, Florida in October 1937. While in high school in Key West, Florida, he developed a strong interest in both mathematics and electrical engineering. He attended the Georgia Institute of Technology in Atlanta, where he met John Nohel, who inspired Sacker to specialize in mathematics. Completing his undergraduate studies in 1959, he entered the graduate program in mathematics at Georgia Tech and continued working with Nohel.



In 1961, Nohel recommended that Sacker transfer to New York University (NYU) and study with Jürgen Moser. Sacker completed his PhD with Moser at NYU in 1964 and then began a postdoctoral fellowship at NYU. In 1966 he began his tenure on the Faculty in the Department of Mathematics at the University of Southern California, his current academic home.

Sacker is one of the pioneers in the study of the longtime dynamics of nonautonomous dynamical systems. We present a description of some of his scientific contributions here.

S. Perturbation Theory for Manifolds. Early on Sacker developed an elegant approach for the study of the perturbation theory for invariant manifolds [19–22]. The statement

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of invariance for a manifold can be reformulated in terms of a first-order partial differential equation (PDE). Sacker's idea was to introduce an 'elliptic regularization' to analyze this PDE. The elliptic problem is readily solvable with good estimates, leading to the smoothness (including the Hölder continuity in the highest derivative) of the original PDE.

It is noteworthy that this methodology has been successfully applied in the study of the infinite dimensional dynamical systems arising in the theory of inertial manifolds [14]. The Sacker methodology is a beautiful idea, and it leads to strong results.

A. Extensions and Lifting Properties. In the study of nonautonomous equations and skew product flows, one encounters two fundamental questions: does a given differential equation with almost periodic coefficients have almost periodic solution? and if not, what kinds of solutions exist? When one transfers such questions into the context of skew product flows, one is seeking a (dynamical) extension of the flow on the base space. In other words, one seeks to describe properties which are lifted from the base flow into the skew product flow. The papers [29,30,33,36] deal with these issues.

(A.1) Extensions. In the paper [33], one studies two spaces M and Y, with a continuous projection $p: M \to Y$ of M onto Y. One assumes that the spaces are compact, invariant sets under the action of a topological group T. The problem is to find sufficient conditions for the space M to be an N-fold covering space for Y, with p as a covering map.

It is assumed that the group T has the property that there is a compact set $K \subseteq T$ such that T is (finitely) generated by any open neighbourhood of K. Topological groups that satisfy this finite generation property include the following – all with the usual topologies: the real line $\mathbb{R}$, the integers $\mathbb{Z}$, the Euclidean space $\mathbb{R}^n$ and the lattice $\mathbb{Z}^n$, where n is finite. However, the real line $\mathbb{R}$ with the discrete topology does not satisfy this finite generation property.

The main theorem states that, if p is a mapping of (fibre) distal-type, and if there exists a point $y_0 \in Y$, where the cardinality of $p^{-1}(y_0)$ is N, a positive integer, then the space M is such an N-fold covering space of Y. In this setting, p^{-1} lifts any equicontinuous flow on Y onto an equicontinuous flow on M.

(A.2): Lifting properties. Before turning to the dynamics in the 1977 AMS Memoir [36], it is useful to recall some aspects of the theory of almost periodic solutions of ordinary differential equations (ODEs) with almost periodic coefficients, as they existed in the early 1970s. There was an apparent paradox in which some theories were based on various stability concepts, and others were based on various distality concepts. A Yin-Yang where some theories required solutions to remain close together and others where solutions stay apart.

In this memoir, the goal was to show that there is no paradox; the two approaches have a common source, namely is a general theory of distality in dynamical systems. In each approach, one is looking for appropriate extensions of an almost periodic minimal set (the hull of the coefficient space). In the memoir, these extensions are referred to as lifting properties.

In this setting, one begins with a skew product flow (or semiflow) $\pi = \pi(t)$ on metric space $Y \times X$, where Y is a compact, minimal set in the flow σ, where $\sigma(y,t) = y \cdot t$. Let $M \subseteq Y \times X$ be a compact, invariant set for π. Assume that the projection mapping $p(y,x) = y$ of M onto Y a mapping of 'distal-type' and that there is a $y_0 \in Y$ and an integer $N \geq 1$ such that $p^{-1}(y_0)$ contains precisely N points. Then M is an N-fold covering space of Y, with covering map being $p|_M$, the restriction of p to M. Also, the flow π on M is almost periodic (resp. distal) if and only if the flow σ on Y is almost periodic (resp. distal). Thus distality and almost periodicity are properties that are lifted to $Y \times X$.

The key point here being that p is of 'distal-type'. This means that if $p(y, x_1) = p(y, x_2) = y$, where $x_1 \neq x_2$, then there is an $\alpha > 0$ such that the metric d on $Y \times X$ satisfies $\mathrm{d}(\pi(t)(y, x_1), \pi(t)(y, x_2)) \geq \alpha$, for all $t \in T$.

C. Linear Skew Product Flows. The story of linear skew product flows is one of the hallmarks of Sacker's career over a 20-year period in which the papers [22–24,32,34,35,37,38–40] appeared.

(C.1) The Splittings Saga. This Saga begins with four papers which include the term 'invariant splittings' in the title. The papers cited above are based on a common theme in the study of nonautonomous differential equations, *videlicet*: exponential dichotomies and invariant splittings of linear differential systems. We use the linear skew product formulation $\pi(t)(y, x) = \pi(y, x, t)$ with: $\pi = \pi(t)$ and

$$\pi(y, x, t) = (\sigma(y, t), \Phi(y, t)x), \quad \text{where } y \in Y, \ x \in X, \ t \in T, \tag{1}$$

X is a finite dimensional Hilbert space, Y is a compact Hausdorff space with a flow $\sigma(y, t) = y{\cdot}t$ and T is the real number space $\mathbb{R}$, or the integers $\mathbb{Z}$. Also Φ is a continuous mapping $\Phi : Y \times T \to \mathcal{L}(X, X)$, where $\mathcal{L}(X, X)$ is the space of bounded, linear operators on X. The bounded set $\mathcal{B}$, the stable set $\mathcal{S}$, and the unstable set $\mathcal{U}$ are defined by

$$\begin{aligned} \mathcal{B} &= \{(y, x) \in Y \times X : \sup_{t \in T} \|\Phi(y, t)x\| < \infty\}, \\ \mathcal{S} &= \{(y, x) \in Y \times X : \|\Phi(y, t)x\| \to 0, \text{ as } t \to +\infty\}, \\ \mathcal{U} &= \{(y, x) \in Y \times X : \|\Phi(y, t)x\| \to 0, \text{ as } t \to -\infty\}. \end{aligned} \tag{2}$$

We let $\mathcal{B}(y)$, $\mathcal{S}(y)$ and $\mathcal{U}(y)$ denote the respective fibres over $y \in Y$. For example, $\mathcal{B}(y)$ is the collection of $x \in X$, with $(y, x) \in \mathcal{B}$. One defines

$$Y_k = \{y \in Y : \dim \mathcal{S}(y) = k \quad \text{and} \quad \dim \mathcal{U}(y) = n - k\}, \quad \text{for } 0 \leq k \leq n.$$

It is shown that each Y_k is a closed set and at least one Y_k is nonempty. Also the skew product flow π has an exponential dichotomy over each nonempty Y_k. Furthermore, π has an exponential dichotomy over Y, and there is an invariant splitting

$$Y \times X = \mathcal{S} \oplus \mathcal{U}, \tag{3}$$

as a Whitney sum, if and only if one has

$$\mathcal{B} = Y \times \{0\} \quad \text{and} \quad Y = Y_k, \quad \text{for a unique } k \in \{0, 1, \ldots, n\}.$$

The Projector. The exponential dichotomy over Y and the splitting (3) comes with a projector $P : Y \times X \to Y \times X$, where $P(y, x) = (y, P(y))$ and $P(y)$ is the linear projection on X that satisfies $P(y)x = x$, when $x \in \mathcal{S}(y)$, and $P(y)x = 0$, when $x \in \mathcal{U}(y)$. The complementary mapping $Q(y) = I - P(y)$ is also a projector. Thus the ranges of the projectors satisfy $\mathcal{R}(P) = \mathcal{S}$ and $\mathcal{R}(Q) = \mathcal{U}$. Also, there exist constants $K \geq 1$ and $\alpha > 0$ such that for all $(y, x) \in Y \times X$ and all $t \in \mathbb{R}$, one has

$$P(y{\cdot}t)\Phi(y, t)x = \Phi(y, t)P(y)x \quad \text{and} \quad Q(y{\cdot}t)\Phi(y, t)x = \Phi(y, t)Q(y)x, \tag{4}$$

and

$$\|\Phi(y,t)P(y)x\| \leq K\|x\|e^{-\alpha t}, \quad \text{for } t \geq 0,$$
$$\|\Phi(y,t)Q(y)x\| \leq K\|x\|e^{\alpha t}, \quad \text{for } t \leq 0. \tag{5}$$

THEOREM 1. (ALTERNATIVE THEOREM) *Assume that* $\mathcal{B} = Y \times \{0\}$, *where* $\mathcal{B}$, $\mathcal{S}$, $\mathcal{U}$ *and* Y_k *are given as above, see* (2). *Then the sets* Y_k *are compact invariant sets, and at least one is nonempty.*

(1) *If precisely one set, say* Y_{k_0} *is nonempty, then* $Y = Y_{k_0}$ *and there is an exponential dichotomy over Y, such that the relations* (3)–(5) *hold.*
(2) *If there are at least two nonempty sets* Y_k, *then the flow* σ *on Y is a Morse flow, where the nonempty sets* Y_k *are the Morse sets; and if* $y \notin \cup_{k=0}^{n} Y_k$, *then there exist* $k_1 > k_2$, *such that the alpha limit set* $\alpha(y)$ *is in* Y_{k_1}, *while the omega limit set* $\omega(y)$ *is in* Y_{k_2}.

We note that, for each minimal set, or for each alpha or omega limit set, or for each chain-recurrent set $M \subseteq Y$, there is an integer $k \in \{0, 1, \ldots, n\}$ such that M lies in Y_k.

Two proofs the alternative theorem are given. One in [32], and a shorter and less complicated version in [34]. Applications include linear differential equations and linear difference equations.

(C.2): Nontrivial bounded sets. The papers [35,23] present a continuation of the analysis of the dynamics of (1), but now one threats the case where the bounded set $\mathcal{B}$ is not trivial, that is, $\mathcal{B} \neq Y \times \{0\}$. These papers include interesting applications to the theory of ODEs and the theory of difference equations.

One begins with a closed invariant subbundle $\mathcal{M}$ for the skew product flow π generated by (1). For example, $\mathcal{M}$ could be the tangent bundle generated by an invariant manifold for a nonlinear differential, or difference equation. In these papers, one encounters various relations between $\mathcal{M}$ and the bounded set $\mathcal{B}$. In order to simplify this treatment, we adopt here the assumption that $\mathcal{M} = \mathcal{B}$. Recall that a subbundle $\mathcal{M}$ in $Y \times X$ is a closed set, where the fibres $\mathcal{M}(y)$ are linear spaces, and the dimension $\dim(\mathcal{M}(y))$ is constant over every connected component of Y (when $T = \mathbb{R}$), or every invariantly connected component of Y (when $T = \mathbb{Z}$).

One uses the inner product on X and the concept of orthogonality to define the orthogonal complement

$$\mathcal{M}^{\perp} = \{(y,x) \in Y \times X : x \perp \mathcal{M}(y)\},$$

and we let P be the projector $P(y,x) = (y, P(y)x)$, where $P(y)$ is the orthogonal projection of X onto the fibre $\mathcal{M}^{\perp}(y)$, for $y \in Y$. Note that $P(y)x = 0$, for $x \in \mathcal{M}(y)$.

While $\mathcal{M}$ is invariant, it is generally not the case that the complementary bundle $\mathcal{M}^{\perp}$ is invariant. Nevertheless, several natural questions arise, to wit, when does there exist an invariant complementary bundle $\mathcal{M}$ such that $Y \times X = \mathcal{M} \oplus \mathcal{N}$, as a Whitney sum? Related questions arise in terms of the stable and unstable sets $\mathcal{S}$ and $\mathcal{U}$. For example, are these sets subbundles in $Y \times X$? If these sets are subbundles, is $\mathcal{N} = \mathcal{S} \oplus \mathcal{U}$ a complementary subbundle for $\mathcal{M}$? If so, then one would have an exponential trichotomy for π over Y with the invariant splitting

$$Y \times X = \mathcal{S} \oplus \mathcal{M} \oplus \mathcal{U}. \tag{6}$$

For the case where the subbundle $\mathcal{M} = \mathcal{B}$ is the tangent bundle for an invariant manifold in a nonlinear flow, the relation (6), with the exponential trichotomy, means that this manifold is 'normally hyperbolic'.

To study these issues, one uses the induced flow $\hat{\pi}$ on the subbundle $\mathcal{M}^{\perp}$, where $\hat{\pi} = \hat{\pi}(t)$,

$$\hat{\pi}(t)(y,x) = (y{\cdot}t, \hat{\Phi}(y,t)x), \quad P(y)x = x \ \text{ and } \ \hat{\Phi}(y,t)x = P(y{\cdot}t)\Phi(y,t)x. \tag{7}$$

By using the invariance of the subbundle $\mathcal{M}$, one shows that $\hat{\pi}$ is a linear skew product flow on the subbundle $\mathcal{M}^{\perp}$. Consequently, the bounded set $\hat{\mathcal{B}}$, the stable set $\hat{\mathcal{S}}$, and the unstable set $\hat{\mathcal{U}}$, for $\hat{\pi}$, are defined as in (2), where $\hat{\Phi}$ replaces Φ. Furthermore, when the bounded set $\hat{\mathcal{B}}$ is trivial; that is, when $\hat{\mathcal{B}}(y) = \{0\}$, for all $y \in Y$; then the theory of exponential dichotomies and invariant splittings, which is described above, is applicable for the induced dynamics on $\mathcal{M}^{\perp}$.

This brings us to the question: What implications does all this have on the original dynamical system π on $Y \times X$? One answer, which is in the theorem below, requires that $\mathcal{M}$ satisfies a distal property, Hypothesis B, which states that there exist positive constants C_0 and C_1 such that: [1]

$$C_0\|x\| \leq \|\Phi(y,t)x\| \leq C_1\|x\|, \quad \text{for every} \quad (y,x) \in \mathcal{M} \text{ with } x \neq 0. \tag{8}$$

THEOREM 2. *Assume that* (*i*) $\mathcal{M} = \mathcal{B}$ *satisfies the distal property* (8)*; that* (*ii*) *the associate bounded set* $\hat{\mathcal{B}}$ *is trivial; and that* (*iii*) *the induced flow* $\hat{\pi}$ *on* $\mathcal{M}^{\perp}$ *has an exponential dichotomy over Y, then the following hold:*

- *the sets* $\mathcal{S}$ *and* $\mathcal{U}$ *are subbundles of* π*;*
- *the flow* π *on* $Y \times X$ *has an exponential trichtomy over Y and*
- *one has* $Y \times X = \mathcal{S} \oplus \mathcal{B} \oplus \mathcal{U}$*, as a Whitney sum.*

(C.3) Without hypothesis B. The distal property (8) is very restrictive. The theory developed in [23] address some issues that arise when (8) fails to hold. It can happen, for example, that the bounded set $\mathcal{B}$ is no longer a subbundle. It need not be a closed set in $Y \times X$, or the dimensions $\dim(\mathcal{B}(y))$ may change as y changes.

The theory developed in [23] is based on the spectral theory appearing in [37,39], see Section K. What typically happens is that the bounded set $\mathcal{M} = \mathcal{B}$ in the decomposition (6) is replaced by a spectral subbundle $\mathcal{V}_0$, where $\mathcal{B} \subseteq \mathcal{V}_0$. One of the goals of this article is, for example, to derive good estimates of $\dim(\mathcal{B}(y))$ and $\dim(\mathcal{V}_0(y))$ in terms of dimensions of $\mathcal{S}$ and $\mathcal{U}$ over $\alpha(y)$ and $\omega(y)$, the alpha and omega limit sets.

(C.4) Banach space theory. As noted in Section C.1, the original theory concerning the existence of exponential dichotomies and invariant splittings is developed for finite dimensional Hilbert spaces X. The extensions of this theory to general Banach spaces, where the solution operator $\Phi(y,t)$ is now a uniformly α-contracting operator on a Banach space X, offers special challenges. (First and foremost, the linear operator $\Phi(y,t)$ is now defined only for $t \geq 0$, and $\pi(t)$ is a *semiflow* and not a flow [45].) Eventually these challenges were met, as seen in [40], and the alternative theorem is valid in this setting. (An important aspect of the extension of the invariant splittings theory to Banach spaces, was the use of Conley theory of chain-recurrence [4], also see [22]).

The theory described above applies in the study of the linearization of the following nonlinear evolutionary equations: (i) parabolic PDEs including systems of reaction diffusion equations and the Navier–Stokes equations; (ii) hyperbolic PDEs, including the nonlinear wave equation, as well as the nonlinear Schrödinger equation, with dissipation; (iii) retarded differential delay equations; and (iv) certain neutral differential delay equations.

K. Spectral Theory. In this section, we present remarks on the theories appearing in [24,31,37,38,39]. In the first section, we examine the linear theory of the dynamical spectrum.[2] In the second section, we examine the application of this Spectral Theory to the nonlinear theory of invariant manifolds. We use here the notation developed in Section C.

(K.1) Dynamical spectrum. The dynamical spectrum theory arises after one takes a small step away from the linear skew product flow π on $Y \times X$ in (1), where X is an n-dimensional linear space and $n \geq 1$. In particular, let λ denote any real number, and define $\pi_\lambda(t)(y,x) = \pi_\lambda(y,x,t)$ with: $\pi_\lambda = \pi_\lambda(t)$,

$$\begin{aligned} \pi_\lambda(t)(y,x) &= \pi_\lambda(y,x,t) = (y{\cdot}t, \Phi_\lambda(y,t)x), \\ \Phi_\lambda(y,t) &= \mathrm{e}^{-\lambda t}\Phi(y,t). \end{aligned} \tag{9}$$

where $y \in Y, x \in X, t \in T$, and $\Phi(y,t)$ is given as in (1). It is easily verified that π_λ is a linear skew product flow, for each $\lambda \in \mathbb{R}$. Next one uses (2), with $\Phi_\lambda(y,t)$ replacing $\Phi(y,t)$ to defined the bounded set $\mathcal{B}_\lambda$, the stable set $\mathcal{S}_\lambda$, and the unstable set $\mathcal{U}_\lambda$.

One says that λ belongs to the resolvent set of π when π_λ has an exponential dichotomy over Y, and one has the invariant splitting $Y \times X = \mathcal{S}_\lambda \oplus \mathcal{U}_\lambda$, as a Whitney sum. The *Dynamical Spectrum* $\Sigma = \Sigma(\pi)$ is defined as the complement in $\mathbb{R}$ of the resolvent set, that is, $\lambda \in \Sigma(\pi)$ when π_λ does *not* have an exponential dichotomy over Y.

The Spectral Theorem for the skew product flow π states the following:

- There is an integer k, with $1 \leq k \leq n$, such that the spectrum $\Sigma = \Sigma(\pi)$ satisfies

$$\Sigma = \bigcup_{i=1}^{k} [a_i, b_i],$$

  where $a_i \leq b_i < a_{i+1} \leq b_{i+1}$, for $i = 1, \ldots, k-1$ and
- there exist k invariant subbundles $\{\mathcal{V}_1, \ldots, \mathcal{V}_k\}$ and k positive integers $\{n_1, \ldots, n_k\}$ such that

$$\dim \mathcal{V}_i = \dim \mathcal{V}_i(y) = n_i, \quad \text{for } i = 1, \ldots, k,$$

  with $n_1 + \cdots + n_k = n$; also,
- one has $\mathcal{V}_i(y) \cap \mathcal{V}_j(y) = \{0\}$, for all $y \in Y$, when $i \neq j$, and

$$Y \times X = \mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_k, \quad \text{as a Whitney sum.}$$

That completes the geometric aspect of the Spectral Theorem. Next, we turn to the dynamical aspects which describe the growth and decay rates in the subbundles $\mathcal{V}_i$. For this purpose, we fix values λ_j, for $j = 0, 1 \ldots k$ in the resolvent set for π, where

$$\lambda_{j-1} < a_j \leq b_j < \lambda_j, \quad \text{for } j = 1, \ldots, k.$$

We then use (2) to define the bounded set $\mathcal{B}_\lambda$, the stable set $\mathcal{S}_\lambda$, and the unstable set $\mathcal{U}_\lambda$, where

$$\lambda \in \Lambda = \{\lambda_0, \lambda_1, \ldots, \lambda_k\}.$$

As λ gets larger, the set $\mathcal{S}_\lambda$ gets larger, while $\mathcal{U}_\lambda$ gets smaller. The subbundles $\mathcal{S}_{\lambda_0}$ and $\mathcal{U}_{\lambda_k}$ are trivial, and one has $Y \times X = \mathcal{U}_{\lambda_0} = \mathcal{S}_{\lambda_k}$, as well as

$$\mathcal{V}_i = \mathcal{U}_{\lambda_{i-1}} \cap \mathcal{S}_{\lambda_i}, \quad \text{for } i = 1, \ldots, k.$$

One also obtains various exponential trichotomies and associate invariant splittings, for example:

$$Y \times X = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{U}_{\lambda_2}, \quad \text{and} \quad Y \times X = \mathcal{S}_{\lambda_2} \oplus \mathcal{V}_3 \oplus \mathcal{U}_{\lambda_3}.$$

as Whitney sums. The projectors P_λ and Q_λ and the ranges satisfy $\mathcal{R}(P_\lambda) = \mathcal{S}_\lambda$ and $\mathcal{R}(Q_\lambda) = \mathcal{U}_\lambda$, for $\lambda \in \Lambda$. The relations (4) and (5) hold for $\lambda \in \Lambda$, as well.

The applications of the Spectral Theory in [2,3] are noteworthy. In these works, one finds a theory of kinematic similarity and asymptotic diagonalization for linear difference and differential equations.

(K.2) Spectrum of an invariant manifold. The major application of the Dynamical Spectrum is to study the longtime dynamics in the vicinity of a compact, invariant manifold for a nonlinear problem. While the spectral theory in [39] is formulated in the context of vector bundles with local metrics, the main ideas can be seen by examining a special nonlinear problem, which is more directly connected with the theory developed in Section C.

We begin with a nonlinear ODE

$$y' = V(y) \quad \text{on} \quad X = \mathbb{R}^n, \tag{10}$$

where V is a C^1-vector field on X. Let Y denote a compact, invariant manifold for this ODE, and let $y{\cdot}t$ denote the solution of this ODE, where $y{\cdot}0 = y$ and $y \in Y$. The $n \times n$ Jacobian matrix $A(y) = DV(y)$ is used to generate the linearized flow on the tangent bundle TY, as follows: In local coordinates $(y, x) \in TY$, we let $\Phi(y, t)x$ denote the solution of the nonautonomous linear equation

$$x' = A(y{\cdot}t)x, \quad \text{with } \Phi(y, 0)x = x.$$

One then has a local skew product flow, which is a fibre preserving flow $\pi = \pi(t)$ on the tangent bundle TY, where

$$\pi(t)(y, x) = (y{\cdot}t, \Phi(y, t)x).$$

Next, we let $\mathcal{M}$ denote a compact, invariant submanifold for the vector field given by (10) with $\mathcal{M} \subseteq Y$. The tangent bundle $T\mathcal{M} = \mathcal{M} \times X$ for $\mathcal{M}$ is an invariant subbundle of π. Wewill let $\pi_T = \pi_T(t)$ denote the restriction of π to $T\mathcal{M}$. A major question in the dynamics of this problem is: when does there exist a complementary subbundle $\mathcal{N}_{\text{inv}}$ that is invariant in the skew product flow π?

To address this question, one lets $\mathcal{N}$ denote any complementary subbundle of $T\mathcal{M}$ in TY. (For example, one might take $\mathcal{N} = (T\mathcal{M})^\perp$, see Section C.2.). Next one lets $P_N(y, x) = (y, P_N(y)x)$ denote the projector on TY that satisfies (i) $P_N(y)x = x$, when $x \in \mathcal{N}(y)$, and (ii) $P_N(y)x = 0$, when $x \in T\mathcal{M}(y)$. One then uses this projector to construct an induced flow $\pi_N = \pi_N(t)$ on $\mathcal{N}$, where

$$\pi_N(t)(y, x) = (y{\cdot}t, \Phi_N(y, t)x), \quad P_N(y)x = x, \quad \text{and} \quad \Phi_N(y, t)x = P_N(y{\cdot}t)\Phi(y, t)x, \tag{11}$$

compare with (7). Since $T\mathcal{M}$ is invariant, π_N is a linear skew product flow on $\mathcal{N}$. In this way, one obtains three linear skew product flows: π, π_T, and π_N. Associated with these flows are three spectra: Σ, Σ_T and Σ_N.[3] One shows that the spectra do *not* depend on the choice of the complementary subbundle $\mathcal{N}$, which brings us to a key theorem.

THEOREM 3. *Let the spectra Σ, Σ_T and Σ_N be given as above. Assume that*

$$\Sigma_T \cap \Sigma_N = \emptyset. \tag{12}$$

Then there is a local coordinate system such that the solution operator $\Phi(y,t)$ can be represented as a block triangular matrix. Also one has $\Sigma = \Sigma_T \cup \Sigma_N$, and there exist a complementary subbundle $\mathcal{N}_{\text{inv}}$ for $T\mathcal{M}$, where $\mathcal{N}_{\text{inv}}$ is an invariant bundle for the flow π. In addition, there is an exponential trichotomy, and one has the invariant splitting (6).

E. Splitting Index. For the theory of the Splitting Index, which is introduced in the paper [24], it is convenient to make a change of notation. In particular, one now studies the dynamics of solutions of the linear problem

$$x'(t) = A(t)x(t), \tag{13}$$

as well as the linear inhomogeneous problem

$$x'(t) = A(t)x(t) + f(t), \tag{14}$$

where $A = A(t) \in \mathfrak{A}$, $x \in X = \mathbb{R}^n$, and $t \in \mathbb{R}$. The topological space $\mathfrak{A}$ is a collection of $n \times n$ matrix-valued functions. We require that $A_\tau \in \mathfrak{A}$, whenever $A \in \mathfrak{A}$ and $\tau \in \mathbb{R}$, where $A_\tau(t) = A(\tau + t)$. We let $\Phi(A,t)$ denote the general solution operator of (13), where $\Phi(A,0) = I$ and $x(t) = \Phi(A,t)x_0$ denotes the solution of (13) that satisfies $x(0) = x_0$. We assume that $\mathfrak{A}$ is compact in the stated topology and that the mapping $\pi = \pi(s)$, where

$$\pi(s)(A,x) = (A_s, \Phi(A,s)x)$$

is continuous, and thereby, a linear skew product flow on $\mathfrak{A} \times X$. Several examples of spaces $\mathfrak{A}$ that satisfy these conditions are presented in [24].

We use the theory of Section C, with (Y, Y_k, y) being replaced with $(\mathfrak{A}, \mathfrak{A}_k, A)$ and use A_s on $\mathfrak{A}$ in place of the flow $y{\cdot}s$ on Y for the flow on the base space. For example, the sets $\mathcal{B}$, $\mathcal{S}$, and $\mathcal{U}$, with the fibres $\mathcal{B}(A)$, $\mathcal{S}(A)$, and $\mathcal{U}(A)$ are defined in (2). However, before defining the Splitting Index, we turn to the equation (14) and the related Fredholm theory.

The object of the Fredholm theory is to develop a methodology to solve the equation (14) for $x(t)$, when $f(t)$ is known. For this purpose, we note that equation (14) can be written in the equivalent form $Lx = f$ where

$$(Lx)(t) = x'(t) - A(t)x(t). \tag{15}$$

The classical Lyapunov–Perron method offers a linkage between certain solutions of equation (14) and the existence of exponential dichotomies for equation (13). For example, if $f = f(t)$ is in the space $\mathfrak{B}(n) = C(\mathbb{R}, X) \cap L^\infty(\mathbb{R}, X)$ of bounded continuous functions with the sup-norm, and equation (13) has an exponential dichotomy with projectors $P(A)$ and $Q(A)$, then (14) has a unique solution φ in $\mathfrak{B}_0(n)$, the space of bounded uniformly continuous, given by

$$\varphi(t) = \int_{-\infty}^{t} \Phi(A, t-s)Q(A)f(s)\mathrm{d}s - \int_{t}^{\infty} \Phi(A, t-s)P(A)f(s)\mathrm{d}s. \tag{16}$$

This pair of spaces $(\mathcal{B}_0(n), \mathcal{B}(n))$ are 'admissible' in the following sense:

1. There exists a dense subset $\mathcal{D} \subseteq \mathcal{B}_0(n)$ on which L is defined;
2. $\varphi \in \mathcal{D}$ implies that $L\varphi \in \mathcal{B}(n)$. Thus, $\mathcal{D} \subseteq \mathcal{D}(L)$, the domain of L;
3. Whenever $A \in \mathfrak{A}$ admits an exponential dichotomy, then the mapping $\hat{L} : \mathcal{B}(n) \to \mathcal{B}_0(n)$, where $\varphi = \hat{L}f$ satisfies (16), is a bounded linear operator;
4. If $\varphi : \mathbb{R} \to X$ satisfies $\|\varphi(t)\| \leq K\mathrm{e}^{-\rho|t|}$, for some $K > 0$, some $\rho > 0$, and all $t \in \mathbb{R}$, then $\varphi \in \mathcal{B}_0(n)$, and
5. If $\varphi : \mathbb{R} \to X$ satisfies either $\lim_{t\to\infty}\|\varphi(t)\| = \infty$, or $\lim_{t\to-\infty}\|\varphi(t)\| = \infty$, then $\varphi \notin \mathcal{B}_0(n)$.

The use of several examples of admissible pairs plays a major role in this paper. These examples lie at the heart of deriving the connection between the Splitting Index and the Fredholm Index.

The Splitting Index $S(A)$ is defined in two cases. The first being the case where $\mathfrak{A}$ is compact and the bounded set $\mathcal{B}$ is trivial, that is $\mathcal{B} = \mathfrak{A} \times \{0\}$. Then the theory in Section C.1 is in play. If $A \notin \cup_{k=0}^{n}\mathfrak{A}_k$, then there exist two integers $k_1 > k_2$, such that $\alpha(A)$, the alpha-limit set of A, is in $\mathfrak{A}_{k_1}$, while $\omega(A)$, the omega-limit set of A, is in $\mathfrak{A}_{k_2}$. The Splitting Index of A is: $S(A) \overset{\text{def}}{=} k_1 - k_2$. If $A \in \mathfrak{A}_k$, for some k, then the alpha and the omega limit sets of A are in $\mathfrak{A}_k$, as well, and $S(A) \overset{\text{def}}{=} 0$.

The second definition of $S(A)$ is for the case where the hull of A ($\mathfrak{A} = H(A)$) is assumed to be compact, and the bounded sets over the alpha and omega limit sets of A are assumed to be trivial, that is

$$\mathcal{B}(\alpha(A)) \overset{\text{def}}{=} \{(y, x) \in \mathcal{B} : y \in \alpha(A)\} \quad \text{and} \quad \mathcal{B}(\omega(A)) \overset{\text{def}}{=} \{(y, x) \in \mathcal{B} : y \in \omega(A)\},$$

satisfy

$$\mathcal{B}(\alpha(A)) = \alpha(A) \times \{0\} \quad \text{and} \quad \mathcal{B}(\omega(A)) = \omega(A) \times \{0\}. \tag{17}$$

It follows from the theory in Section C.1, that the equation (13) has exponential dichotomies over each set $\alpha(A)$ and $\omega(A)$. The Splitting Index is defined as

$$S(A) \overset{\text{def}}{=} \dim \mathcal{S}(\alpha(A)) - \dim \mathcal{S}(\omega(A)).$$

One then obtains the following result concerning the Splitting Index and the Fredholm Index,

$$i(L) \overset{\text{def}}{=} \dim(\text{Null-space}(L)) - \operatorname{codim}(\text{Range}(L)).$$

THEOREM 4. *Let* $(\mathcal{B}_0, \mathcal{B})$ *be an admissible pair for* (14) *and let L satisfy* (15), *where the hull* $H(A)$ *is compact and* (17) *is satisfied. Then L is a Fredholm operator with*

$$i(L) = -S(A).$$

R. Other Topics. During the last several years, Sacker has turned his attention to the theory and applications of discrete dynamics and difference equations. In this context, it is important to mention the role of the Sacker bifurcation theorem. This theorem has become known as the Neimark–Sacker Theorem, see [26] and [27]. The theorem had been announced in late 1950s by Neimark in the Soviet Doklady, but no detailed proof was ever offered. This theorem, with the proof by Sacker was included in 1964 in his NYU PhD thesis. We are very pleased that this very important bifurcation theorem is now being reprinted in this Issue of the Journal. This should serve the community well.

(R.1) Mathematical biology and genetics. In the papers [41–44] one finds the theory of discrete dynamics applied to a class of problems dealing with genetics in mathematical biology, including applications for genetically altered mosquitoes. In [44] the concept of *Dynamic Reduction* is introduced. This concept is easily described for the periodic difference equation:

$$x_{n+1} = f_n(x_n), \quad x \in \mathbb{R}^d, \quad f_{n+p} = f_n, \quad n = 0, 1, 2, \ldots \tag{18}$$

Let $\mathcal{P}$ denote the class of periodic functions from $\mathbb{Z}$ to $\mathbb{R}^d$, where

$$v = \{v_0, v_1, \ldots, v_{p-1}\} \in \mathcal{P}.$$

In certain cases the dependency on x in the function f can be re-distributed so that

$$f_n(x) = F_n(x, g(x)), \quad g : \mathbb{R}^d \to \mathbb{R}^d,$$

and the *reduced* equation

$$x_{n+1} = F_n(x_n, g(v_n))$$

has a globally attracting asymptotically stable p-periodic solution $\hat{v} \in \mathcal{P}$. This establishes a mapping

$$\mathcal{T} : \mathcal{P} \to \mathcal{P}, \quad v \mapsto \hat{v}.$$

Under certain smallness conditions on the derivative of g, $\mathcal{T}$ is shown to be a contraction. This technique is used in [44] to establish a globally attracting asymptotically stable p-periodic solution of a system in $(\mathbb{R}^+)^2$ modelling the interaction of genetically altered and wild type mosquitoes.

In [28] it is shown that the class of mappings from $\mathbb{R}^+ \to \mathbb{R}^+$ that are either *convex* or *concave*, the latter satisfying some minimality conditions, and have a non-negative Schwarzian on $\mathbb{R}^+$ form a semigroup. This reduces the search for globally attracting asymptotically stable periodic solutions of such periodic systems to the observation that each mapping in the class automatically possesses a globally attracting asymptotically stable fixed point. This technique is applied to the decoupling of large dimensional nonlinear systems and rational delay difference equations, which are referred to in [1], pp. 492–512, for example. This in turn greatly expands the class of rational equations currently being studied in the literature.

(R.2) Periodic phenomena for mappings. In the papers [7–13,17,25,28], one finds various theories of periodic behaviour appearing in nonautonomous difference equations and population biology. For these papers one is dealing with discrete-time semiflows generated by a (generally) nonautonomus family of mappings, or difference equations, $x_{n+1} = F(n, x_n)$ on a metric space X, where $F = F(n, x) \in \mathcal{C}$, and $\mathcal{C} = \mathcal{C}(\mathbb{N}_0 \times X)$ is the space of sequences of continuous mappings of X into X, $n \in \mathbb{N}_0$, with $\mathbb{N}_0$ being the set of nonnegative integers.

One constructs the solution x_{n+1} of the system $x_{n+1} = F(n, x_n)$ and the associate skew product semiflow $\pi(m)$ by means of a sequence of composition mappings. For example,

$$x_1 = F(0, x_0), \quad x_2 = F(1, x_1) = F(1, F(0, x_0)), \text{ etc.}$$

However, a better notation is to use the composition operator Φ, which is defined inductively as follows: $x_1 = \Phi(F, 1)x_0 = F(0, x_0)$ and

$$x_2 = \Phi(F, 2)x_0 = \Phi(F_1, 1)\Phi(F, 1)x_0 = \Phi(F_1, 1)x_1.$$

In this way one obtains a family of mappings $\Phi(F_m, n)$, which satisfy

$$x_{m+n} = \Phi(F, m + n)x_0 = \Phi(F_m, n)\Phi(F, m)x_0 = \Phi(F_m, n)x_n,$$

for all $m, n \in \mathbb{N}_0$, and all $F \in \mathcal{C}$, and one has the co-cycle property:

$$\Phi(F, m + n) = \Phi(F_m, n)\Phi(F, m), \quad \text{for all } m, n \in \mathbb{N}_0 \text{ and all } F \in \mathcal{C}. \tag{19}$$

By using the topology of uniform convergence on compact sets in X for the space $\mathcal{C}$, one obtains a skew product semiflow

$$\pi(m)(F, x_0) = (F_m, \Phi(F, m)x_0),$$

on $\mathcal{C} \times X$, where $F_m(n, x) = F(m + n, x)$ and Φ is a continuous mapping that satisfies $\Phi(F, 0)x = x$, for all $(F, x) \in \mathcal{C} \times X$), and (19) holds.

The Periodic Problem. Of special interest here is the case where the sequence $\{F_n(m, x)\}$ is periodic in n with minimal period $p \geq 1$. One then has

$$F(p + m, x) = F_p(m, x), \quad \text{for all } m \in \mathbb{N}_0 \text{ and all } x \in X. \tag{20}$$

A periodic motion with period r in the skew product semiflow $\pi(m)$ occurs when there is an $x_0 \in X$ such that

$$\pi(r + m)(F, x_0) = \pi(m)(F, x_0), \quad \text{for all } m \in \mathbb{N}_0. \tag{21}$$

As usual, we require r to be positive, and that it be a minimal period, which brings us to the question: What are the permissible periods r for periodic solutions of equation (21)?

For $r \leq p$, one defines an r-periodic cycle as an ordered set

$$C_r = \{x_0, x_1, \ldots, x_{r-1}\}$$

with the property that

$$\Phi(F_m, nr)x_m = x_{m+1}, \quad \text{for all } n \geq 0 \text{ and all } m \text{ with } 0 \leq m \leq r - 1. \tag{22}$$

Let $s = \operatorname{lcm}[p, r]$, the least common multiple of p and r. One uses (22) with the co-cycle identity (19) to extend the representation in C_r to obtain an ordered set $C_s = \{x_0, \ldots, x_{s-1}\}$ with s members, where $\Phi(F_m, 0)x_m = x_{m+1}$, for $0 \leq m \leq s-1$. Furthermore, one has

$$\pi(s+m)(F, x_0) = \pi(m)(F, x_0), \quad \text{for all } m \geq 0.$$

Thus $\pi(m)(F, x_0)$ is an s-periodic orbit for the skew product flow $\pi(m)$.

By introducing appropriate stability concepts for the periodic orbits of $\pi(m)$, one can use the theory of distal dynamics (see [33,36] and Sections A.1 and A.2) to obtain an N-fold covering of the base space $\{F_0, F_1, \ldots, F_{p-1}\}$ in $\mathcal{C}$. For example, in the setting described above, one would have exactly $N = s/p$ points in each fibre over F_j, for $0 \leq j \leq p-1$, and each of these points spawns an s-periodic orbit for the skew product semiflow.

The Beverton–Holt equations. In the papers [9,10] it is shown that the p-periodic Beverton–Holt equation

$$x_{n+1} = \frac{\mu K_n x_n}{K_n + (\mu - 1)x_n}, \quad n \in \mathbb{N}_0, \tag{23}$$

where the carrying capacities K_n are periodic with period p, has a globally asymptotically p-periodic solution $\Phi(F, m)x_0$ that satisfies the attenuation relation

$$\bar{x} = \frac{1}{p}\sum_{i=0}^{p-1} x_i < \frac{1}{p}\sum_{i=0}^{p-1} K_i = \bar{K}. \tag{24}$$

This result is a proof of a conjecture of Cushing and Henson, see [5] and [6].

However, the attenuation relation can fail to hold in a variation of (23), where both the parameter μ_n, as well as the carrying capacity K_n, are periodic in n. A very interesting example, which illustrates such complexity, is presented in [10,12].

Our Meetings. In January 2009, we recalled on our earliest years as researchers in the mathematical community, including the conference in Mayaguez, Puerto Rico in 1965. Bob pointed out that: His entire career, his future for the next 40 + years, was set in motion during that one week. He met the world leaders in dynamical systems (Lefschetz, Hale and LaSalle). He met Henry Antosiewicz, who offered him a position in the Mathematics Department at the University of Southern California (USC). And he met his future wife, Marti, on the flight back to New York!

Marti was a flight attendant for Pan American World Airways, which had a base in Los Angeles. As Bob put it: that made the decision to accept the USC offer very simple!

Antosiewicz had another major role, offering me a position in the Mathematics Department at USC. I accepted, and our family moved to Los Angeles for the 1967–1968 academic year. This was a time that will never be forgotten. It was a year of profound sadness and shock over the assassinations of Martin Luther King and Robert Kennedy. But it was also a time of excitement and satisfaction over the beginning of the Sacker–Sell mathematical collaboration, an effort which grew out of the Seminar on Dynamical Systems organized by Henry Antosiewicz and Tom Kyner. Many visitors were invited for the Seminar, including Charles Conley and, a one-month visit by, Victor A Pliss, from Russia.

Bob and Marti visited Firenze in 1972, where both Jack Hale and I were spending sabbatical leaves. In 1973–1974, Bob spent his sabbatical at the University of Minnesota, and he visited me in Palermo, Italy in 1975. This latter trip was especially important because it was at that time and in that place where the Splittings Saga began. Due to our interest in the then popular Hollywood Saga: 'the Godfather', we took a drive to Corleone in Sicily to check it out!

Bob is a man of many talents. Among other things, he is a small aircraft pilot. We have had flights on several occasions around the Los Angeles area. One memorable flight was over Catalina Island, where we had the opportunity to observe from the air a pod of 15–20 whales in the ocean. He also loves the opera. A beautiful photo of Guiseppe Verdi graces his office at USC, and he has an impressive collection of Verdi tapes from the operas at the Met in New York.

Our working in Italy was especially noteworthy because we discovered that there are major advantages of combining hard work, good food and great wines, to aid in the discovery of new mathematics. Nor did it hurt to replay an opera from time-to-time.

I value highly my working relation with Bob. *Grazie a molto.*

Notes

1. A variation of (8) is used by Favard in his theory of almost periodic equations, [15].
2. This is sometimes referred to as the Sacker–Sell spectrum.
3. Note that if $\dim \mathcal{M} \geq 1$, then $\lambda = 0 \in \Sigma_T$.

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