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PERIODIC UNIMODAL ALLEE MAPS, THE SEMIGROUP PROPERTY AND THE λ-RICKER MAP WITH ALLEE EFFECT

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Abstract. The λ-Ricker equation has, for certain values of the parameters, an unstable fixed point giving rise to the Allee effect, and an attracting fixed point, the carrying capacity. The k-periodic λ-Ricker equation is studied and parameter intervals are determined for which there exist a k-periodic Allee state and a k-periodic attracting state.

1. Introduction.

The λ-Ricker equation (1) was introduced in [13] and later studied in [24]. It is a modification of the well known Ricker equation that produces a zero slope to the graph at the origin and for certain values of the parameters gives rise to an unstable fixed point, the Allee threshold as well as a stable one, the carrying capacity. This same effect was also achieved in the Sigmoid Beverton Holt equation [18] and later in [17] where periodic equations were considered and a periodic Allee state and a periodic attracting state were shown to exist. While the Sigmoid Beverton Holt equation exhibits a depensatory effect, the λ-Ricker equation has one added feature that complicates the study, namely over-depensation that causes very large populations to collapse to levels below the Allee threshold and thus go extinct.

Understanding the Allee effect is of paramount importance in the management of fisheries and establishment of safeguards against overfishing [2], [25]. In [29], Stephens and Sutherland described several scenarios that cause the Allee effect in animals. For example, cod and many freshwater fish species have high juvenile mortality when there are fewer adults. Fewer red sea urchin give rise to worsening feeding conditions of their young and less protection from predation. In some mast flowering trees, such as smooth cordgrass, *Spartina alterniflora*, low population density results in lower probability of seed production and germination, [7].

For an in-depth review of the various occurrences of the Allee effect, see [31]. A mapping having essentialy the same graph as the λ-Ricker map was studied in [28] and later in [23].

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See [13] for a discussion of some new examples of models exhibiting the Allee effect and, similar to the Beverton-Holt model, having important biological quantities as parameters, e.g. intrinsic growth rate, carrying capacity, Allee threshold and a new parameter, the shock recovery parameter. Partial results were derived in each case. For the λ-Ricker equation the threshold beyond which period doubling occurred was explored and a condition was given guaranteeing asymptotic stability of the carrying capacity fixed point.This condition will be re-derived in Section 5 using an equivalent mapping that allows us to simultaneously consider the Allee effect. This new equation is particularly useful in considering the composition of these maps by explicitly exhibiting the carrying capacity as one of the parameters in the mapping while the Allee threshold appears as the symmetric point of a parabola-like graph, Figure 1. The topic of period doubling and chaos are not explored here. Further references pertaining to the Allee effect can be found in [1], [24], [3], [4], [8], [23], [30], [19], [28], [32], [34], [14], [15], [16] and for references to the general theory of Difference Equations, see [9] and [26].

In this work we consider the λ-Ricker equation for fixed parameter λ,

$$x_{n+1} = \mathcal{R}_\rho(x_n) \qquad x_0 \in \mathbb{R}^+, \tag{1}$$

where

$$\mathcal{R}_\rho(x) = x^\lambda e^{\rho - x}, \quad \lambda > 1, \quad \rho > 0. \tag{2}$$

We allow ρ to vary periodically within certain bounds and show there exist a periodic unstable Allee state and a periodic asymptotically stable attracting state, both with the same period as ρ and both contained respectively within the envelopes of the Allee thresholds and carrying capacities of the individual maps in the periodic sequence.

2. A mapping equivalent to the $\lambda-$Ricker map.

For ρ very small the origin is the only fixed point of (2) and it attracts all solutions of (1) having $x_0 \geq 0$. As ρ increases a saddle-node bifurcation takes place and two new fixed points emerge. One of them is unstable and it represents the Allee threshold and the other fixed point is asymptotically stable and represents the *carrying capacity* of the population governed by the model (1). Since we wish to consider periodic environmental fluctuations of the carrying capacity, we shall assume this bifurcation has taken place and rewrite the equation in a form in which the carrying capacity is a parameter in the model. Thus we consider the following modified λ-Ricker Equation for $x \in \mathbb{R}^+$,

$$x_{n+1} = \frac{x_n^\lambda}{p^{\lambda-1}} e^{p - x_n} = x_n^\lambda e^{p-(\lambda-1)\log(p) - x_n}, \qquad n = 0, 1, \cdots, \tag{3}$$

where $\lambda > 1$, $x_n, p \in \mathbb{R}^+$. The results obtained will be interpreted for (2) in Section 6.

Define

$$R_p(x) = x^\lambda e^{p-(\lambda-1)\log(p)-x}, x, p \in \mathbb{R}^+, \lambda > 1. \tag{4}$$

By simple computation, we obtain

$$R_p(p) = p \tag{5}$$

and

$$R'_p(x) = (\lambda - x)x^{\lambda-1}e^{p-(\lambda-1)\log(p)-x} = (\lambda - x)(\frac{x}{p})^{\lambda-1}e^{p-x}. \tag{6}$$

It is clear that any positive fixed point of $R_p(x)$ is a solution of the following equation:

$$x - (\lambda - 1)\log(x) = p - (\lambda - 1)\log(p). \tag{7}$$

Define the function $u : \mathbb{R}^+ \to \mathbb{R}^+$, see Figure 1,

$$u(x) = x - (\lambda - 1)\log(x). \tag{8}$$

Thus (7) can be rewritten as

$$u(x) = u(p). \tag{9}$$

The proofs of Lemmas 2.1 and 2.2 follow from

$$u'(p) = 1 - \frac{\lambda - 1}{p}, \tag{10}$$

see Figure 1.

Lemma 2.1. *If $1 < \lambda < e + 1$, then $u(x) > 0$ for $x > 0$.*

Note. In the following statements, we always assume that $1 < \lambda < e + 1$.

Lemma 2.2. *p is a fixed point of $R_p(x)$. Morever, if $p = \lambda - 1$, then p is a unique fixed point of $R_p(x)$.*

Proof. Its follows from (10) that u is 2-to-1 on $(1, e + 1)$ except at $p = \lambda - 1$. □

For $x \neq \lambda - 1$ (see Figure 1), define $S_x \neq x$ to be the second solution of

$$u(S_x) = u(x). \tag{11}$$

Lemma 2.3. *For all $p \in \mathbb{R}^+$, $p \neq \lambda - 1$, there exists a unique solution S_p of Eq.(9) which is distinct from p. In other words, S_p is a unique fixed point of $R_p(x)$ which is distinct from p. Moreover,*
(1) if $p > \lambda - 1$, then $S_p < \lambda - 1$.
(2) If $p < \lambda - 1$, then $S_p > \lambda - 1$.

Proof. Clearly, p and S_p are the fixed points of $R_p(x)$ which follows from (5) and (7). By (8), we have

$$\lim_{x \to 0^+} u(x) = \lim_{x \to +\infty} u(x) = +\infty. \tag{12}$$

Notice that

$$u'(x) = 1 - \frac{\lambda - 1}{x}x, \tag{13}$$

from which follows that $u(x)$ has a unique critical point $\lambda - 1$ such that $u'(\lambda - 1) = 0$.

In addition, $u'(x) < 0$ for $x < \lambda - 1$, $u'(x) > 0$ for $x > \lambda - 1$ which means that $u(x)$ is decreasing for $x < \lambda - 1$ and increasing for $x > \lambda - 1$. Then $u(x)$ attains its minimum $u(\lambda - 1)$ when $x = \lambda - 1$. If $p > \lambda - 1$, by the above argument, we have $S_p \notin [\lambda - 1, +\infty)$. As $p \neq \lambda - 1$, we obtain

$$u(\lambda - 1) < u(p) < +\infty. \tag{14}$$

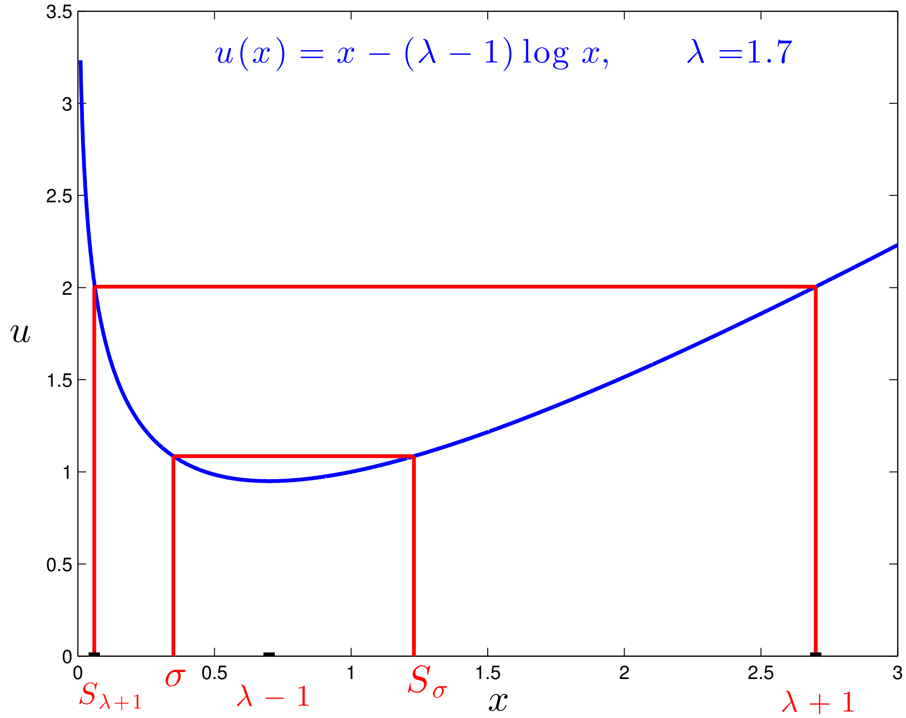

FIGURE 1. Shows the relation between the fixed points of $R_p(x)$. When $x = p > \lambda - 1$ then p is the stable fixed point or carrying capacity and $S_p < \lambda - 1$ is the unstable fixed point or Allee threshold. Figure shown with $p = \lambda + 1$.

Combining (12) and (14) with continuity and monotonicity of $u(x)$, there exists a unique solution $S_p \in (0, \lambda - 1)$ of Eq.(9).

By a similar argument, if $p < \lambda - 1$, there exists a unique solution $S_p \in (\lambda - 1, +\infty)$ and the lemma is proved. □

Recall that S_λ and $S_{\lambda+1}$ satisfy

$$u(S_\lambda) = u(\lambda) \tag{15}$$

$$u(S_{\lambda+1}) = u(\lambda + 1), \tag{16}$$

and from Lemma 2.3 it follows that $S_\lambda < \lambda - 1$ and $S_{\lambda+1} < \lambda - 1$.

The next theorem identifies p and S_p as the Carrying Capacity-Allee Threshold pair.

Theorem 2.4. *The fixed point* 0 *of* $R_p(x)$ *is always asymptotically stable. If the fixed point* $p \in (\lambda - 1, \lambda + 1)$, *then* p *is asymptotically stable, while the other fixed point* $S_p \in (S_{\lambda+1}, \lambda - 1)$ *of* $R_p(x)$ *is unstable.*

Proof. The first statement follows from $R'_p(0) = 0$. Notice that $|R'_p(p)| < 1$ for $\lambda - 1 < p < \lambda + 1$ which implies p is asymptotically stable. By Lemma 2.3, we get $S_p < \lambda - 1$ for $p > \lambda - 1$. We next show $S_p > S_{\lambda+1}$. If not, assume $S_p \leq S_{\lambda+1}$. Since $u(x)$ is decreasing for $x < \lambda - 1$, we have $u(S_p) \geq u(\lambda + 1)$. By Lemma 2.3 (exchange S_p with p), $p \geq \lambda + 1$. This contradicts the assumption that $p < \lambda + 1$ which implies

$$S_{\lambda+1} < S_p < \lambda - 1. \tag{17}$$

By (6) and (7),we have

$$R'_p(S_p) = (\lambda - S_p)(\frac{S_p}{p})^{\lambda-1} e^{p-S_p}, \tag{18}$$

and

$$p - S_p = (\lambda - 1)(\log(p) - \log(S_p)). \tag{19}$$

Combining (18) with (19), we obtain

$$R'_p(S_p) = \lambda - S_p. \tag{20}$$

This plus (17) leads to that

$$R'_p(S_p) > 1.$$

Hence S_p is unstable. □

Theorem 2.5. *If $p = \lambda - 1$, then the unique fixed point p of $R_p(x)$ is semi-stable. More precisely, p is stable from the right and unstable from the left.*

The proof consists of verifying that

$$R'_{\lambda-1}(\lambda - 1) = 1 \quad \text{and} \quad R''_{\lambda-1}(\lambda - 1) = 1 - \lambda < 0,$$

and is omitted.

By interchanging the roles of p and S_p in Theorem 2.4, we obtain the following

Theorem 2.6. *If $p \in (S_{\lambda+1}, \lambda - 1)$, one fixed point p of $R_p(x)$ is unstable, the other fixed point $S_p \in (\lambda - 1, \lambda + 1)$ of $R_p(x)$ is asymptotically stable.*

Definition 2.7. Let $I = [0, b] \subset \mathbb{R}^+$. A continuous function $f : I \to I$ is called an *Allee map* if the following hold:
(a) $f(0) = 0$.
(b) There are positive points A_f and K_f such that
$f(x) < x$ for $x \in (0, A_f) \bigcup (K_f, b)$ and $f(x) > x$ for $x \in (A_f, K_f)$.

Note. b can be equal to $+\infty$.

In addition, if the map is unimodal, then it is called a *unimodal Allee map*. Explicitly, unimodal is defined as follows:

Definition 2.8. A *unimodal map* is a continuous function $f : \mathbb{R}^+ \to \mathbb{R}^+$ for which there exists $c \in \mathbb{R}^+$ such that f is strictly increasing on $[0, c]$ and strictly decreasing on $[c, +\infty]$. Moreover, we require that $f(0) = 0$.

It then follows that

$$\lim_{x\to+\infty} f(x) = M \text{ where } M \in [0, f(c)).$$

Theorem 2.9. *If $p \in (S_{\lambda+1}, \lambda - 1) \bigcup (\lambda - 1, \lambda + 1)$, then $R_p(x)$ is a unimodal Allee map.*

Proof. First, we will show $R_p(x)$ is a unimodal Allee map for $p \in J \doteq (S_{\lambda+1}, \lambda-1)$.

It is clear from (6) that $R_p(x)$ is a unimodal map. By Theorem 2.6, if $p \in (S_{\lambda+1}, \lambda-1)$, then the companion fixed point S_p lies in $(\lambda-1, \lambda+1)$. From (8) and (9), when $0 < x < p$ and $p \in J$, we have

$$\begin{aligned} u(p) < u(x) \quad &\Rightarrow \quad p - (\lambda-1)\log(p) < x - (\lambda-1)\log(x) \\ &\Rightarrow \quad \log(R_p(x)) < \log(x) \Rightarrow R_p(x) < x. \end{aligned} \tag{21}$$

Using a similar argument,

$$R_p(x) > x \text{ for } p < x < S_p \text{ and } R_p(x) < x \text{ for } x > S_p. \tag{22}$$

Combining (21) and (22) with Definition 2.7, we obtain $R_p(x)$ is the unimodal Allee map for $p \in J$.

Following the same reasoning one can show that $R_p(x)$ is the unimodal Allee map for $\lambda - 1 < p < \lambda + 1$. □

Theorem 2.10. *Assume $S_\lambda < p < \lambda$, then the following statements are true:*

$$(i)\ 0 < R_p(x) < \lambda. \ (ii) \ \min_{S_\lambda < p < \lambda} \max_{x \in \mathbb{R}^+} R_p(x) > \lambda - 1.$$

Proof. First we will establish (i). From (4), $R_p(x) > 0$.

It is easy to verify that $R_p(x)$ attains its maximum when $x = \lambda$.

Thus,

$$\max_{x\in\mathbb{R}^+} R_p(x) = R_p(\lambda) = \lambda^\lambda e^{p-(\lambda-1)\log(p)-\lambda} \doteq h(p). \tag{23}$$

Taking the derivative of the function $h(p)$,

$$h'(p) = \lambda^\lambda e^{p-(\lambda-1)\log(p)-\lambda}(1 - \frac{\lambda-1}{p}). \tag{24}$$

Consider the function $h(p)$ for $S_\lambda < p < \lambda - 1$.

By (24), $h(p)$ is decreasing for $0 < p < \lambda - 1$ and we obtain

$$\sup_{S_\lambda < p < \lambda-1} h(p) = h(S_\lambda) = \lambda^\lambda e^{S_\lambda - (\lambda-1)\log(S_\lambda)-\lambda} = \lambda^\lambda e^{u(\lambda)-\lambda}, \tag{25}$$

where u is defined in (8). By (15), we have

$$h(S_\lambda) = h(\lambda) = \lambda^\lambda e^{\lambda-(\lambda-1)\log(\lambda)-\lambda} = \lambda. \tag{26}$$

From (23), (25) and (26), we obtain

$$R_p(x) < \lambda \text{ for } S_\lambda < p < \lambda - 1. \tag{27}$$

In a similar fashion we obtain,

$$R_p(x) < \lambda \text{ for } \lambda - 1 \le p < \lambda. \tag{28}$$

From (27) and (28), we have

$$R_p(x) < \lambda \text{ for } S_\lambda < p < \lambda,$$

and from (23) and (24), we obtain

$$\min_{S_\lambda < p < \lambda} \max_{x\in\mathbb{R}^+} R_p(x) = h(\lambda-1) = \frac{\lambda}{e}(\frac{\lambda}{\lambda-1})^{\lambda-1}. \tag{29}$$

Next we will compare $\frac{\lambda}{e}(\frac{\lambda}{\lambda-1})^{\lambda-1}$ with $\lambda - 1$. Observe that

$$\log(\frac{\lambda}{e}(\frac{\lambda}{\lambda-1})^{\lambda-1}) - \log(\lambda-1) = \lambda\left[\log(\frac{\lambda}{\lambda-1}) - \frac{1}{\lambda}\right]. \tag{30}$$

Let
$$f(x) = \log \frac{x}{x-1} \text{ and } g(x) = \frac{1}{x}.$$
Differentiating $f - g$, we obtain
$$(f(x) - g(x))' = -\frac{1}{x(x-1)} + \frac{1}{x^2} < 0 \text{ for } x > 1, \tag{31}$$
which implies that $f - g$ is decreasing for $x > 1$. By a simple computation, we have
$$\lim_{x\to+\infty} (f(x) - g(x)) = 0. \tag{32}$$
This plus (31) lead to
$$f(x) > g(x) \text{ for } 1 < x < +\infty. \tag{33}$$
From (29), (30) and (33), we have
$$\min_{S_\lambda < p < \lambda} \max_{x\in\mathbb{R}^+} R_p(x) > \lambda - 1,$$
and the proof is complete. □

3. **The composition λ-Ricker map for constant λ.** In this section we study the k-periodic λ-Ricker map,
$$x_{n+1} = R_{p_n}(x_n) = x_n^\lambda e^{p_n - (\lambda-1)\log(p_n) - x_n} \qquad n = 0, 1, \cdots, \tag{34}$$
where $\lambda > 1$, $x_n, p_n \in \mathbb{R}^+$ and p_n is a k-periodic sequence $\{p_0, p_1, \cdots, p_{k-1}\}$, i.e. $p_{i+k} = p_i$, for all $i \geq 0$.

We will find that only for very restrictive intervals of p is it possible that the composition of unimodal λ-Ricker maps is again of that type. These values are tabulated later, but only after we interpret the results for the original mapping (2).

Henceforth, "increasing" and "decreasing" shall always mean strictly increasing and strictly decreasing respectively.

The second derivative of $R_p(x)$ is
$$R_p''(x) = ((x-\lambda)^2 - \lambda)x^{\lambda-2} e^{p-(\lambda-1)\log(p)-x}. \tag{35}$$
Thus $R_p(x)$ has only two inflection points, $\lambda - \sqrt{\lambda}$ and $\lambda + \sqrt{\lambda}$ for $x > 0$.

Also recall the definition of S_x in (11) and the fact that $S_{\lambda-\sqrt{\lambda}}$ satisfies
$$u(S_{\lambda-\sqrt{\lambda}}) = u(\lambda - \sqrt{\lambda}). \tag{36}$$

Lemma 3.1. *$S_\lambda < \lambda - \sqrt{\lambda}$ and $\lambda - 1 < S_{\lambda-\sqrt{\lambda}} < \lambda$ for $1 < \lambda < \lambda_0 \approx 3.08439$ where S_λ and $S_{\lambda-\sqrt{\lambda}}$ are defined by (15) and (36) respectively.*

Proof. Clearly, $S_\lambda < \lambda - 1$ and $\lambda - \sqrt{\lambda} < \lambda - 1$ for $\lambda > 1$. Since $u(x)$ is decreasing for $0 < x < \lambda - 1$, we have
$$S_\lambda < \lambda - \sqrt{\lambda} \iff u(S_\lambda) = u(\lambda) > u(\lambda - \sqrt{\lambda}).$$
Writing this out, we obtain
$$-(\lambda-1)\log\lambda + \sqrt{\lambda} + (\lambda-1)\log(\lambda - \sqrt{\lambda}) > 0,$$
or, noting that $\lambda - 1 > 0$, we need the following to be true,
$$\varphi(\lambda) \doteq \frac{\sqrt{\lambda}}{\lambda-1} - \log\frac{\sqrt{\lambda}+\lambda}{\lambda-1} > 0. \tag{37}$$

It is easy to see that for $0 < \varepsilon << 1$ and $\lambda \in (1, 1+\varepsilon)$, $\varphi(\lambda) > 0$. After a lengthy computation, one sees that

$$\varphi'(\lambda) = -\frac{3 - \lambda + \sqrt{\lambda} - 1/\sqrt{\lambda}}{2(\lambda + \sqrt{\lambda})(\lambda - 1)^2}.$$

It is easy to see that $\varphi'(\lambda) < 0$ at least on $(1, 4]$. Producing a graph in Matlab bears this out and the graph is seen to decrease and cross the axis near 3. Again, using Matlab's "zero" finder *fsolve* in double precision with an initial guess 3, we find $\varphi(\lambda_0) = 0$ where $\lambda_0 \approx 3.08439$.

Hence if $1 < \lambda < \lambda_0$, (37) holds and $S_\lambda < \lambda - \sqrt{\lambda}$. From the definition of S_* and noting that $S_\lambda < \lambda - \sqrt{\lambda} < \lambda - 1$, it is easy seen from Figure 1 that $S_{\lambda-\sqrt{\lambda}} < \lambda$. Finally, $S_{\lambda-\sqrt{\lambda}} > \lambda - 1$ follows from Lemma 2.3. □

Note. By Lemma 2.1, the allowable λ interval is $(1, e+1)$ where $e + 1 \approx 3.71828$. So the λ restriction on the interval $(1, \lambda_0)$ does not seem too restrictive.

As previously discussed, p_n and S_{p_n} are the fixed points of $R_{p_n}(x)$ for $n \in \{0, 1, \cdots k-1\}$.

Let

$$P = \{p_i | i \in \{0, 1, \cdots, k-1\} \text{ and } p_i \neq \lambda - 1\} = \mathcal{U} \cup \mathcal{V}, \tag{38}$$

where

$$\mathcal{U} = P \cap (0, \lambda - 1) \text{ and } \mathcal{V} = P \cap (\lambda - 1, \infty). \tag{39}$$

Next define A_p and K_p as follows:

$$\begin{aligned} p \in \mathcal{U} \to A_p = p \text{ and } K_p = S_p \\ p \in \mathcal{V} \to A_p = S_p \text{ and } K_p = p. \end{aligned} \tag{40}$$

By what has been shown earlier, A_p is the *Allee threshold* and K_p is the *carrying capacity* for the mapping $R_p(x)$.

Define

$$A_{\min} = \min_{p \in P}\{A_p\}, \quad A_{\max} = \max_{p \in P}\{A_p\};$$

and

$$K_{\min} = \min_{p \in P}\{K_p\}, \quad K_{\max} = \max_{p \in P}\{K_p\}.$$

Then we have the following

Theorem 3.2. *Assume* $1 < \lambda < \lambda_0 \approx 3.08439$ *and* p_n *satisfies either all* $p_n \in (S_\lambda, \lambda - \sqrt{\lambda}) \cup (S_{\lambda-\sqrt{\lambda}}, \lambda)$ *or all* $p_n \in [\lambda - \sqrt{\lambda}, S_{\lambda-\sqrt{\lambda}}] \backslash \{\lambda - 1\}$ *for* $n = 0, 1, \cdots k-1$. *Then the Allee-Ricker equation (34) has two positive k-periodic orbits: one unstable orbit* $\alpha = \{\alpha_0, \alpha_1, \cdots, \alpha_{k-1}\} \subset [A_{\min}, A_{\max}]$ *the other stable orbit* $\beta = \{\beta_0, \beta_1, \cdots, \beta_{k-1}\} \subset [K_{\min}, K_{\max}]$.

The proof will be given in Section 4.2.

4. **A general theorem.** The following comes from [17, Section 3] and deals with functions that are concave and increasing on an interval containing infinity.

Definition 4.1. Given $r \geq 0$, define $\mathcal{F}_r$ as the set of all continuous functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ that have the following properties:
(i) $f : [r, \infty) \to [r, \infty)$.
(ii) There exists a number $B \geq r$ such that $f(B) > B$ and f is increasing and

concave on (B, ∞).
(iii) There exists a number $x^* > B$ such that $f(x^*) < x^*$.

For $f \in \mathcal{F}_r$ define $B_f = \inf\{B\}$, where the infimum is taken over all B satisfying (ii). Notice that $B_f \geq r$, $f(B_f) \geq B_f$, and f is increasing and concave on (B_f, ∞).

Lemma 4.2. [17, Lemma 3.1] *For each function $f \in \mathcal{F}_r$ the iterated mapping given by*

$$x_{n+1} = f(x_n)$$

has a unique fixed point L_f on the interval (B_f, ∞). This point is asymptotically stable and attracts all orbits starting at $x \in (B_f, \infty)$.

For any function $f \in \mathcal{F}_r$, it follows from (ii) of Definition 4.1 that $x < f(x) < L_f$ for $x \in (B_f, L_f)$ and $L_f < f(x) < x$ for $x \in (L_f, \infty)$. Given $r \geq 0$ and $\ell \in [r, \infty)$ we define the class

$$\mathcal{U}_{r,\ell} \doteq \{f \in \mathcal{F}_r | B_f \leq \ell < L_f\}.$$

Theorem 4.3. [17, Theorem 3.2] *$\mathcal{U}_{r,\ell}$ is a semigroup under the operation of composition of maps. Moreover, for any $f, g \in \mathcal{U}_{r,\ell}$, $B_{f\circ g} \leq \max\{B_f, B_g\}$ and if $L_f \neq L_g$, $L_{f\circ g}$ lies on the open interval with endpoints L_f and L_g. Otherwise, $L_f = L_g = L_{f\circ g}$.*

Note. From Lemma 4.2, $L_{f\circ g}$ is a unique fixed point of $f \circ g$ for $x > B_{f\circ g}$.

It is this semigroup property that is key to studying periodic equations. In this next section we will set forth conditions that guarantee that the unimodal Allee maps form a semigroup with respect to composition.

4.1. **A new class of mappings.** We next define the class $\mathcal{G}$ to be all functions which satisfy
(A_1) $f \in C^1$ is a unimodal Allee map (Definitions 2.7 and 2.8),
(A_2) All $f \in \mathcal{G}$ have the same critical point $\gamma \in (0, \infty)$ at which a maximum takes place so that f is increasing for $x < \gamma$ and f is decreasing for $x > \gamma$.
(A_3)

$$\sup_{f\in\mathcal{G}}\{K_f\} < \gamma.$$

(A_4)

$$\sup_{f\in\mathcal{G}}\{A_f\} < \inf_{f\in\mathcal{G}}\{K_f\}.$$

Define

$$\mathcal{I}_f = \{x \in (0, \gamma) \mid x \text{ is an inflection point of } f\}.$$

(A_5) Either

$$\sup_{f\in\mathcal{G}}\{A_f\} \leq \inf_{f\in\mathcal{G}} \inf \mathcal{I}_f \leq \sup_{f\in\mathcal{G}} \sup \mathcal{I}_f \leq \inf_{f\in\mathcal{G}}\{K_f\} \qquad (A_5\text{-}1)$$

or

$$\sup_{f\in\mathcal{G}} \sup \mathcal{I}_f \leq \inf_{f\in\mathcal{G}}\{A_f\}. \qquad (A_5\text{-}2)$$

Note that due to A_4 there must be at least one strict inequality in $(A_5$-1$)$.

Theorem 4.4. *Let $f, g \in \mathcal{G}$,*

(a) The class of functions $\mathcal{G}$ is a semigroup under the operation of composition of maps, i.e. $f \circ g \in \mathcal{G}$,
(b) If $A_f \neq A_g$ then $A_{f\circ g}$ lies on the open interval with endpoints A_f and A_g. Otherwise $A_{f\circ g} = A_f = A_g$.

(c) If $K_f \neq K_g$ then $K_{f\circ g}$ lies on the open interval with endpoints K_f and K_g. Otherwise $K_{f\circ g} = K_f = K_g$.

Proof. Let $f, g \in \mathcal{G}$ be given.

Claim 1. *The composition $f \circ g$ is a unimodal map and satisfies condition (A_2) and (A_3) in the definition of the class $\mathcal{G}$.*

Since both f and g satisfy conditions (A_1) and (A_2) of the class $\mathcal{G}$, it follows that

$$f'(\gamma) = g'(\gamma) = 0, \tag{41}$$

and thus f and g attain its maximum when $x = \gamma$.

From condition (A_3), $\gamma > K_f$ and $\gamma > K_g$ which implies $f(\gamma) < \gamma, g(\gamma) < \gamma$. Hence

$$f(x) < \gamma \text{ and } g(x) < \gamma \text{ for } x > 0. \tag{42}$$

Taking the derivative of $f \circ g(x)$,

$$[f \circ g(x)]' = f'(g(x))g'(x), \tag{43}$$

and from (42), $f'(g(x)) > 0$ for all $x > 0$.

From (41) and (43), γ is the critical point of $f \circ g$. Since f and g are increasing for $x < \gamma$ and decreasing for $x > \gamma$ and (43), $f \circ g$ is a unimodal map that increases for $x < \gamma$ and decreases for $x > \gamma$. Thus $f \circ g$ attains its maximum when $x = \gamma$. This establishes (A_2).

For all $x < \max\{K_f, K_g\} < \gamma$, we have

$$f \circ g(x) < f \circ g(\gamma) = f(g(\gamma)) < f(\gamma) < \gamma.$$

which implies that

$$K_{f\circ g} < \gamma. \tag{44}$$

Hence $f \circ g$ satisfies condition (A_3) and Claim 1 is established.

Claim 2. *(b) is true.*

Since f and g are Allee maps, by Definition 2.7 and monotonicity of f and g on $[0, \gamma]$, if $A_g < A_f$, then

$$f(g(A_g)) = f(A_g) < A_g. \tag{45}$$

Combining with condition (A_4),we have,

$$A_g < A_f < K_g \Longrightarrow g(A_f) > A_f.$$

This plus (42) lead to

$$f(g(A_f)) > f(A_f) = A_f. \tag{46}$$

Similarly if $A_g > A_f$, we get

$$f(g(A_g)) = f(A_g) > A_g, \tag{47}$$

and

$$g(A_f) < A_f \Longrightarrow f(g(A_f)) < f(A_f) = A_f. \tag{48}$$

From (45) through (48), there exists a fixed point $A_{f\circ g}$ of $f \circ g$ such that $A_{f\circ g}$ lies strictly between A_f and A_g, thus establishing *(b)*. Since $f \circ g$ is increasing, $A_{f\circ g}$ is indeed an Allee fixed point.

The case $A_g = A_f$ is trivial since a common fixed point is also a fixed point of the composition. Thus Claim 2 is established.

Claim 3. *(c) is true.*

If $K_g < K_f$, by condition (A_4) and monotonicity of f and g, we get

$$\max\{A_f, A_g\} < K_g < K_f \Longrightarrow f(g(K_g)) = f(K_g) \geq K_g, \tag{49}$$

and

$$g(K_f) < K_f \Longrightarrow f(g(K_f)) < f(K_f) = K_f. \tag{50}$$

Similarly if $K_g > K_f$, we have

$$f(g(K_g)) < K_g \text{ and } f(g(K_f)) > K_f. \tag{51}$$

From (49) through (51), there exists a fixed point $K_{f\circ g}$ of $f \circ g$ such that $K_{f\circ g}$ lies strictly between K_f and K_g.

The case $K_f = K_g$ is treated as in Claim 2. Thus Claim 3 is established.

Claim 4. (A_4) *is true.*

Since it is true for each map,

$$A_{f\circ g} < \max\{A_f, A_g\} < \min\{K_f, K_g\} < K_{f\circ g},$$

and therefore it is true for the composition, thus establishing Claim 4.

Claim 5. *The composition $f \circ g$ is an Allee map and (A_1) and (A_5) are satisfied.*

We first lay some groundwork before considering the cases $(A_5$-1) and $(A_5$-2).

Let

$$\overline{I} = \sup(\mathcal{I}_f \cup \mathcal{I}_g) \text{ and } \underline{I} = \inf(\mathcal{I}_f \cup \mathcal{I}_g).$$

Define the extensions,

$$F_-(x) = \begin{cases} f(x), & x \in [0, \underline{I}] \\ f(\underline{I}) + f'(\underline{I})(x - \underline{I}), & x \in (\underline{I}, \infty) \end{cases}$$

and

$$G_-(x) = \begin{cases} g(x), & x \in [0, \underline{I}] \\ g(\underline{I}) + g'(\underline{I})(x - \underline{I}), & x \in (\underline{I}, \infty). \end{cases}$$

These are increasing convex functions on $\mathbb{R}^+$. Clearly $F_-^{-1}, G_-^{-1} \in \mathcal{U}_{0,0}$ where F_-^{-1} and G_-^{-1} represent the inverse function of F_- and G_- respectively and are increasing concave functions on $\mathbb{R}^+$. Recall that $\mathcal{U}_{0,0}$ is a semigroup under composition. By Lemma 4.2 and Theorem 4.3, the following statements are true (recall the definition of B in Definition 4.1):

(B_1) $G_-^{-1} \circ F_-^{-1} \in \mathcal{U}_{0,0}$.

(B_2) $B_{G_-^{-1}\circ F_-^{-1}} \leq \max\{B_{F_-^{-1}}, B_{G_-^{-1}}\} = 0 \Longrightarrow B_{G_-^{-1}\circ F_-^{-1}} = 0$.

(B_3) There exists at most one fixed point of $f \circ g$ for $0 < x < \underline{I}$.

By the condition (A_5) and (B_1), if $\max\{A_f, A_g\} \leq \underline{I}$, then

$$\max\{A_f, A_g\} \leq \underline{I} \leq \inf \mathcal{I}_{f\circ g}. \tag{52}$$

Recall that γ is the common critical point at which f and g attain their maximums and define

$$F_+(x) = \begin{cases} f(x), & x \in [\overline{I}, \gamma - \varepsilon] \\ f(\gamma - \varepsilon) + f'(\gamma - \varepsilon)(x - \gamma + \varepsilon), & x \in (\gamma - \varepsilon, \infty) \end{cases}$$

and

$$G_+(x) = \begin{cases} g(x), & x \in [\overline{I}, \gamma - \varepsilon] \\ g(\gamma - \varepsilon) + g'(\gamma - \varepsilon)(x - \gamma + \varepsilon), & x \in (\gamma - \varepsilon, \infty) \end{cases}$$

where ε is a sufficiently small positive real number so that $\max\{K_f, K_g\} < \gamma - \varepsilon$.

Similarly we have $F_+, G_+ \in \mathcal{U}_{\overline{I},\ell}$. By Lemma 4.2 and Theorem 4.3 , the following statements are true:
(C_1) $F_+ \circ G_+ \in \mathcal{U}_{\overline{I},\ell}$.
(C_2) $B_{F_+ \circ G_+} \leq \max\{B_{F_+}, B_{G_+}\}$.
(C_3) There exists a unique fixed point $K_{f\circ g}$ on the closed interval with endpoints K_f and K_g.

By condition (A_5) and (C_1), if $\overline{I} \leq \min\{K_f, K_g\}$, then

$$\sup \mathcal{I}_{f\circ g} \leq \overline{I} \leq \min\{K_f, K_g\}. \tag{53}$$

If $\overline{I} \leq \min\{A_f, A_g\}$, then

$$\sup \mathcal{I}_{f\circ g} \leq \overline{I} \leq \min\{A_f, A_g\}. \tag{54}$$

From (52) $-$ (54), $f \circ g$ satisfies condition (A_5).

We next prove that the composition $f \circ g$ is an Allee map. Since we know by *Claim 2* and *Claim 3* that there exists two fixed points $A_{f\circ g}$ and $K_{f\circ g}$ of $f \circ g$ that lie on the two closed intervals with endpoints A_f and A_g, K_f and K_g respectively, we first show the uniqueness of $A_{f\circ g}$ and $K_{f\circ g}$. There are two cases.

Case $(A_5$-1). $\max\{A_f, A_g\} \leq \underline{I} \leq \overline{I} \leq \min\{K_f, K_g\}$.

In this case $f(x) > x$ and $g(x) > x$ for $\underline{I} \leq x < \min\{K_f, K_g\}$. Since f and g are increasing for $0 < x < \gamma$, we get

$$f \circ g(x) > f(x) > x \text{ for } \underline{I} \leq x < \min\{K_f, K_g\}. \tag{55}$$

Hence $B_{F_+ \circ G_+} = \overline{I}$. This plus condition (C_3) and (55) imply $K_{f\circ g}$ is the unique fixed point of $f \circ g$ for $x \in [\underline{I}, \gamma)$. Furthermore $f \circ g(x) > x$ for $\underline{I} \leq x < K_{f\circ g}$ and $f \circ g(x) < x$ for $K_{f\circ g} < x < \gamma$. If $0 < x < \min\{A_f, A_g\}$, then $f(x) < x$ and $g(x) < x$. Similarly we have

$$f \circ g(x) < f(x) < x \text{ for } 0 < x < \min\{A_f, A_g\}. \tag{56}$$

From (55) and (56) and (B_3), there exists the unique fixed point $A_{f\circ g}$ of $f \circ g$ for $x \in (0, \underline{I}]$. In addition $f \circ g(x) < x$ for $0 < x < A_{f\circ g}$ and $f \circ g(x) > x$ for $A_{f\circ g} < x \leq \underline{I}$.

Case $(A_5$-2). $\overline{I} \leq \min\{A_f, A_g\}$.

In this case, by (56) we have

$$f \circ g(x) < x \text{ for } 0 < x < \overline{I} \tag{57}$$

Combing Lemma 4.2 and Theorem 4.3, we obtain exactly two fixed points $A_{f\circ g}$ and $K_{f\circ g}$ of $f \circ g$. Furthermore $f \circ g(x) < x$ for $x \in (0, A_{f\circ g}) \bigcup (K_{f\circ g}, \gamma)$ and $f \circ g(x) > x$ for $x \in (A_{f\circ g}, K_{f\circ g})$.

As previously discussed, we have $f \circ g(x) < x$ for $x \in (0, A_{f\circ g}) \bigcup (K_{f\circ g}, \gamma)$ and $f \circ g(x) > x$ for $x \in (A_{f\circ g}, K_{f\circ g})$. Since $\gamma > \max\{K_f, K_g\}$, $f(x) < x, g(x) < x$ for $x > \gamma$. Thus $f \circ g(x) = f(g(x)) < f(x) < x$ for $x > \gamma$ which follows since f and g are decreasing for $x > \gamma$. Thus $f \circ g$ is an Allee map and Claim 5 is established.

From Claim 1 and Claim 5, $f \circ g$ satisfies condition (A_1). Thus the composition $f \circ g$ satisfies conditions $(A_1) - (A_5)$ which establishes *(a)* and the Theorem is proved. $\square$

4.2. **Proof of theorem 3.2.**

Proof. Assume $1 < \lambda < \lambda_0$ and $p_n \in (S_\lambda, \lambda - \sqrt{\lambda}) \bigcup (S_{\lambda-\sqrt{\lambda}}, \lambda)$ for $n = 0, 1, \cdots k-1$. Thus for each $R_i(x)$ where $i \in P$, defined in (38), we obtain the following conclusions:

(D_1) $R_p(x)$ is a unimodal Allee map and there exist only two nonzero fixed points A_p and K_p which follows from Theorem 2.9.

(D_2) All $R_p(x)$ have the same critical point γ and $R_p(x)$ is increasing for $x < \gamma$ and decreasing for $x > \gamma$. Moreover, it follows from Theorem 2.10 that $K_p < \gamma$.

(D_3) From (38), (39) and (40) it follows that

$$\max_{p\in P}\{A_p\} < \min_{p\in P}\{K_p\}.$$

(D_4) From the discussion following (35), each R_p has a unique inflection point I_{R_p} in the interval $0 < x < \gamma$. Moreover,

$$\max_{p\in P}\{A_p\} \le \min_{p\in P}\{I_{R_p}\} \le \max_{p\in P}\{I_{R_p}\} \le \min_{p\in P}\{K_p\},$$

where at least one of the inequalities is strict.

Recall the definition of $\mathcal{G}$, and notice that $R_i(x) \in \mathcal{G}$ for $i \in P$, where P is defined in (38). It follows by Theorem 4.4 that $F_0 = R_{p_{k-1}} \circ R_{p_{k-2}} \circ \cdots \circ R_{p_0} \in \mathcal{G}$ which implies that F_0 is an Allee map and has only two nonzero fixed points A_{F_0} and K_{F_0}. Moreover, A_{F_0} is unstable and K_{F_0} is asymptotically stable. To show A_{F_0} and K_{F_0} lie in $(A_{\min}, A_{\max})$ and $(K_{\min}, K_{\max})$ respectively, we use induction in the case where at least one pair of maps are unequal. From Theorem 4.4,

$$\min\{A_{p_0}, A_{p_1}\} < A_{R_{p_1}\circ R_{p_0}} < \max\{A_{p_0}, A_{p_1}\},$$

and

$$\min\{K_{p_0}, K_{p_1}\} < K_{R_{p_1}\circ R_{p_0}} < \max\{K_{p_0}, K_{p_1}\}.$$

Assume as an induction hypothesis,

$$\min_{0\le i\le m}\{A_{p_i}\} < A_{R_{p_m}\circ\cdots\circ R_{p_0}} < \max_{0\le i\le m}\{A_{p_i}\},$$

and

$$\min_{0\le i\le m}\{K_{p_i}\} < K_{R_{p_m}\circ\cdots\circ R_{p_0}} < \max_{0\le i\le m}\{K_{p_i}\}.$$

Applying Theorem 4.4 to $R_{p_{m+1}}$ and $R_{p_m} \circ \cdots \circ R_{p_0}$ we get

$$\min\{A_{p_{m+1}}, A_{R_{p_m}\circ\cdots\circ R_{p_0}}\} < A_{R_{p_{m+1}}\circ\cdots\circ R_{p_0}} < \max\{A_{p_{m+1}}, A_{R_{p_m}\circ\cdots\circ R_{p_0}}\},$$

and

$$\min\{K_{p_{m+1}}, K_{R_{p_m}\circ\cdots\circ R_{p_0}}\} < K_{R_{p_{m+1}}\circ\cdots\circ R_{p_0}} < \max\{K_{p_{m+1}}, K_{R_{p_m}\circ\cdots\circ R_{p_0}}\}.$$

From the induction hypothesis it follows that

$$\min_{0\le i\le m+1}\{A_{p_i}\} < A_{R_{p_{m+1}}\circ\cdots\circ R_{p_0}} < \max_{0\le i\le m+1}\{A_{p_i}\},$$

and

$$\min_{0\le i\le m+1}\{K_{p_i}\} < K_{R_{p_{m+1}}\circ\cdots\circ R_{p_0}} < \max_{0\le i\le m+1}\{K_{p_i}\}.$$

This shows that $A_{F_0} = A_{R_{p_{k-1}}\circ R_{p_{k-2}}\circ\cdots\circ R_{p_0}} \in (A_{\min}, A_{\max})$ and $K_{F_0} = K_{R_{p_{k-1}}\circ R_{p_{k-2}}\circ\cdots\circ R_{p_0}} \in (K_{\min}, K_{\max})$ which implies $K_{F_0} < \lambda$. To show that the two entire periodic orbits lie in $(A_{\min}, A_{\max})$ and $(K_{\min}, K_{\max})$ respectively, notice that $A_{F_i} = A_{R_{p_{i-1}}\circ\cdots R_{p_0}\circ R_{p_{k-1}}\circ\cdots\circ R_{p_{i+1}}\circ R_{p_i}}$ and $K_{F_i} = K_{R_{p_{i-1}}\circ\cdots R_{p_0}\circ R_{p_{k-1}}\circ\cdots\circ R_{p_{i+1}}\circ R_{p_i}}$ and apply a similar argument.

The remaining case in which all the mappings are the same follows since they all share the same fixed points.

This proves Theorem 3.2 in the case where all the $p_n \in (S_\lambda, \lambda-\sqrt{\lambda}) \cup (S_{\lambda-\sqrt{\lambda}}, \lambda)$. A similar argument establishes the result when all the $p_n \in [\lambda-\sqrt{\lambda}, S_{\lambda-\sqrt{\lambda}}] \backslash \{\lambda-1\}$. □

5. **Attenuation and resonance.** We wish to determine whether periodic fluctuations in the parameters in a dynamical system produce *attenuation* or *resonance* which we now define. Let $\{\bar{x}\} = \{\bar{x}_0, \bar{x}_1, \cdots, \bar{x}_{k-1}\}$ be an asymptotically stable k-periodic solution of (34) and $K = \{K_0, K_1, \cdots, K_{k-1}\}$ be the carrying capacities of the individual maps R_{p_n}, $n = 0, 1, \cdots, k-1$.

Definition 5.1. A periodic solution $\{\bar{x}\}$ of equation (34) is said to be *attenuant* or *resonant* if

$$av(\bar{x}) < av(K) \text{ or } av(\bar{x}) > av(K),$$

respectively, where "av" represent the average of any n-periodic sequence $t = \{t_0, t_1, \cdots, t_{n-1}\}$,

$$av(t) = \frac{1}{n}\sum_{i=0}^{n-1} t_i.$$

The interpretation, at least in population biology, is quite straightforward. Attenuation means the periodic environmental fluctuations have a deleterious effect on the average population whereas resonance has the exact opposite enhancing effect. In [5] they considered the 2-periodic Beverton-Holt equation and proved there exists a globally attracting asymptotic stable periodic solution that exhibited attenuation. They further conjectured [6] the same would be true for higher periods. In 2003 a complete solution was announced [10] and later appeared in [11]. Thereafter followed solutions using different techniques [20], [21, 22]. In [12] the resonance question was solved in the 2-periodic case where a pair of parameters, the intrinsic growth rate and the carrying capacities, were allowed to vary periodically. In fact a formula was derived giving an exact expression for the difference,

$$av(\bar{K}) - av(a).$$

The expression was rather involved for such a simple situation and the second author even commented "the calculations, even for period 4, seem daunting at best." In 2011 the first author took up the challenge and there resulted [33] where the expression was derived for arbitrary period. In addition an informal conjecture as to the root cause of resonance was proven to be wrong.

In [27] the periodic Ricker maps $f_i = xe^{p_i - x}$, $p_i \in (0, 2)$, $i = 0, 1, \cdots, k-1$ was studied and it was shown that the periodic difference equation $x_{n+1} = f_n(x_n)$ always has a periodic solution $\bar{x} = \{\bar{x}_0, \bar{x}_1, \cdots, \bar{x}_{k-1}\}$ that globally attracts all solutions with $x_0 > 0$ and there is neither attenuation nor resonance.

Next observe that for each Allee-Ricker map defined by (34),

$$\frac{x_{i+1}}{x_i} = x_i^{\lambda-1} e^{p_i - (\lambda-1)\log(p_i) - x_i}, \; i = 0, 1, \cdots k-1,$$

and if $\{x_i\}$ is a periodic orbit then $x_k = x_0$ and therefore,

$$\prod_{i=0}^{k-1} \frac{x_{i+1}}{x_i} = \prod_{i=0}^{k-1} x_i^{\lambda-1} e^{p_i - (\lambda-1)\log(p_i) - x_i} = \frac{x_k}{x_0} = 1.$$

Thus, the natural logarithm of the second product must be zero, from which it follows that

$$av(\bar{x}) - av(p) = (\lambda - 1)\frac{1}{k}\sum_{i=0}^{k-1}(\log(x_i) - \log(p_i)). \tag{58}$$

According to the discussion in Section 2, p_i are not alway equal to K_i, but we still wish to compute,

$$av(\bar{x}) - av(K) = (\lambda - 1)\frac{1}{k}\sum_{i=0}^{k-1}(\log(x_i) - \log(K_{p_i})). \tag{59}$$

Two cases occur here. First if all the carrying capacities K_i are known *a priori* and satisfy the conditions of Theorem 3.2, then one simply chooses $p_i = K_i$ in (58). The second case is more interesting when we suppose for certain values of the index i, say

$$i \in \mathcal{U} \subset \{0, 1, 2, \ldots, k-1\},$$

the Allee threshold is the only fixed point of

$$R_{p_i}(x) = x^{\lambda} e^{p_i - (\lambda-1)\log(p_i) - x},$$

that can be measured (aside from 0).Then for $i \in \mathcal{U},\ p_i = A_i < \lambda - 1$. In order to compute K_i, let $\sigma = p_i$ and form $a = u(\sigma)$, see Figure 1. Then find S_σ, the other solution of $u(S_\sigma) = a$. Then $K_i = S_\sigma$. We know from Theorem 3.2 that $S_\sigma \in (\lambda - 1, \lambda)$. This requires a solver that won't stray from that interval, e.g. simple bisection method.

6. **Interpretation of results for the original λ-Ricker equation.**

We began this study of the λ-Ricker equation (1)-(2) which we repeat here

$$x_{n+1} = \mathcal{R}_\rho(x_n), \quad \text{where} \quad \mathcal{R}_\rho(x) = x^\lambda e^{\rho - x}, \quad \lambda > 1, \quad \rho > 0. \tag{60}$$

Theorem 3.2 tells us that if $1 < \lambda < \lambda_0 \approx 3.08439$ and *exactly* one of the following conditions $(\mathcal{C}_1, \mathcal{C}_2)$ hold, then there is an unstable periodic Allee state and a periodic attracting state each lying within the envelopes of the Allee thresholds and carrying capacities respectively of the individual maps, $\mathcal{R}_{\rho_n}(x_n)$:

$\mathcal{C}_1$: All $p_n \in (S_\lambda, \lambda - \sqrt{\lambda}) \cup (S_{\lambda-\sqrt{\lambda}}, \lambda) \quad n = 0, 1, \ldots, k-1,$

$\mathcal{C}_2$: All $p_n \in [\lambda - \sqrt{\lambda}, S_{\lambda-\sqrt{\lambda}}] \backslash \{\lambda - 1\} \quad n = 0, 1, \ldots, k-1.$

The relation between the p_n and the ρ_n comes from (4) and (8) and is given by

$$\rho = u_\lambda(p) = p - (\lambda - 1)\log(p),$$

where we have now indicated by a subscript the dependence on λ to avoid confusion in interpreting Table 6.

But the function $u_\lambda(p)$, for $p \neq \lambda - 1$, is 2-to-1 (see Figure 1) and S_p was defined so that $u_\lambda(S_p) = u_\lambda(p)$ whenever $p \neq \lambda - 1$. Thus each interval in $\mathcal{C}_1$ gives rise to the same interval of ρ values. Likewise the ρ values determined by $\mathcal{C}_2$ are completely determined by either of the two p-intervals $[\lambda - \sqrt{\lambda}, \lambda - 1)$ or $(\lambda - 1, S_{\lambda-\sqrt{\lambda}}]$, see Table 1.

Table 1: For various admissible values of λ, $1 < \lambda < \lambda_0 \approx 3.08439$, the boundaries of the intervals of validity of Theorem 3.2 are shown. The valid ρ values are found using the last three columns and the fact that $u_\lambda(*) = u_\lambda(S_*)$, e.g. for $\lambda = 2$, all ρ values must be in $(u_\lambda(\lambda - 1), u_\lambda(\lambda - \sqrt{\lambda})] = (1.0, 1.12059]$ or all ρ values must be in $(u_\lambda(\lambda - \sqrt{\lambda}), u_\lambda(\lambda)) = (1.12059, 1.30865)$.

λ	$\lambda - \sqrt{\lambda}$	S_λ	$S_{\lambda-\sqrt{\lambda}}$	$u_\lambda(\lambda)$	$u_\lambda(\lambda - \sqrt{\lambda})$	$u_\lambda(\lambda - 1)$
1.05	0.025301	[a]	0.08712	1.04756	0.20914	0.19979
1.1	0.05119	[a]	0.17291	1.09047	0.34841	0.33026
1.2	0.10455	0.00302	0.34097	1.16354	0.55616	0.52189
1.3	0.15982	0.01812	0.50492	1.22129	0.70992	0.66119
1.4	0.21678	0.04762	0.66535	1.26541	0.82832	0.76652
1.5	0.27526	0.08928	0.82270	1.29727	0.92028	0.84657
1.6	0.33509	0.14051	0.97737	1.31799	0.99110	0.90650
1.7	0.39616	0.19922	1.12965	1.32856	1.04432	0.94967
1.8	0.45836	0.26384	1.27981	1.32977	1.08244	0.97851
1.9	0.52160	0.33319	1.42805	1.32233	1.10737	0.99482
2.0	0.58579	0.40638	1.57457	1.30865	1.12059	1.00000
2.1	0.65086	0.48272	1.71951	1.28387	1.12327	0.99516
2.2	0.71676	0.56169	1.86301	1.25385	1.11638	0.98121
2.3	0.78342	0.64288	2.00519	1.21722	1.10073	0.95893
2.4	0.85081	0.72594	2.14615	1.17434	1.07701	0.92894
2.5	0.91886	0.81061	2.28599	1.12556	1.04579	0.89180
2.6	0.98755	0.89667	2.42478	1.07118	1.00760	0.84799
2.7	1.05683	0.98393	2.56260	1.01147	0.96286	0.79793
2.8	1.12668	1.07225	2.69951	0.94669	0.91198	0.74198
2.9	1.19706	1.16150	2.83557	0.87705	0.85531	0.68048
3.0	1.26795	1.25157	2.97083	0.80278	0.79315	0.61370
3.08	1.32501	1.32415	3.07849	0.74015	0.73966	0.55667

[a] values not reliable

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