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RESEARCH ARTICLE

Semigroups of Maps and Periodic Difference Equations

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A collection $\mathcal{M}$ of monotonic maps from the positive reals to the positive reals is defined. Each map is linearly bounded, has non-negative Schwarzian and is either concave increasing or convex decreasing. It is shown that $\mathcal{M}$ is a semigroup under composition that contains the sub-semigroup of fractional linear maps and each function in $\mathcal{M}$ that is uni-linearly bounded has a globally attracting exponentially asymptotically stable fixed point. Thus we obtain a condition under which a periodic difference equation (mapping system) will have a periodic solution having the same properties. Certain restricted algebraic operations are valid in $\mathcal{M}$ and the structure of $\mathcal{M}$ is explored together with conjectures regarding the interlacing of roots of a rational function in $\mathcal{M}$.

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1. Introduction

It is impossible to overestimate the importance of the *semigroup* in the study of periodic difference equations, or as we prefer to call them,"periodic mapping systems". It is understood that the semigroup operation is composition of maps. This notion was explored in [4] where it was shown that the class of continuous concave maps $f : \mathbb{R}^+ \to \mathbb{R}^+$ (and thus increasing) whose graph crosses the diagonal form a semigroup and each map in the semigroup has a globally attracting exponentially asymptotically stable fixed point. No differentiability was assumed. In applications to rational difference equations, see Section 5, the narrow class of concave maps proves to be too burdensome and one is forced to admit convex decreasing maps $\mathcal{X}$ as well. These maps do not form a semigroup but taken together with the concave maps $\mathcal{V}$ plus a condition on the Schwarzian, $S(f) \geq 0$ one does obtain a semigroup $\mathcal{M} = \mathcal{V} \cup \mathcal{X}$, Section 3. The semigroup $\mathcal{M}$ is also invariant under the formation of the reciprocal provided it is defined and each of $\mathcal{V}$ and $\mathcal{X}$ is invariant under addition and positive scalar multiplication.

In Section 2 we study the special class of *fractional linear* maps $\mathcal{F} \subset \mathcal{M}$ where the Schwarzian need not be considered at all. In fact, in the general case the importance of the Schwarzian was not realized until certain counterexamples to proposed theorems were discovered, see (18). In the study of maps of an interval one often sees the Schwarzian used as an *arbiter* to settle the stability question at a non-hyperbolic fixed point [3] and in the case of Singer's Theorem [8], [1], to predict the maximum number of stable fixed points for a map with multiple critical points. In both cases it is the negative Schwarzian that implies stability. It is thus counterintuitive, at least to the author, that the non-negative Schwarzian plays such an important role in the present work.

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In Subsection 3.2 we explore the structure of $\mathcal{M}$ and give several examples and a conjecture that a rational function is in $\mathcal{M}$ if, and only if the roots of the numerator and denominator are interlaced. Examples suggest that all the roots must be real.

2. Fractional Linear Maps

Throughout this work, $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{R}_0^+ = (0, \infty)$.

We begin with a study of properties of a certain subclass $\mathcal{F}$ of fractional linear maps

$$\mathcal{F} = \{ f : \mathbb{R}^+ \to \mathbb{R}^+ \mid f(x) = \frac{ax+b}{x+d}, \qquad a, b \geq 0, \quad a+b > 0, \quad d > 0 \}, \tag{1}$$

and the subset

$$\mathcal{F}_0 = \{ f \in \mathcal{F} \mid b > 0 \}. \tag{2}$$

Lemma 2.1:

(1) $\mathcal{F}$ and $\mathcal{F}_0$ are semigroups under composition,
(2) if $f \in \mathcal{F}$ and $g \in \mathcal{F}_0$ then $f \circ g$ and $g \circ f \in \mathcal{F}_0$,
(3) $f \in \mathcal{F} \implies f$ is bounded,
(4) $f \in \mathcal{F} \implies f$ is
a) strictly concave and strictly increasing $\iff$ $ad - b > 0$ or
b) strictly convex and strictly decreasing $\iff$ $ad - b < 0$ or
c) constant $\neq 0$ when $ad - b = 0$.

Proof: Let

$$f(x) = \frac{ax+b}{x+d}, \qquad \text{and} \qquad g(x) = \frac{\alpha x + \beta}{x + \delta}.$$

Then (1) and (2) follow from

$$f \circ g(x) = \frac{(a\alpha + b)x + (a\beta + b\delta)}{(\alpha + d)x + (\beta + d\delta)} = \frac{Ax + B}{x + D} \tag{3}$$

upon division of numerator and denominator by $(\alpha + d)$ and the inequalities in (1) and (2). Item (3) follows from the fact that the coefficient of x in the denominator is positive. Item (4) uses the fact that f is defined on *all* of $\mathbb{R}^+ \to \mathbb{R}^+$ and is a trivial calculus exercise. Note that $a + b > 0$ excludes the identically zero function. ■

Corollary 2.2: *If at least one of the maps $\{f_0, f_1, \ldots, f_n\} \subset \mathcal{F}$ satisfies $f_j(0) > 0$, then the composite $f_n \circ f_{n-1} \circ \cdots \circ f_0(0) > 0$. This follows from (2) of Lemma 2.1.*

The following Lemma is an elementary exercise.

Lemma 2.3: *Referring to (3), one has*

$$AD - B = \frac{(ad - b)(\alpha\delta - \beta)}{(\alpha + d)^2} \ .$$

Definition 2.4: Define, for $f \in \mathcal{F}$

$$x_- = \inf\{x \in \mathbb{R}^+ : |f'(x)| < 1\}.$$

Lemma 2.5: *For $f \in \mathcal{F}$*

(1) $|f'(t)| < 1 \ \forall t > x_-$ and f has a unique fixed point x_f,
(2) For the case $ad - b \leq 0$, one has $0 \leq x_- < x_f$,
(3) For the case $ad - b > 0$, if $b > 0$ or $a/d > 1$, one has $0 \leq x_- < x_f$.

Proof: (1): In either case $ad - b \leq 0$ or > 0 it is easily shown that $|f'(t)|$ is either identically zero or a strictly decreasing function.

(2): If $ad - b = 0$ then $f \equiv k > 0$ and the conclusion follows with $0 = x_- < x_f = k$. For the case $ad - b < 0$, if $x_- = 0$ we are done since we must have $b > 0$ and thus $f(0) > 0$. This, together with Lemma 2.1, (4b) implies $x_f > 0$. We are thus left with the case $x_- > 0$. Solving $f'(x_-) = -1$ for x_- and $f(x_f) = x_f$ for x_f we wish to satisfy

$$x_- = \sqrt{b - ad} - d < \frac{a - d + \sqrt{(a-d)^2 + 4b}}{2} = x_f \ .$$

By a reversible set of steps one arrives at

$$-4ad < 2a^2 + 2d^2 + 2(a+d)\sqrt{(a-d)^2 + 4b},$$

which is trivially true.

(3): Solving $f'(x_-) = 1$ for x_- one has

$$x_- = -d + \sqrt{ad - b} \ .$$

The conditions $b > 0$ or $a/d > 1$ insures that the graph of f lies above the diagonal in some neighborhood $(0, \epsilon)$. Boundedness assures that the graph crosses the diagonal and finally concavity assures that $0 < f'(x_f) < 1$ and thus $0 \leq x_- < x_f$. ■

Remark 1: The case (3) was treated in [4] where it was shown that arbitrary continuous concave maps $f : \mathbb{R}^+ \to \mathbb{R}^+$ that *cross* the diagonal $y = x$ in the x, y plane form a semigroup under composition and have a unique globally attracting exponentially asymptotically stable fixed point.

Remark 2: The requirement $d > 0$ in the definition of $\mathcal{F}$ is essential to the argument in the convex case. For otherwise the map $f(t) = b/t$ has $x_- = x_f = \sqrt{b}$.

We next consider the discrete dynamical system

$$x_{n+1} = f(x_n), \qquad x_0 \in \mathbb{R}^+ \ , \qquad f \in \mathcal{F} \ . \tag{4}$$

The next lemma applies to arbitrary continuous f and its proof is a simple exercise.

Lemma 2.6: *Let X be a metric space and assume $f : X \to X$ is continuous. Assume*

(1) f has a unique fixed point x_f and
(2) $f \circ f$has a globally asymptotically stable fixed point y.

Then $y = x_f$ is a globally asymptotically stable fixed point of f.

Theorem 2.7: **(Stability and Invariant Intervals)**

(1) The fixed point x_f of (4) guaranteed by Lemma (2.5) is globally attracting exponentially asymptotically stable with respect to $\mathbb{R}_0^+$,
(2) (Concave case, $ad - b > 0$ with $b > 0$ or $a/d > 1$): Any interval $[t_1, t_2]$ containing x_f is invariant under f.
(3) (Convex case, $ad - b \leq 0$): For any $\tau \in (0, x_f)$ the interval $I = [\tau, f(\tau)]$, contains x_f on its interior and is invariant under f.

Proof: We assume $ad - b \neq 0$ since otherwise all the statements are trivially true. (1) The local exponential asymptotic stability is guaranteed by Lemma (2.5), (1). It remains to establish globally attracting.

(2) Under application of f, points in $(0, x_f)$ move monotonically to the right and remain in $(0, x_f)$ while points in (x_f, ∞) move monotonically to the left and remain in (x_f, ∞). Uniqueness of x_f implies globally attracting.

(3) From Lemma 2.3, $f \circ f \in \mathcal{F}$ is a concave map and by (2) has a fixed point s_0 that is globally attracting exponentially asymptotically stable with respect to $\mathbb{R}_0^+$. Since f has a unique fixed point, Lemma 2.6 gives $x_f = s_0$. To obtain the invariance note that f maps $[\tau, x_f)$ one-to-one onto $(x_f, f(\tau)]$. Thus, for points $t \in (x_f, f(\tau)]$, $f(t) = f \circ f(s)$ for some $s \in [\tau, x_f)$ and thus by (2), $f(t) \in [\tau, x_f)$. ■

As a corollary to the above, one has

Theorem 2.8 : *Let $\{f_0, f_1, \ldots, f_{p-1}\}$ be a p-periodic system of Fractional Linear Maps*

$$f_j(x) = \frac{a_j x + b_j}{x + d_j}, \qquad a_j, b_j \geq 0, \quad a_j + b_j > 0, \quad d_j > 0,$$

and assume at least one of the $b_j > 0$. Then there exists a p-periodic orbit,

$$\{\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_{p-1}\}, \qquad \hat{x}_{n+1 \mod p} = f_n(\hat{x}_n)$$

which is exponentially asymptotically stable and globally attracting on $(0, \infty)$.

3. General Convex or Concave Functions

From this point on we assume all maps $f \in C^3(\mathbb{R}^+, \mathbb{R}^+)$.

In Section 2 we treated a very narrow class $\mathcal{F}$ of Fractional Linear mappings and saw that members of $\mathcal{F}$ were concave or convex in addition to bounded. From the very form of the mappings it was easily shown that $\mathcal{F}$ is a semigroup under composition and this, in turn, led to a smooth treatment of p-periodic difference equations

$$x_{n+1} = f_n(x_n), \qquad f_{n+p} = f_n, \qquad f_n \in \mathcal{F}.$$

In this section we will extend the results of Section 2 to the much wider class of maps. By f *increasing*($\nearrow$) we mean $f'(x) > 0\ \forall\ x \in \mathbb{R}^+$ and by *concave* we mean $f''(x) \leq 0\ \forall\ x \in \mathbb{R}^+$ and similarly for *decreasing*($\searrow$) and *convex*. By $\mathbb{R}^*$ we mean the extended reals $\{-\infty\} \cup \mathbb{R} \cup \{\infty\}$.

Central to our investigation is the Schwarzian, [5, Ch. 10] of a C^3 function f,

$$S(f) \doteq \frac{f'''}{f'} - \frac{3}{2}(\frac{f''}{f'})^2.$$

Since only the sign of S concerns us, we define the *modified Schwarzian*,

$$\hat{S}(f) \doteq (f')^2 \; S(f) = f'f''' - \frac{3}{2}(f'')^2 \tag{5}$$

which is much easier to deal with. From the Chain Rule for S, [2],

$$S(f \circ g)(x) = S(f)(g(x)) \; [g'(x)]^2 + S(g)(x),$$

we obtain the Chain Rule for the modified Schwarzian,

$$\hat{S}(f \circ g)(x) = \hat{S}(f)(g(x)) \; [g'(x)]^4 + \hat{S}(g)(x) \; [f'(g(x))]^2. \tag{6}$$

It is easy to verify that if f is Fractional Linear then $\hat{S}(f) \equiv 0$.

In subsection 3.1 we define a special sub-class of monotonic functions $\mathcal{M}$ which contains the fractional linear maps $\mathcal{F}$ and investigate some of its properties. In subsection 3.2 we give several examples which, together with Lemmas of subsection 3.1, give us methods for examining the structure of $\mathcal{M}$.

3.1. *Definitions and Properties*

Definition 3.1: $f : \mathbb{R}^+ \to \mathbb{R}^+$ is called *linearly bounded* if there exist $a, b \in \mathbb{R}^+$ such that

$$f(x) \leq ax + b \qquad \text{for all } x \in \mathbb{R}^+,$$

and f is called *uni-linearly bounded* if in addition, $a < 1$.

Lemma 3.2: *Let $f \in C^3(\mathbb{R}^+, \mathbb{R}^+)$ and assume that for all $x \in \mathbb{R}^+$,*

(1) $|f'(x)| > 0$,
(2) $f'(x)f'''(x) \geq 0$ and
(3) f is linearly bounded.

Then f is either concave increasing or convex decreasing on all of $\mathbb{R}^+$ and in either case $\lim_{x\to\infty} f''(x) = 0$.

Proof: First consider the case $f'(x) > 0 \; \forall x$. Then from (2), $f'''(x) \geq 0$ and thus $f''(x)$ is increasing. Let $\lim_{x\to\infty} f''(x) = c \in \mathbb{R}^*$.

If $c < 0$ then

$$f'(x) = f'(0) + \int_0^x f''(s)ds < f'(0) + cx \to -\infty$$

as $x \to \infty$, contradicting $f'(x) > 0$.

If $c > 0$ then there exists $x_0 > 0$ such that $x \geq x_0 \implies f''(x) > c/2$ and thus integrating as above from x_0 to x, one obtains $f'(x) > f'(x_0) + c(x - x_0)/2$. Therefore given *any* $M > 0$ there is an x_1 such that $f'(x) > M \; \forall x > x_1$ contradicting (3).

Thus $f''(x)$ is increasing with limit 0 and is therefore $f''(x) \leq 0$ and f is concave and increasing.

Next consider the case $f'(x) < 0 \; \forall x$. Then we must have $f'''(x) \leq 0$. Thus $f''(x)$ is decreasing and therefore $\lim_{x\to\infty} f''(x) = c \in \mathbb{R}^*$. If $c < 0$ then arguing as above we obtain $f(x) \to -\infty$ as $x \to \infty$ contradicting the fact that $f \geq 0 \; \forall x \geq 0$. If

$c > 0$ then arguing as before, $f''(x) > c/2$ for $x > x_0$ and thus $f' \to \infty$ ultimately contradicting (3).

Thus $f''(x)$ is decreasing with limit 0 and is therefore $f''(x) \geq 0$ and f is convex and decreasing. ∎

Remark 1 : Lemma 3.2 says that the first three derivatives have constant signs and they alternate:

$$sign\{f', f'', f'''\} = \{+, -, +\}, \quad \text{or}$$
$$sign\{f', f'', f'''\} = \{-, +, -\}.$$

We next define a class of functions

$$\Sigma = \{f : \mathbb{R}^+ \to \mathbb{R}^+ \mid f \in C^3,\ \hat{S}(f) \geq 0 \text{ and } f \text{ is linearly bounded}\},$$

and the two subsets

$$\mathcal{V} = \Sigma \cap \{f'(x) \equiv 0 \text{ or } f'(x) > 0\ \forall\ x\} \tag{7}$$
$$\mathcal{X} = \Sigma \cap \{f'(x) \equiv 0 \text{ or } f'(x) < 0\ \forall\ x\}.$$

By Lemma 3.2, $\mathcal{V}$ consists of functions that are constant or strictly increasing and "conca$\mathcal{V}$e" while $\mathcal{X}$ consists of functions that are constant or strictly decreasing and "conve$\mathcal{X}$".

Finally we define the class that is the sought after enlargement of the class $\mathcal{F}$ discussed in Section 2,

$$\mathcal{M} = \mathcal{V} \cup \mathcal{X},$$

a sub-class of monotonic maps.

Theorem 3.3 : *$\mathcal{M}$ is a semi-group under the operation of composition. More precisely, if $f, g \in \mathcal{V}$ or $f, g \in \mathcal{X}$ then $f \circ g \in \mathcal{V}$ and if $f \in \mathcal{V}$ and $g \in \mathcal{X}$ then $f \circ g \in \mathcal{X}$ and $g \circ f \in \mathcal{X}$.*

Proof: Let $f,\ g \in \mathcal{M}$. If either of f or g is constant then $f \circ g$ is constant and we are done. From (6) it follows that $\hat{S}(f \circ g) \geq 0$. The linear boundedness is immediate. Thus $f \circ g \in \Sigma$. To determine whether $f \circ g \in \mathcal{V}$ or $\mathcal{X}$ one need only look at the sign of

$$[f \circ g]'(x) = f'(g(x))\ g'(x).$$

∎

The next lemma tells us that $\mathcal{V}$, $\mathcal{X}$ and $\mathcal{M}$ are each closed under certain algebraic operations.

Lemma 3.4: *Assume $\alpha \geq 0$. Then*

$$f, g \in \mathcal{X} \implies f + g \text{ and } \alpha f \in \mathcal{X}$$
$$f, g \in \mathcal{V} \implies f + g \text{ and } \alpha f \in \mathcal{V}$$

and in each case $\hat{S}(f+g) \geq \hat{S}(f) + \hat{S}(g)$.

Further, if $g \in \mathcal{M}$ and $\dfrac{1}{g}$ is defined and linearly bounded then $\dfrac{1}{g} \in \mathcal{M}$.

Proof: Multiplication by α follows since it leaves all the membership properties of classes $\mathcal{V}$ and $\mathcal{X}$ unchanged. Linearly bounded is immediate. Let f and g be in the same class. Then from Remark **1**,

$$f''g'' \geq 0 \qquad \text{and} \tag{8}$$

$$f', f''', g', g''' \quad \text{are all } \geq 0 \text{ or all } \leq 0 \tag{9}$$

from which it is clear that the sum of two convex's is convex and likewise for concave maps. We next prove the positivity condition on the Schwarzian. Suppressing the arguments of the individual functions,

$$\hat{S}(f+g) = (f'+g')(f'''+g''') - \frac{3}{2}(f''+g'')^2 \tag{10}$$

$$= \hat{S}(f) + \hat{S}(g) + R, \qquad \text{where} \tag{11}$$

$$R = f'g''' + f'''g' - 3f''g''. \tag{12}$$

Since $\hat{S}(f) \geq 0$ and $\hat{S}(g) \geq 0$ one has

$$f'f''' \geq \frac{3}{2}(f'')^2 \geq 0 \qquad \text{and} \tag{13}$$

$$g'g''' \geq \frac{3}{2}(g'')^2 \geq 0. \tag{14}$$

Multiplying these we obtain

$$f'f'''g'g''' \geq (\frac{3}{2})^2(f''g'')^2$$

and taking the square root

$$3f''g'' \leq 2\sqrt{f'g'''}\sqrt{f'''g'}$$

or equivalently,

$$-3f''g'' \geq -2\sqrt{f'g'''}\sqrt{f'''g'}.$$

Substituting this in R,

$$R \geq f'g''' + f'''g' - 2\sqrt{f'g'''}\sqrt{f'''g'} = (\sqrt{f'g'''} - \sqrt{f'''g'})^2 \geq 0.$$

Thus $\hat{S}(f+g) \geq \hat{S}(f) + \hat{S}(g)$.

Finally, let $g \in \mathcal{M}$. Although $f(x) = \dfrac{1}{x} \notin \mathcal{M}$, it is fractional linear in the general sense and therefore the Schwarzian $\hat{S}(f) \equiv 0$. Thus $f \circ g \in \mathcal{M}$ *provided* $f \circ g$ is linearly bounded. ■

3.2. *Examples and Techniques for Examining the Structure of $\mathcal{M}$*

Recall that

$$\mathcal{M} = \mathcal{V} \cup \mathcal{X},$$

where $\mathcal{V}$, the concave increasing and $\mathcal{X}$, the convex decreasing functions are defined in (7). We now give some examples:

(1) $f(x) = \log(\alpha + x)$ with $\alpha \geq 1$. Then $\hat{S}(f)(x) = \dfrac{1}{2(\alpha+x)^4}$ and $f \in \mathcal{V}$.

(2) If $f(x) = (x+\alpha)^\beta$ with $\alpha > 0$ and $|\beta| < 1$, then $\hat{S}(f)(x) = \dfrac{1}{2}\beta^2(1-\beta^2)$ and $f \in \mathcal{M}$.

(3) If $f \in \mathcal{M}$ then for $c \in \mathbb{R}^+$, $f_c \in \mathcal{M}$ where $f_c(x) = f(c+x)$.

(4) $f(x) = \dfrac{ax^2+bx}{x+c}$ where $a > 0, b > c > 0$ and $b > ac$. Then

$$f'(x) = \frac{2ax^2 + 2acx + bc}{(x+c)^2} > 0 \quad \text{and} \quad \hat{S}(x) = \frac{6ac(b-ac)}{(x+c)^4} > 0.$$

Thus, $f \in \mathcal{V}$.

(5) This example illustrates a method of building members of $\mathcal{M}$ from other simpler functions in $\mathcal{M}$ as well as verifying whether or not a given function belongs to $\mathcal{M}$. Let

$$f(x) = \frac{x^2+6x+7}{(x+1)(x+2)} = 1 + \frac{2}{x+1} + \frac{1}{x+2}.$$

One may compute, using f,

$$f' = -\frac{3x^2+10x+9}{(x+1)^2(x+2)^2} \quad \text{and} \tag{15}$$

$$\hat{S}(f) = \frac{12}{(x+1)^4(x+2)^4}. \tag{16}$$

From $\hat{S}(f) > 0$ we see that (2) of Lemma 3.2 holds. Then (1) of Lemma 3.2 follows from (15) and we conclude $f \in \mathcal{X}$.

Or one may conclude directly from the partial fraction decomposition that f is the sum of three functions in $\mathcal{X}$ and apply Lemma 3.4.

(6) In a given partial fraction decomposition one may have to re-distribute the positive constant (assuming one occurs) so that the terms all represent functions with values in $\mathbb{R}^+$, e.g.

$$4 - \frac{1}{x+1} - \frac{2}{x+4} = 2 + (1 - \frac{1}{x+1}) + (1 - \frac{2}{x+4}),$$

the sum of three functions in $\mathcal{V}$.

(7) The occurrence of double, non-removable roots can be fatal, e.g.

$$f(x) = \frac{ax^2+bx+c}{(x+d)^2}, \quad a,b,c \geq 0, a+b+c > 0, d > 0, \quad \text{gives us}$$

$$\hat{S}(f)(x) = \frac{-6(ad^2-bd+c)^2}{(x+d)^8}\ .$$

But, if $c > 0$ and $a+b > 0$, $\dfrac{1}{f}$ is defined and linearly bounded so that Lemma

3.4 tells us $\frac{1}{f} \notin \mathcal{M}$. Thus, *fractional quadratics* with double roots in $(-\infty, 0)$ in the numerator or denominator *never* occur in $\mathcal{M}$.

(8) We next consider the fractional quadratic

$$f(x) = \frac{(x+a)(x+b)}{(x+c)(x+d)}, \quad a, b, c, d > 0.$$

The following is based solely on numerical experimentation. It is curious to note that when $a < c < b < d$, i.e. the roots of the numerator and denominator appear in alternate fashion, then $\hat{S}(f) > 0$, but whenever both roots of the numerator are between the roots of the denominator, i.e. $c < a < b < d$, then $\hat{S}(f) < 0$.

(9) Complex roots can also be troublesome. Let

$$f(x) = \frac{(x^2 + \alpha x + 1)(x+1)(x+3)}{(x^2 + \beta x + 2)(x+2)(x+4)}.$$

When $\alpha = \beta = 0$, $\hat{S}(f)$ changes sign in $\mathbb{R}^+$and when $\alpha = \beta = 1$, $\hat{S}(f) < 0$.

Based on Examples (7), (8) and (9) we make the following

Definition 3.5: Let $A, B \subset \mathbb{R}$ be subsets. We say A *isolates* B,

$$A \sqsubset B,$$

if given any two points in B, say $b_1 < b_2$ there is a point $a \in A$ such that $b_1 < a < b_2$. We also say A and B are *interlaced* if $A \sqsubset B$ and $B \sqsubset A$.

Conjecture 1 Given a rational function $f(x) = \frac{P(x)}{Q(x)} : \mathbb{R}^+ \to \mathbb{R}^+$, assume the roots of P, R_P and the roots of Q, R_Q are all in $\mathbb{R}^-$. Then R_P and R_Q are interlaced if, and only if $\hat{S}(f) \geq 0$.

Example (9) shows requiring the *real* roots to be interlaced is not enough to guarantee $\hat{S}(f) \geq 0$. Thus we make

Conjecture 2 Given a rational function $f(x) = \frac{P(x)}{Q(x)} : \mathbb{R}^+ \to \mathbb{R}^+$, assume $\hat{S}(f) \geq 0$. Then the roots of P, R_P and the roots of Q, R_Q are all real and interlaced.

If the conjectures are true then for every rational function in $\mathcal{M}$ the roots must be real and the degrees of the numerator and denominator differ by no more than 1. Then for every rational function $f \in \mathcal{M}$, $\frac{1}{f} \in \mathcal{M}$ provided only that $\frac{1}{f}$ is defined since linearly bounded is automatic..

4. Existence and Global Stability of Periodic Points

The existence of a p-periodic solution $P = \{\hat{x}_{p-1}, \ldots, \hat{x}_1, \hat{x}_0\}$ of the p-periodic difference equation

$$x_{n+1} = f_n(x_n), \qquad f_{n+p} = f_n, \qquad f_n \in \mathcal{M}. \tag{17}$$

is equivalent to the problem of finding a fixed point $\hat{x}_0$ of the composite map

$$F(x) \doteq f_{p-1} \circ f_{p-2} \circ \cdots \circ f_1 \circ f_0(x).$$

Likewise, P is globally asymptotically stable (GAS) if, and only if, $\hat{x}_0$ is GAS as a fixed point of F.

Lemma 4.1: *Define*

$$\mathcal{V}_u = \{f \in \mathcal{V} \mid f \text{ is uni-linearly bounded (Definition 3.1) }\}.$$

Then $\mathcal{V}_u$ is a semigroup under composition and $f \in \mathcal{V}_u$ has a globally asymptotically stable, and hence unique, fixed point, x_f. If $f(0) > 0$ or $f'(0) > 1$ then $x_f > 0$.

Proof: If $f(x) \le a_f x + b_f$ and $g(x) \le a_g x + b_g$, then $f \circ g(x) \le a_f a_g x + a_f b_g + b_f$ and the first statement follows. Uni-linearly bounded implies that the graph of f crosses the diagonal at say, x_f and $f \nearrow$ implies $0 \le f'(x_f) < 1$. Global attraction of x_f follows from the concavity. ■

The positivity condition on the Schwarzian plays a subtle role in showing stability in the convex case as the following example shows. We construct f as follows:

$$f(x) = 5 - 2x, \qquad 0 \le x \le 2$$

and continue f to $(2, \infty)$ as a C^3, decreasing, convex function with asymptotic value 0 at ∞. Clearly $f'(x_f) = -2$ and thus x_f is unstable. The graph must turn rapidly to avoid colliding with the x-axis. Thus the second derivative goes from 0 at 0 to some large positive value in an interval to the right of $x = 2$, then approaches 0 as $x \to \infty$. Thus f''' changes sign.

It is thus tempting to build the theory around the assumption that functions satisfy either the sign distribution $sign\{f', f'', f'''\} = \{-, +, -\}$ or $\{+, -, +\}$. The following example should put an end to that thought.

Define

$$f(x) = \frac{1}{(2x+5)^3}, \qquad \text{and} \tag{18}$$

$$g(x) = \frac{4}{1 + \arctan 5x}. \tag{19}$$

Then $sign\{f', f'', f'''\} = sign\{g', g'', g'''\} = \{-, +, -\}$, but $h \doteq f \circ g$ fails to satisfy either one; $sign\{h', h'', h'''\} = \{+, 0, 0\}$ where "0" indicates the derivative changes sign (computer recommended). Note that $\hat{S}(f) < 0$ and $\hat{S}(g) < 0$.

Lemma 4.2: *Let $f \in \mathcal{X}$. Then f has a unique fixed point x_f, $-1 < f'(x_f) \le 0$ and x_f is globally attracting. If $f(x) > 0$ for some x then $x_f > 0$.*

Proof: If $f \equiv$ constant then the statements are trivially true. The proof employs Lemma 2.6. Note that $f(0) > 0$ since otherwise $f \equiv 0$. Also, $f \searrow$ implies that the graph of f crosses the diagonal at a unique point x_f. By Theorem 3.3, $f \circ f \in \mathcal{V}$. Since f is bounded $f \circ f \in \mathcal{V}_u$. Then by Lemma 4.1, $f \circ f$ has a globally attracting exponentially asymptotically stable fixed point y. Then by Lemma 2.6, $y = x_f$ is a globally asymptotically stable fixed point of f. ■

Theorem 4.3: *Let $\{f_n,\ n = 0, 1, \ldots, p-1\}$ be a collection of functions in $\mathcal{M}$, and assume one of the following is true*

(1) $f_n \in \mathcal{V}$ and uni-linearly bounded for all n and $f_n(0) > 0$ for some n

(2) $f_n \in \mathcal{V}$, $f_n(0) = 0$ *and linearly bounded for all* n
and $f'_{p-1}(0) \cdots f'_1(0) f'_0(0) > 1$

(3) For some n, $f_n \in \mathcal{X}$ *and* $f_n(x) > 0$ *for some* x.

Then (17) has a globally attracting exponentially asymptotically stable periodic solution

$$\{\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_{p-1}\}, \qquad \hat{x}_{n+1 \mod p} = f_n(\hat{x}_n)$$

that is not the identically zero sequence.

Proof: Parts (1) and (2) are covered by Lemma 4.1: For (1), we apply Theorem 3.3 to obtain

$$F \doteq f_{p-1} \circ f_{p-2} \circ \cdots \circ f_1 \circ f_0 \ \in \ \mathcal{V}_u.$$

Assume $f_k(0) > 0$ and cyclically permute the functions so that f_k is the last to act. Then

$$\hat{F}(0) = f_k \circ f_{k-1} \circ \cdots \circ f_{k+1}(0) > 0$$

where all subscripts are understood " mod p". Then apply Lemma 4.1 to $\hat{F}$.

For (2), apply Lemma 4.1 directly to F.

For (3), assume $f_k \in \mathcal{X}$ and $f_k(x) > 0$ for some x. Since $f'_k < 0$, one must have $f_k(0) > 0$. Again cyclically permute to obtain $\hat{F} \in \mathcal{M}$. If $\hat{F} \in \mathcal{V}$ then since f_k is bounded, $\hat{F} \in \mathcal{V}_u$. Now apply Lemma 4.1. If $\hat{F} \in \mathcal{X}$ apply Lemma 4.2. ■

Remark 1 : In Theorem 4.3 one could replace "uni-linearly bounded for all n" by the weaker condition that the f_n are just linearly bounded: $f_n(x) \leq a_n x + b_n$ where $\alpha \doteq a_0 a_1 \cdots a_{p-1} < 1$ since then the composite F of all the maps would then satisfy $F(x) \leq \alpha + b_F$.

5. An application

Dynamic reduction was introduced in [7] where several examples of its applicability were given. One was to Rational Difference Equations with periodic parameters. In this section we will explore this application in greater detail and obtain some new results which include, in the autonomous case, some of the results given in [6]. The starting point will be delay difference equations of the form

$$x_{n+1} = \frac{\alpha + \sum_{i=0}^{k} \beta_i x_{n-i}}{A + \sum_{i=0}^{k} B_i x_{n-i}}, \tag{20}$$

using the notation of [6]. We next allow the parameters to be periodic of period p and separate out all the delayed terms

$$x_{n+1} = \frac{\alpha_n + \beta_{0,n} x_n + g_{1,n}(x_{n-1}, \ldots, x_{n-k})}{A_n + B_{0,n} x_n + g_{2,n}(x_{n-1}, \ldots, x_{n-k})}. \tag{21}$$

In the process of dynamic reduction, as applied to periodic difference equations, one then defines a class of periodic sequences $\mathcal{P}_\lambda$ where λ is an appropriate multiple of p. For each $v \in \mathcal{P}_\lambda$ one then solves the "reduced" equation,

$$\begin{aligned} x_{n+1} &= \frac{\alpha_n + \beta_{0,n} x_n + g_{1,n}(v_{n-1}, \ldots, v_{n-k})}{A_n + B_{0,n} x_n + g_{2,n}(v_{n-1}, \ldots, v_{n-k})} \\ &= \frac{\hat{\alpha}_n + \beta_{0,n} x_n}{\hat{A}_n + B_{0,n} x_n}, \end{aligned} \tag{22}$$

for an exponentially asymptotically stable λ-periodic solution $\hat{v} \in \mathcal{P}_\lambda$. This establishes a mapping

$$\mathcal{T} : \mathcal{P}_\lambda \to \mathcal{P}_\lambda, \qquad \hat{v} = \mathcal{T}(v). \tag{23}$$

A fixed point of $\mathcal{T}$ then yields a periodic solution of (21) and hence of (20).

The question, how to define the "reduction" function g in any particular problem, has to be guided by ones ability to solve the reduced equation (22) for a solution having the specified properties. The choice made in (21) yields a periodic family of Fractional Linear Maps whose properties were investigated in Section 2. In particular, Theorem 2.8 guarantees a exponentially asymptotically stable and globally attracting λ-periodic solution of the reduced equation (22) thus establishing the mapping (23). Then what remains to be done is to impose (a) conditions on the range of the sequences in $\mathcal{P}_\lambda$ and (b) a smallness condition on the first derivatives of the g's in order to show $\mathcal{T}$ is a contraction. See [7] for details.

Remark 1: The theorems of Section 3 allow us to extend this to "non-linear" rational differential equations, i.e. ones in which the numerator and denominator are non-linear. For example, borrowing from the examples of Subsection 3.2 one could consider

$$x_{n+1} = \frac{a_n x_n^2 + b_n x_n + g_{1,n}(\text{delayed terms})}{x_n + c_n + g_{2,n}(\text{delayed terms})},$$

or

$$\frac{x_n^2 + 6x_n + 7 + g_{1,n}(\text{delayed terms})}{(x_n+1)(x_n+2) + g_{2,n}(\text{delayed terms})}.$$

This will be pursued in later publications.

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