

Skew-product dynamical systems: Applications to difference equations ¹

Saber Elaydi
Department of Mathematics
Trinity University
San Antonio, Texas 78212, USA
E-mail: selaydi@trinity.edu
<http://www.trinity.edu/selaydi>

Robert J. Sacker
Department of Mathematics
University of Southern California
Los Angeles, CA 90089-2532 USA
E-mail: rsacker@math.usc.edu
<http://math.usc.edu/~rsacker>

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1 Introduction

One of the earliest difference equations, the Fibonacci sequence, was introduced in 1202 in “Liberabaci,” a book about the abacus, by the famous Italian Leonardo di Pisa, better known as Fibonacci. The problem may be stated as follows: how many pairs of rabbits will there be after one year when starting with one pair of mature rabbits, if each pair of rabbits give birth to a new pair each month starting when it reaches its maturity age of two months? If $F(n)$ is the number of pairs of rabbits at the end of n months,

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then the difference equation that represents this model is given by the second order difference equation

$$F(n+1) = F(n) + F(n-1), \text{ with } F(0) = 1, F(1) = 1.$$

This very early connection between biology and difference equations has been, until recently, fairly dormant due largely to the creation of differential Calculus by Isaac Newton in 1732 and the dominance of differential equations in all natural sciences and economics.

Two great mathematicians have contributed significantly to the modern theory of difference equations; Henri Poincaré and George Birkhoff. In “Sur les équations linéaires aux différentielles ordinaires et aux différences finies” [61] Poincaré studied the asymptotic representation of solutions of nonautonomous linear difference equations. The Poincaré-Perron Theorem [20] is the extension of Poincaré Theorem by Oscar Perron in 1921 [58]. Walter Gautschi [31] showed how to use Poincaré–Perron Theorem to find minimal (recessive) solutions of linear nonautonomous difference equations. This has been a significant observation in the area of orthogonal polynomials and special functions since every sequence of a monic polynomial must satisfy a second order difference equations [8, 20, 73]. The importance of the existence of minimal solutions has been observed by Pincherle [60] much earlier in 1894, where he showed a continued fraction converges if and only if the associated second order difference equation has a minimal solution [20]. In a series of papers by George Birkhoff and his collaborators [6, 7], the formal theory of analytic difference equations has been established. Jet Wimp and Doran Zeilberger [74] applied Birkhoff Theory to compute the asymptotics of special functions and combinatorial identities. The study of the asymptotics of combinatorial identities was greatly facilitated by Zeilberger’s algorithm by which one can produce a second order difference equation that represents a given combinatorial identity [74].

Another area where difference equations have played a prominent role is numerical analysis. Here one would approximate a given differential equation, through a discretization method, by a difference equation. However, these discretization methods may lead to instabilities and sometimes chaotic behavior. To remedy this situation, Ronald Mickens introduced a nonstandard discretization scheme which is dynamically consistent [2, 51, 55].

The modern theory of difference equations can be traced back to the 60’s and 70’s. In 1964, Alexander Nikolai Sharkovsky introduced his fundamental

theorem on the periods of continuous maps on the real line [21, 69]. Part of Sharkovsky Theorem was later discovered in 1975, independently, by T.Y. Li and James Yorke in “Period three implies chaos” [50]. In addition to introducing “chaos” in mathematics, the Li–Yorke paper was instrumental in introducing Sharkovsky Theorem in English which made it accessible to scientists in the west. In 1978, Mitchell Feigenbaum [28] discovered a universal constant, the “Feigenbaum number,” that is shared by unimodal continuous maps on the real line [21].

Since then, chaos theory has been in the frontier of science, particularly in physics and economics. In biology, however, progress in the use of difference equations as models has been rather slow. In 1976, difference equations received a huge boost through the fundamental work of Robert May who publicized the use of difference equations as biological models in his ever popular paper “Simple mathematical models with very complicated dynamics” [53]. The last twenty years witnessed an exponential growth in the area of mathematical biology with several new scientific journals and books devoted to the subject. But until recently, the favorite models for biologists have been the continuous ones, i.e., differential equations [19, 43, 59, 70].

Let us now move on to more recent developments. There are by now two schools in difference equations. The first school views difference equations as the discrete analogue of differential equations and analysis. It is no wonder, that the majority in this school came over from differential equations. Books representing this school are those of Lakschmikantham and Trigiante [48], Agarwal [1], Kelley and Peterson [37], Mickens [54], Kocic and Ladas [39], Kulenovic and Ladas [45], Sedaghat [65], Elaydi [20].

The second school views difference equations as iterations of maps or as discrete dynamical systems. The questions raised here are concerned with stability, bifurcation and chaos. Representing this school are the books by Devaney [16], Holmgren [34], Strogatz [71], Alligood et al [3], Martelli [52], Sandefur [64], Cushing [11, 12], Kulenovic and Merino [46], Elaydi [21]. The first two authors do not even mention “difference equations” in their books. The remaining books, however, are clear about the interplay between difference equations and discrete dynamical systems. It should be noted that the last two books [21, 46] made a lot of efforts to bridge the gap between the two schools. However, a larger and more sustained efforts are needed to integrate both schools and consolidate their centrality in mathematics.

As research in difference equations takes root and more and more prominent mathematicians join in, boundaries between the two schools are increas-

ingly blurred. Intrinsic discrete models in biology and economics are being studied without going through differential equations [12, 57, 65]. It has been argued recently that difference equations are the right medium to model physical phenomena [4, 22, 72, 75]. The question of whether time is discrete has been settled affirmatively for quite some time in quantum mechanics, and recently advocated by the above-mentioned group of mathematicians.

In this article, our main objective is to introduce to specialists and non-specialists alike the notion of discrete skew-product dynamical systems. We then specialize this to the case of periodic difference equations and apply the general results to nonautonomous Beverton–Holt equations [5, 13, 23, 24, 25, 26, 38, 41]. In addition, we will survey the recent literature on periodic difference equations [10, 9, 29, 30, 32, 33, 35, 36, 40, 49, 67, 76, 77].

The final section includes some further developments and open questions.

2 Discrete Dynamical Systems

Let X be a topological space, T a topological group, and let $\pi : X \times T \rightarrow X$. Then the triple (X, T, π) or just π is called a dynamical system if π is continuous and

(i)

$$\pi(x, 0) = x \quad \text{for all } x \in X, \text{ where } 0 \text{ is the identity of } T, \quad (2.1)$$

(ii)

$$\pi(\pi(x, s), t) = \pi(x, s + t). \quad (2.2)$$

If T is a topological semigroup, then π is called a semi-dynamical system.

There are two important examples that are of general interest.

- a. $T = \mathbb{R}(\mathbb{R}^+)$, the space of real numbers (nonnegative real numbers). Consider the autonomous differential equation

$$x' = f(x), \quad x(0) = x_0. \quad (2.3)$$

We assume that Eq. (2.3) has a unique solution $x(t, x_0)$. Define $\pi(x, t) \doteq x(t, x_0)$. Then $\pi : X \times \mathbb{R} \rightarrow X$ defines a continuous dynamical system. If, however, the solution $x(t, x_0)$ is defined only on the set of nonnegative real numbers, then $\pi : X \times \mathbb{R}^+ \rightarrow X$ defines a continuous semi-dynamical system.

- b. $T = \mathbb{Z}(\mathbb{Z}^+)$, the space of integers (nonnegative integers). Consider the difference equation

$$x(n+1) = f(x(n)), \quad x(0) = x_0, \quad (2.4)$$

where $f : X \rightarrow X$ is at least continuous.

Let $\pi(x_0, n) = x(n)$. Then π defines either a discrete dynamical system $\pi : X \times \mathbb{Z} \rightarrow X$ or a discrete semi-dynamical system $\pi : X \times \mathbb{Z}^+ \rightarrow X$. Equation (2.4) may be generated by the map $f : X \rightarrow X$, by putting $x(n) = f^n(x_0)$, and thus $\pi(x_0, n) = f^n(x_0)$, where $f^n = f \circ f \circ \cdots \circ f$ is the n th composition of f .

Notice that a fixed point x^* of the map f is called an equilibrium point of Eq. (2.4). Similarly, the (positive) orbit of $x_0 \in X$, under the map f , $O^+(x_0) = \{x_0, f(x_0), f^2(x_0), \dots\}$ is the same as the solution curve $\{x(n) : n \in \mathbb{Z}^+\}$ of Eq. (2.4).

One of the most popular, and still fascinating, examples is the logistic map/difference equation

$$x(n+1) = \mu x(n)(1 - x(n))$$

$$\text{or the map } f(x) = \mu x(1 - x)$$

defined on the closed interval $[0, 1]$. Due to the centrality of this example in the modern theory of difference equations, I have presented it, with great detail, in my two books [20, 21]. The reader may find other interesting expositions in many of the other books mentioned above.

3 Skew-Product Dynamical Systems

To motivate the notion of skew-product dynamical systems, let us look at the following simple but illustrative example.

Example 3.1. *Consider the nonautonomous difference equation*

$$x(n+1) = (-1)^n \left(1 + \frac{1}{n+1} \right) x(n), \quad x(0) = x_0, \quad (3.1)$$

where $n \in \mathbb{Z}^+$.

The solution of Eq. (3.1) is given by [20]

$$x(n) = (-1)^{\frac{n(n-1)}{2}}(n+1)x_0. \quad (3.2)$$

If we let $\pi(x_0, n) = x(n) = (-1)^{\frac{n(n-1)}{2}}(n+1)x_0$, then

$$\pi(\pi(x_0, s), t) = (-1)^{\frac{t(t-1)}{2}}(-1)^{\frac{s(s-1)}{2}}(t+1)(s+1)x_0.$$

However

$$\begin{aligned} \pi(x_0, s+t) &= (-1)^{\frac{(s+t)(s+t-1)}{2}}(s+t+1)x_0 \\ &\neq \pi(\pi(x_0, s), t). \end{aligned}$$

Hence π is not a semi-dynamical system as the semigroup property (2.2) fails to hold.

Let us write $F(n, x) = (-1)^n \left(1 + \frac{1}{n+1}\right) x$. For each $t \in \mathbb{Z}^+$, let $F_t(n, x) = F(n+t, x) = (-1)^{n+t} \left(1 + \frac{1}{n+t+1}\right) x$. In the space of continuous functions $C(\mathbb{Z}^+ \times X, X)$, $X = \mathbb{R}$, the hull of F is defined as the closure of the translates of $F(n, \cdot)$: $\mathcal{H}(F) = cl\{F_t(n, \cdot) : t \in \mathbb{Z}^+\}$. Now $\mathcal{H}(F) = \{F_t(n, \cdot) : t \in \mathbb{Z}^+\} \cup \{G(n, \cdot)\}$, where $G(n, x) = (-1)^n x$ is the omega limit set of $\{F_t(n, \cdot) : t \in \mathbb{Z}^+\}$. Let us now switch to the friendly notation: for each $n \in \mathbb{Z}^+$, put $f_n(x) = F(n, x)$, $g_n(x) = G(n, x)$. Then g_n is periodic: $g_0 = g_{2n}$, $g_1 = g_{2n+1}$, for all $n \in \mathbb{Z}^+$, $g_0(x) = x$, $g_1(x) = -x$.

Define $\pi : \mathcal{H}(F) \times \mathbb{Z}^+ \rightarrow \mathcal{H}(F)$ as

$$\pi((x, f_i), n) = (f_{i+n-1} \circ \cdots \circ f_{i+1} \circ f_i(x), f_{i+n}).$$

Then

$$\begin{aligned} \pi(\pi((x, f_i), s), t) &= \pi((f_{i+s-1} \circ \cdots \circ f_{i+1} \circ f_i(x), f_{i+s}), t) \\ &= (f_{i+s+t-1} \circ \cdots \circ f_{i+s} \circ f_{i+s-1} \circ \cdots \circ f_{i+1} \circ f_i(x), f_{i+s+t}) \\ &= \pi((x, f_i), s+t). \end{aligned}$$

And similarly for g_i .

The above scheme illustrates the essence of the notion of skew-product. To explain the general situation, consider the nonautonomous difference equation

$$x(n+1) = F(n, x(n)) \quad (3.3)$$

where $F(n, \cdot) \in C(\mathbb{Z}^+ \times X, X) = C$. The space C is equipped with the topology of uniform convergence on compact subsets of $\mathbb{Z}^+ \times X$. Let $F_t(n, \cdot) = F(t + n, \cdot)$ and $\mathcal{A} = \{F_t(n, \cdot) : t \in \mathbb{Z}^+\}$ be the set of translates of F in C . Then $G(n, \cdot) \in \omega(\mathcal{A})$, the omega limit set of \mathcal{A} , if for each $n \in \mathbb{Z}^+$,

$$|F_t(n, x) - G(n, x)| \rightarrow 0$$

uniformly for x in compact subsets of X , as $t \rightarrow \infty$ along some subsequence $\{t_{g_n}\}$. The closure of \mathcal{A} in C is called the hull of $F(n, \cdot)$ and is denoted by $Y = \mathcal{H}(\mathcal{A})$. Let $\sigma : Y \times \mathbb{Z} \rightarrow Y$ be defined as the shift map : $\sigma(G(n, \cdot), 1) = G(n + 1, \cdot)$, or more generally, $\sigma(G(n, \cdot), m) = G(n + m, \cdot)$. Then σ is a discrete dynamical system on Y .

The skew product dynamical system is now defined as $\pi : X \times Y \times \mathbb{Z}^+ \rightarrow X \times Y$, such that

$$\pi((x, G(s, \cdot)), t) = (\Phi_t(s)(x), G_t(s, \cdot)),$$

where $\Phi_t(s) = g_{s+t-1} \circ \dots \circ g_{s+1} \circ g_s$, $G_t(s, \cdot) = g_{s+t}$. If p is the projection map, $p(a, b) = b$, then the following commuting diagram illustrates the notion of skew product discrete semi-dynamical system.

$$\begin{array}{ccc} X \times Y \times \mathbb{Z}^+ & \xrightarrow{\pi} & X \times Y \\ \downarrow p \times id & & \downarrow p \\ Y \times \mathbb{Z}^+ & \xrightarrow{\sigma} & Y \end{array}$$

For each $G(n, \cdot) \in Y$, we define the fiber \mathcal{F}_g over G as $\mathcal{F}_g = p^{-1}(g)$. If $g = f_i$, then we write \mathcal{F}_g as \mathcal{F}_i .

4 Periodic Difference Equations

In this section we turn our attention to the case when the map $F(n, x)$ is periodic of minimal period say $p > 1$, that is $F(n + p, \cdot) = F(n, \cdot)$ for all $n \in \mathbb{Z}^+$. We found it rather informative and useful if we switch to writing $F(n, x)$ as $f_n(x)$ and hence Eq. (3.2) may be written in the convenient form

$$x(n + 1) = f_n(x(n)), \quad n \in \mathbb{Z}^+ \tag{4.1}$$

where it is assumed that $f_{n+p} = f_n$ for all $n \in \mathbb{Z}^+$. A point x^* is a fixed point of Eq. (4.1) if $f_n(x^*) = x^*$ for all $n \in \mathbb{Z}^+$. For periodic points or cycles we

have to be extra careful in defining them. For there are two main types of periodic points, one in the space X and another in the product space $X \times Y$. A point (x_0, g) is a periodic point with period k in the skew product system π if $\pi((x_0, g), k + n) = \pi((x_0, g), n)$ for all $n \in \mathbb{Z}^+$. And by the semigroup property this is equivalent to $\pi((x_0, g), k) = (x_0, g)$.

We now give the definition of periodicity in the space X that is consistent with that in the product space.

Definition 4.1. *Let $c_r = \{\bar{x}_0, \bar{x}_1, \dots, \bar{x}_{r-1}\}$ be a set of points in X , with $r \geq 1$. Then c_r is said to be a geometric r -cycle if for $i = 0, 1, \dots, r - 1$*

$$f_{(i+nr) \bmod p}(\bar{x}_i) = \bar{x}_{(i+1) \bmod r}, \quad n \in \mathbb{Z}^+. \quad (4.2)$$

The following example illustrates the “layout” in $X \times Y$ of the orbit (\bar{x}_0, f_0) under the action of the skew product semi-dynamical system π .

Example 4.2. *Define*

$$\begin{aligned} f_0(x) &= 1 - x, \\ f_2(x) &= f_0(x) \text{ for } x \leq 1 \text{ and } = 0 \text{ otherwise,} \\ f_4(x) &= f_0(x) \text{ for } x \leq 1 \text{ and } = 0 \text{ otherwise,} \\ f_1(x) &= x, \\ f_3(x) &= f_1(x) \text{ for } x \leq 1 \text{ and } = 0 \text{ otherwise,} \\ f_5(x) &= f_1(x) \text{ for } x \leq 1 \text{ and } = 0 \text{ otherwise,} \\ f_n(x) &= f_{n \bmod 6}, \quad n \geq 6 \end{aligned}$$

Then the system σ on the base $Y = \{f_0, f_1, \dots, f_5\}$ is defined by $\sigma(f_i, n) = f_{(i+n) \bmod 6}$. Notice that $c_4 = \{0, 1, 1, 0\}$ is a geometric 4-cycle, which gives rise to a 12-cycle in the skew-product system: $\{(0, f_0), (1, f_1), (1, f_2), (0, f_3), (0, f_4), (1, f_5), (1, f_0), (0, f_1), (0, f_2), (1, f_3), (1, f_4), (0, f_5)\}$. Observe that the period of the cycle in the skew-product system is the least common multiple $[4, 6] = 12$ of $r = 4$ and $p = 6$. Furthermore, the number of distinct points in each fiber is $[4, 6]/6 = 2$ each of which is of period 12 in the skew product π (Figure 1).

The following crucial lemma formalizes the above discussion.

Lemma 4.3. *[23] Let $c_r = \{\bar{x}_0, \bar{x}_1, \dots, \bar{x}_{r-1}\}$ be a geometric r -cycle. Then the π -orbit intersects each fiber \mathcal{F}_i , $0 \leq i \leq p - 1$, in exactly $l = s/p$, where $s = [r, p]$ is the least common multiple of r and p , and each of these points is periodic under the skew-product dynamical system with period s .*

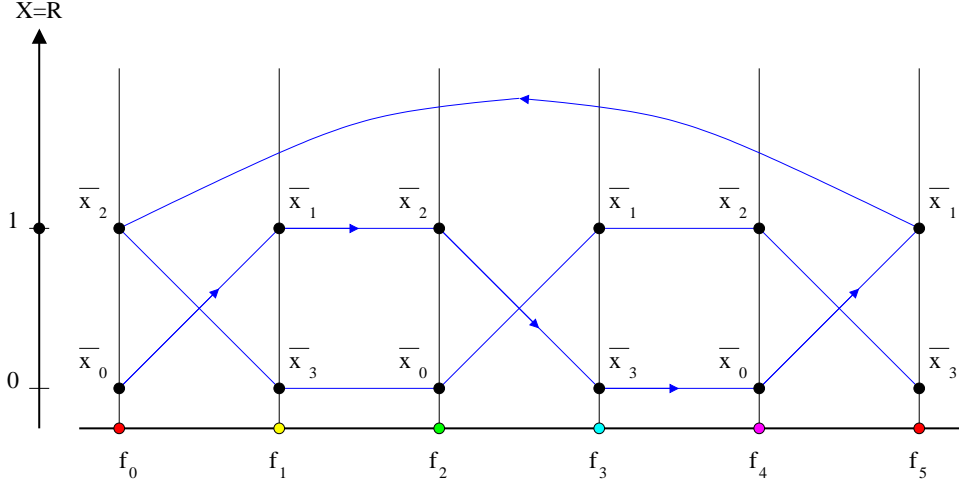


Figure 1: The period of the cycle in the skew-product is given by $s = [r, p]$, the least common multiple of r and p , and the number of distinct points from c_r in each fiber is $l = [r, p]/p$.

As a consequence of the preceding Lemma, we now state the second fundamental result in this survey, the anticipated extension of Elaydi-Yakubu Theorem to periodic difference equations.

Theorem 4.4 (Elaydi-Sacker Theorem). [23] *Consider the periodic difference Eq. (4.1) with minimal period p such that each $f_i : X \rightarrow X$ is a continuous function on a connected metric space X . If c_r is a geometric r -cycle which is globally asymptotically stable, then r divides p .*

The next example shows how to utilize Theorem 4.4 to prove that a given cycle is not globally asymptotically stable.

Example 4.5. *Consider the two-dimensional system*

$$\begin{aligned} x(n+1) &= \frac{1}{2}x(n) - \frac{\sqrt{3}}{2}y(n) + (x^2(n) + y^2(n) - 1) \cos\left(\frac{2\pi n}{9}\right) \\ y(n+1) &= \frac{\sqrt{3}}{2}x(n) + \frac{1}{2}y(n) + (x^2(n) + y^2(n) - 1) \sin\left(\frac{2\pi n}{9}\right). \end{aligned}$$

This is a periodic system of period $p = 9$. The solution $(x(n), y(n)) = (\cos \frac{n\pi}{3}, \sin \frac{n\pi}{3})$ is of period 6.

Since 6 does not divide 9, it follows by Theorem 4.4 that this periodic solution is not globally asymptotically stable.

To this end, we have investigated the permissible globally asymptotically stable r -cycle. The next question that we will address is what are the permissible cycles, with or without any stability properties, in a given p -periodic difference equation?

The next result provides the definitive answer to the above question.

Theorem 4.6. [25] *Consider the p -periodic difference Eq. (4.1). Let $c_r = \{\bar{x}_0, \bar{x}_1, \dots, \bar{x}_{r-1}\}$ be a set of r points in a metric space X , $d = (r, p)$, the greatest common divisor of r and p , and $m = p/d$. Then the following statements are equivalent.*

- (a) c_r is a geometric r -cycle of equation (4.1),
- (b) $f_{(i+nd) \bmod p}(\bar{x}_i) = \bar{x}_{i+1}$, $0 \leq i \leq r-1$ and $n = 0, 1, \dots, (m-1)$,
- (c) the graphs of the functions $f_i, f_{i+d}, f_{i+2d}, \dots, f_{i+(m-1)d}$ intersect at the points $(\bar{x}_i, \bar{x}_{(i+1) \bmod r})$, $(\bar{x}_{(i+d) \bmod r}, \bar{x}_{(i+d+1) \bmod r})$, \dots , $(\bar{x}_{(i+(m-1)d) \bmod r}, \bar{x}_{(i+(m-1)d+1) \bmod r})$, for $0 \leq i \leq r-1$.

We remark here that if the graphs of the maps f_i , $0 \leq i \leq p-1$, are disjoint (with the exception of possibly possessing a common fixed point), then by the preceding theorem, the only possible geometric cycles are those of period p or multiples of p .

In the next section we will discuss the implications of this fundamental observation for nonautonomous Beverton-Holt equations.

5 Nonautonomous Beverton–Holt Equations

Cushing and Henson [13] considered a periodically forced Beverton–Holt equation of the form

$$x(n+1) = \frac{\mu K_n x(n)}{K_n + (\mu - 1)x(n)} \quad (5.1)$$

where μ is the intrinsic growth rate and $K_n = K_{n+p}$, $n \in \mathbb{Z}^+$, is the periodic carrying capacity of a population.

Equation (5.1) is a perturbation of the original autonomous Beverton–Holt model

$$x(n+1) = \frac{\mu K x(n)}{K + (\mu - 1)x(n)}. \quad (5.2)$$

It is well known (see [20] or [15]) that if $K > 0$ and $0 < \mu < 1$, then the zero solution is globally asymptotically stable on $[0, \infty)$. Moreover, if $K > 0$ and $\mu > 1$, the positive equilibrium $x^* = K$ is globally asymptotically stable on $(0, \infty)$. The case $\mu = 1$ is trivial since then every point is a fixed point.

The following two conjectures were proposed by Cushing and Henson [13]. From now on we assume that $p \geq 2$, $K_n > 0$, $\mu > 1$.

Conjecture 5.1. *Equation (5.1) has a positive p -periodic solution which is globally asymptotically stable on $(0, \infty)$.*

Conjecture 5.2. *If $c_p = \{\bar{x}_0, \bar{x}_1, \dots, \bar{x}_{p-1}\}$ is a p -periodic solution of Eq. (5.1), then*

$$av(\bar{x}_n) < av(K_n)$$

where

$$av(\bar{x}_n) = \frac{1}{p} \sum_{i=0}^{p-1} \bar{x}_i,$$

and similarly for $av(K_n)$.

These two conjectures were proved by Elaydi and Sacker in [23, 24]. In fact, Conjecture 5.1 was proved for a more general class of maps, called class \mathcal{K} . Independently, Kocic [38] solved the above two conjectures where he also considered the more general case when K_n is bounded, that is $0 < \alpha < K_n < \beta < \infty$. In [41, 42], Kon proved the second conjecture for a class of systems that include the periodic Beverton–Holt equation. He considered the following class of difference equations of the form

$$x(n+1) = g\left(\frac{x(n)}{K_n}\right) x(n), \quad x(0) = x_0, \quad n \in \mathbb{Z}^+, \quad (5.3)$$

where $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is continuous and satisfies the following conditions:

- (i) $g(1) = 1$,
- (ii) $g(x) > 1$, for all $x \in (0, 1)$, and $g(x) < 1$ for all $x \in (1, \infty)$. It is also assumed that $K_n > 0$, $K_{n+p} = K_n$ for all $n \in \mathbb{Z}^+$.

Notice that Eq. (5.3) includes Ricker's equation

$$x(n+1) = x(n) \exp \left[r \left(1 - \frac{x(n)}{K_n} \right) \right] \quad (5.4)$$

where $f_n(x) = x \exp \left[r \left(1 - \frac{x(n)}{K_n} \right) \right]$ is not monotonic (in contrast to the Beverton–Holt maps which are monotonic).

Let $K_M = \max\{K_i : 0 \leq i \leq p-1\}$, $K_m = \min\{K_i : 0 \leq i \leq p-1\}$. Suppose that the following inequality holds

$$\frac{K_M}{K_m} \exp(r-1) \leq 2. \quad (5.5)$$

Zhou and Zou [77] showed that under condition (5.5), Eq. (5.4) has a globally asymptotically stable periodic solution of period p . In fact, the authors proved only the existence of a p -periodic solution which is globally attracting. However, by a Theorem of Sedaghat [66], a globally attracting fixed point in \mathbb{R} is necessarily stable, and hence the above statement.

Kon [41] used this result to show that, under condition (5.5), Eq. (5.4) has a p -periodic solution for which Conjecture 5.2 holds. In 1990, Clark and Gross [9] discussed a discrete analogue of the nonautonomous Pearl-Verhulst logistic differential equation

$$N'(t) = r(t)N(t)[1 - N(t)/K(t)] \quad (5.6)$$

where $r(t)$ and $K(t)$ are positive, bounded periodic functions of period T . Their discrete model is of the form

$$x(n+1) = \frac{a_n x(n)}{1 + b_n x(n)} \quad (5.7)$$

where a_n and b_n are positive, bounded and periodic of integer period p . The authors then showed that if

$$\sum_{i=0}^{p-1} a_i > 1, \quad (5.8)$$

then Eq. (5.7) has a globally asymptotically stable p -periodic solution. Observe that the periodic Beverton–Holt equation may be written in the form

$$x(n+1) = \frac{\mu x(n)}{1 + \frac{(\mu-1)}{K_n} x(n)} \quad (5.9)$$

which is of the form (5.7) with $a_n = \mu$ and $b_n = (\mu - 1)/K_n$. Thus if $\mu > 1$, condition (5.8) is automatically satisfied, and the result of Clark and Gross proves Conjecture 5.1 as well. Surprisingly, most of the authors who recently tackled Cushing–Henson Conjectures, including Cushing and Henson and the authors of this paper, were not aware of this early work!

To this end, we have shown that Eq. (5.1), with $\mu > 1$, $K_n > 0$, has a globally asymptotically stable p -periodic solution. By the remarks after Theorem 4.6, we can now confirm that this periodic solution has a minimal period p . For otherwise let $r|p$, $r < p$, be the minimal period of this periodic solution. Then by Theorem 4.6

$$K_i = K_{(i+r) \bmod p} = \cdots = K_{(i+(m-1)r) \bmod p}, \quad i \in \mathbb{Z}^+$$

where $m = p/r$. This implies that Eq. (5.1) is of period r , a contradiction.

The situation, however, is drastically different if we assume also that $\mu = \mu_n$ is periodic with a common minimal period p .

Consider the equation

$$x(n+1) = \frac{\mu_n K_n x(n)}{K_n + (\mu_n - 1)x(n)}, \quad \mu_n > 1, \quad K_n > 0, \quad (5.10)$$

where both intrinsic growth rate μ_n and the carrying capacity K_n are of minimal common period $p \geq 2$.

It was shown in [25] that Eq. (5.10) has a globally asymptotically stable p -cycle $\{\bar{x}_0, \bar{x}_1, \dots, \bar{x}_{p-1}\}$, where

$$\begin{aligned} \bar{x}_0 &= \frac{L_{p-1}(Q_{p-1} - 1)}{E_{p-1}}, \\ L_{p-1} &= K_{p-1} \cdots K_0, \\ Q_{p-1} &= \mu_{p-1} \cdots \mu_0, \\ E_{p-1} &= K_{p-1} E_{p-2} + (\mu_{p-1} - 1) \mu_{p-2} \mu_{p-3} \cdots \mu_0 K_{p-2} K_{p-3} \cdots K_0. \end{aligned}$$

Moreover, this p -periodic orbit may not be of minimal period p . This is in contrast to the case of Eq. (4.1) when the intrinsic growth μ_n is constant. In fact, Eq. (5.10) has a p -periodic orbit of minimal period $r < p$ if and only if

$$\frac{L_{p-1}(Q_{p-1} - 1)}{E_{p-1}} = \frac{L_{r-1}(Q_{r-1} - 1)}{E_{r-1}}.$$

The following example from [11] produces a periodic cycle whose minimal period is less than the period of the given difference equation.

Example 5.3. Let $\mu_0 = 3$, $\mu_1 = 4$, $\mu_2 = 2$, $\mu_3 = 5$; $K_0 = 1$, $K_1 = \frac{6}{17}$, $K_2 = 2$, $K_3 = \frac{4}{17}$, where μ_n and K_n are periodic of minimal period $p = 4$. The equation

$$x(n+1) = \frac{\mu_n K_n}{K_n + (\mu_n - 1)x_n} x_n = f_n(x_n)$$

has the geometric 2-cycle $c_2 = \left\{ \frac{2}{5}, \frac{2}{3} \right\}$. Notice that the graphs of the maps f_0 and f_2 intersect at the point $\left(\frac{2}{5}, \frac{2}{3} \right)$, while the graphs of the maps f_1 and f_3 intersect at the points $\left(\frac{2}{3}, \frac{2}{5} \right)$.

6 Further Developments

In [25], the authors investigated the extension of the second Cushing–Henson conjecture to Eq. (5.10). They obtained the following inequality:

$$av(\bar{x}_n) < \frac{\mu^*}{\mu_*} \cdot \frac{(\mu^* - 1)}{(\mu_* - 1)} av(K_n), \quad (6.1)$$

where $\mu^* = \max\{\mu_n\}$, $\mu_* = \min\{\mu_n\}$. And for the case $p = 2$, we have the following sharp result.

$$av(x_n) = av(K_n) + \sigma \left(\frac{K_0 - K_1}{2} \right) - \gamma \left(\frac{\mu_0 + \mu_1}{2} \right) (K_0 - K_1)^2, \quad (6.2)$$

where

$$\sigma = \frac{\mu_1 - \mu_0}{\mu_0 \mu_1 - 1}, \quad \gamma = \frac{(\mu_0 - 1)(\mu_1 - 1)}{(\mu_0 \mu_1 - 1)^2}.$$

It would be interesting to extend Eq. (6.2) to the case $p > 2$.

Recently, several authors [17, 18] considered the following autonomous difference equation with delay.

$$x(n) = f(x(n - k)), \quad k > 1. \quad (6.3)$$

If we let

$$y_j(n) = x(nk - j), \quad j = 1, 2, \dots, k,$$

then

$$y_j(n) = f(y_j(n - 1)), \quad j = 1, 2, \dots, k$$

is a set of k uncoupled first order difference equations. Hence one gains information about Eq. (6.3) by considering the associated first order difference equation

$$x(n) = f(x(n - 1)). \quad (6.4)$$

In particular, [18] considered the set

$$S_k(p') = \{lp' : l|k \text{ and } (k/l, p') = 1\}.$$

They showed that if p is a period of Eq. (6.4) and $p' \supseteq p$ in Sharkovsky ordering of positive integers [21], i.e., p' is either equal to p or to the left of p in the Sharkovsky ordering, then each number in the set $S_k(p')$ is a period of Eq. (6.3).

This is a nice extension of the famous Sharkovsky Theorem [21] from continuous maps on the real line to higher-order difference equations.

This leads to several questions:

1. What is the natural extension of Sharkovsky's Theorem to first order periodic difference equations?
2. What is the natural extension of Sharkovsky's Theorem to periodic difference equations with delay such as

$$x(n) = f_n(x(n - k)) \quad (6.5)$$

3. What is the natural extension of Elaydi–Sacker Theorem to periodic equations of the form (6.5)?

4. In particular, what are the extensions of the Cushing–Henson Conjectures to the equations

$$x(n) = \frac{\mu K_n x_{n-k}}{K_n + (\mu - 1)x_{n-k}} \quad \text{and} \quad x(n) = \frac{\mu K_n x_{n-k}}{K_n + (\mu_n - 1)x_{n-k}}?$$

Question 1 has been successfully addressed in [4].

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