

13th International Workshop on
Dynamics & Control
DaimlerChrysler Training Center, Wiesensteig, Germany
May 22 – 26, 2005

Some Stability Results in a Model for Genetically Altered Mosquitoes

Robert J. Sacker

Department of Mathematics, University of Southern California,
Los Angeles, CA, USA
rsacker@math.usc.edu

Hubertus F. von Bremen

Department of Mathematics and Statistics, California State Polytechnic University,
Pomona, CA, USA
hfvonbremen@csupomona.edu

Abstract. The stability of the positive fixed points of a discrete-time mathematical model for populations consisting of wild and genetically altered mosquitos proposed by Jia Li [1] is considered in this paper. The model is studied for proportional and constant mating rates. Under certain conditions the model with proportional mating rate has two positive fixed points, one unstable and the other locally asymptotically stable. For the constant mating rate case we show that under certain conditions the positive fixed point is *globally* asymptotically stable with respect to initial populations in which both species are present.

1 Introduction

The discrete-time mathematical model for populations consisting of wild and genetically altered mosquitos proposed by Jia Li [1] is considered in this paper. In [1] a two species model having a hybrid Ricatti/Ricker type nonlinearity is developed for proportional and constant mating rates. In this paper we explore the stability of the positive fixed points of the model with these two types of mating rates.

For the proportional mating rate case under certain conditions the system has two positive fixed points. The one with the smaller x value is unstable and the other is *locally* asymptotically stable provided some conditions are satisfied.

For the constant mating rate case Li [1] gives sufficient conditions guaranteeing the existence of a *locally* asymptotically stable fixed point. It is shown below that under less restrictive conditions the positive fixed point is actually *globally* asymptotically stable (GAS) with respect to initial populations in which both species are present.

2 Model for population of mosquitos

The following description closely follows [1] and [2]. Let x_n be the number of wild mosquitos present at generation n . The population dynamics of the wild mosquitos is described by the difference equation

$$x_{n+1} = f(x_n)s(x_n)x_n, \quad (1)$$

where f is the birth function (per-capita rate of offspring production) and s is the survival probability (fraction of the off-spring that survive). The survival probability is assumed to have a Ricker-type form $s(x_n) = e^{-d-kx_n}$.

Let y_n be the number of genetically altered mosquitos present at generation n , and assume that before the wild and altered mosquitos interact, the dynamics of the altered mosquito population is similar to that of the wild type. Once the altered mosquitos are released into the wild mosquito habitat, the populations are governed by the system of difference equations

$$x_{n+1} = f_1(x_n, y_n)x_n e^{-d-k(x_n+y_n)}, \quad y_{n+1} = f_2(x_n, y_n)y_n e^{-d-k(x_n+y_n)}. \quad (2)$$

It is assumed that both wild and altered mosquitos have the same survival probability $e^{-d-k(x_n+y_n)}$. For $x_n > 0$, $y_n > 0$ the birth rate functions f_1 and f_2 are given by

$$f_1(x_n, y_n) = c(N_n) \frac{\alpha_1 x_n + \beta_1 y_n}{x_n + y_n}, \quad f_2(x_n, y_n) = c(N_n) \frac{\alpha_2 x_n + \beta_2 y_n}{x_n + y_n}, \quad (3)$$

where $c(N_n)$ is the number of matings per individual, per unit time with $N_n = x_n + y_n$. At generation n the number of matings, per individual, with wild mosquitos is $c(N_n)x_n/(x_n + y_n)$ and with altered mosquitos, $c(N_n)y_n/(x_n + y_n)$. Let α_1 be the number of wild offspring that a wild mosquito produces through mating with a wild mosquito, and β_1 be the number of wild mosquitos produced through mating with an altered mosquito. Similarly, α_2 and β_2 are the number of altered mosquitos produced by the mating of altered mosquitos with wild and altered mosquitos respectively.

Combining (2) and (3) gives the following set of difference equations that govern the interacting populations of wild and altered mosquitos

$$x_{n+1} = c(N_n) \frac{\alpha_1 x_n + \beta_1 y_n}{x_n + y_n} x_n e^{-d-k(x_n+y_n)}, \quad y_{n+1} = c(N_n) \frac{\alpha_2 x_n + \beta_2 y_n}{x_n + y_n} y_n e^{-d-k(x_n+y_n)}. \quad (4)$$

The mating rate depends on the population density. When the population is relatively small the mating rate will be assumed to be proportional to the total population, N_n , that is, $c(N_n) = c_0 N_n$. Once the population size exceeds a certain level, we expect the number of matings to saturate, and we assume the mating rate is constant, that is, $c(N_n) = c$.

Using (4), and assuming a proportional mating rate for a small population size of $c(N_n) = c_0(x_n + y_n)$ gives

$$x_{n+1} = (a_1 x_n + b_1 y_n)x_n e^{-d-k(x_n+y_n)}, \quad y_{n+1} = (a_2 x_n + b_2 y_n)y_n e^{-d-k(x_n+y_n)}, \quad (5)$$

with $a_i = c_0 \alpha_i$ and $b_i = c_0 \beta_i$, for $i = 1, 2$.

Using (4), and assuming a constant mating rate for a large population size of $c(N_n) = c$ gives

$$x_{n+1} = \frac{a_1 x_n + b_1 y_n}{x_n + y_n} x_n e^{-d-k(x_n+y_n)}, \quad y_{n+1} = \frac{a_2 x_n + b_2 y_n}{x_n + y_n} y_n e^{-d-k(x_n+y_n)}, \quad (6)$$

where $a_i = c\alpha_i$ and $b_i = c\beta_i$, for $i = 1, 2$.

In (5) and (6) we assume that $x_n > 0$, $y_n > 0$, $n \geq 0$.

3 Proportional Mating Rate

In this section we will focus on the proportional mating rate case (for small population size). We will study the ratio $z_n = x_n/y_n$ and we will show that under certain conditions (5) has two positive fixed points. Among the two positive fixed points, the one with the smaller x value is unstable and the other with the larger x value is *locally* asymptotically stable. The origin is also a stable fixed point giving rise to a “threshold” effect, i.e., very small initial populations tend to die out. First we will present a Lemma that establishes the global asymptotic stability of a fixed point of a difference equation that will arise in the study of the ratio ($z_n = x_n/y_n$) dynamics.

Lemma 1 *Suppose $a, b, c, d > 0$, $z \geq 0$ and consider the difference equation*

$$z_{n+1} = f(z_n), \quad \text{with} \quad f(z) = z \frac{az + b}{cz + d}. \quad (7)$$

If $b/d > 1$ and $c/a > 1$, then (7) has a unique positive, GAS fixed point.

Proof. From the hypothesis we have that $bc/ad > 1$ and so $bc - ad > 0$. Through direct computation we have that

$$f'(z) = \frac{acz^2 + 2adz + bd}{(cz + d)^2} > 0 \text{ for } z \geq 0. \quad (8)$$

We have that $f'(0) = b/d > 1$ and

$$f'' = \frac{2d(ad - bc)}{(cz + d)^3}$$

Therefore $f''(z) < 0$ for all $z > 0$, and (7) has the unique positive fixed point $z^* = (d - b)/(a - c)$. It is easily seen from the increasing and concave properties of f that the positive fixed point is unique and GAS (for $z > 0$). See [3] for more details. ■

Considering the ratio $z_n = x_n/y_n$ and using (5) we get

$$z_{n+1} = \frac{a_1 z_n + b_1}{a_2 z_n + b_2} z_n. \quad (9)$$

Due to the decay survival probability term in (5) the populations can not grow indefinitely ($x_n + y_n \rightarrow \infty$ is not possible).

The nonzero fixed point of (9) is

$$\hat{z} = \frac{b_2 - b_1}{a_1 - a_2}. \quad (10)$$

With $a = a_1$, $b = b_1$, $c = a_2$, and $d = b_2$ (9) is just (7). If $b_1/b_2 > 1$ and $a_2/a_1 > 1$, then the conditions of Lemma 1 apply and the positive fixed point of (9) given in (10) is GAS (for $z > 0$).

Note that a fixed *point* for (9) represents a fixed or invariant line in the (x, y) plane, i.e. the line $S = \{(x, y) : y/x = r\}$ is invariant where $r = (a_2 - a_1)/(b_1 - b_2)$. From the GAS of this fixed point we then have that $y_n/x_n \rightarrow 1/\hat{z} = r$, i.e. the ω -limit set of any point (x, y) with $x > 0$, $y > 0$ lies in the line S . Then to study the solutions in S we can set $y_n = rx_n$ in the first equation of (5), and in the second equation we can set $x_n = (1/r)y_n$. Using these substitutions we get the following two uncoupled equations (on S).

$$x_{n+1} = (a_1 + b_1r)x_n^2 e^{-d-k(1+r)x_n}, \quad y_{n+1} = (a_2(1/r) + b_2)y_n^2 e^{-d-k(1+1/r)y_n}. \quad (11)$$

The x -equation in (11) is of the form

$$x_{n+1} = \alpha x_n^2 e^{-\beta x_n}, \quad (12)$$

with $\alpha = (a_1 + b_1r)e^{-d}$ and $\beta = k(1 + r)$. Note that the y -equation in (11) has the same general form as (12).

If $\alpha > \beta e$, then (12) will have two positive fixed points. An unstable fixed point will occur at $x < 1/\beta$, and a stable fixed point will occur at $x > 1/\beta$. The fixed point, x_* , with $x_* > 1/\beta$ will be *locally* asymptotically stable if $1 < \beta x_* < 3$.

This shows the following result.

Theorem 1 *If $\alpha > \beta e$, the positive fixed point of (5) with $x_* > 1/\beta$ is locally asymptotically stable in the first open quadrant provided that $b_1/b_2 > 1$, $a_2/a_1 > 1$ and $1 < \beta x_* < 3$.*

4 Constant Mating Rate

In this section we will again consider the ratio $z_n = x_n/y_n$, but applied to the constant mating rate case given by (6). We will show that under certain conditions the positive fixed point of (6) is *globally* asymptotically stable. In fact, we will show that this takes place under less stringent conditions than those imposed in [1] to obtain *local* asymptotic stability.

As in the proportional mating rate case, the the ratio $z_n = x_n/y_n$ of (6) yields a difference equation of identical form to (9). The only difference is that now the values of the coefficients come from (6) and not (5). The exact same argument used in the proportional mating rate case applies and the positive fixed point (10) of (9) is GAS provided $b_1/b_2 > 1$ and $a_2/a_1 > 1$.

As before, to study the solutions in S we can set $y_n = rx_n$ in the first equation of (6), and in the second equation we can set $x_n = (1/r)y_n$. Using these substitutions we get the following two uncoupled Ricker's equations (on S).

$$x_{n+1} = \frac{a_1 + b_1 r}{1 + r} x_n e^{-d-k(1+r)x_n}, \quad y_{n+1} = \frac{a_2 + b_2 r}{1 + r} y_n e^{-d-k(1+1/r)y_n}. \quad (13)$$

The above two equations are of the general form of the Ricker's equation

$$w_{n+1} = R(w_n), \quad \text{where} \quad R(w) \doteq \rho w e^{p-\alpha w}. \quad (14)$$

The nonzero fixed point of (14) is $w^* = (p + \ln(\rho))/\alpha$, and w^* is GAS for $0 < p + \ln(\rho) < 2$, or $e^{-p} < \rho < e^{2-p}$ with

$$\rho = \frac{a_1 + b_1 r}{1 + r} = \frac{a_2 + b_2 r}{1 + r} = \frac{a_1 b_2 - b_1 a_2}{a_1 - a_2 + b_2 - b_1}. \quad (15)$$

The fixed points of the decoupled system (13) are

$$\hat{x} = (-d + \ln(\rho))/(k(1+r)), \quad \hat{y} = r(-d + \ln(\rho))/(k(1+r)), \quad (16)$$

and these fixed points are stable provided $0 < -d + \ln(\rho) < 2$.

This shows the following result.

Theorem 2 *The positive fixed point of (6) (given in (16)) is globally asymptotically stable in the first open quadrant provided that $b_1/b_2 > 1$, $a_2/a_1 > 1$ and $0 < -d + \ln(\rho) < 2$.*

Let

$$N = \frac{(a_1 - a_2)(b_2 - b_1)}{a_1 b_2 - b_1 a_2}, \quad \text{and} \quad P = -d + \ln(\rho). \quad (17)$$

Thus if $b_1/b_2 > 1$, $a_2/a_1 > 1$ and $0 < P < 2$, then (6) has a positive fixed point and it is GAS.

In terms of N and P , Li's result (Theorem 3.3, [1]) requires $b_1/b_2 > 1$, $a_2/a_1 > 1$, $0 < P$, and

$$N(P - 1) < P < 2 + (N/2)(P - 2) \quad (18)$$

for *local* asymptotic stability of the positive fixed point. The right portion ($P < 2 + (N/2)(P - 2)$) of the inequality in (18) is equivalent to $P < 2$. Note that $N < 0$, and thus our condition $0 < P < 2$ is less restrictive than (18). If $P < 1$ then $N(P - 1)$ is a finite positive quantity and therefore more restrictive than just $P > 0$.

In [2] we show that when the mortality rates (fixed as well as density dependent) are different for the two species, the constant mating rate model becomes very sensitive to small changes in the reproductive parameters.

5 Conclusions

In this paper we explored the stability of the positive fixed points of a discrete-time mathematical model for populations consisting of wild and genetically altered mosquitos proposed by Jia Li [1]. We studied a model with proportional mating rate and a model

with a constant mating rate. For the proportional mating rate case, we showed that under a set of conditions the system will have two positive fixed points. The positive fixed point with the largest x value is *locally* asymptotically stable, provided certain conditions are satisfied, and the other positive fixed point is unstable. The origin is also a *locally* asymptotically stable fixed point. This creates a "threshold" effect so that very small initial populations tend to die out.

We also show that under certain conditions the fixed point of the system with constant mating rate is *globally* asymptotically stable. The *global* stability is achieved under less stringent conditions than those imposed by [1] to obtain *local* asymptotic stability.

References

- [1] J. Li: Simple mathematical models for mosquito populations with genetically altered mosquitos. *Math. Bioscience*, 189:39–59, 2004.
- [2] R.J. Sacker and H.F. von Bremen: Global Asymptotic Stability in the Jia Li Model for Gentically Altered Mosquitoes. (*to appear*).
- [3] S. Elaydi and R. J. Sacker: Global stability of periodic orbits of nonautonomous difference equations. *J Differential Eq*, 208(1):258-273, 2005.