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Global stability in the 2D Ricker equation

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We improve a previous result for the 2D Ricker equation by reducing an infinite number of topological conditions to a finite number. We also give sufficient conditions in terms of the parameters where many of these topological conditions are satisfied. We also discuss the various pathologies that occur for other parameter choices.

Keywords: global stability; proper maps; 2D competition model

AMS Subject Classification: 39A30; 92D25; 92D40

1. Introduction

The scalar equation has been studied for a long time. W. E. Ricker proposed the equation in 1954 to model fishery stocks. One approach is to fit the Ricker model to data gathered over a long period of time and then use this model to gain insight into future stocks. For example the best Ricker's model for the Skeena sockeye salmon population from 1908 to 1952 is done in the book by Mahaffy and Chavez-Ross [9] based on data collected by Shepard and Withler [15],

$$P_{n+1} = 1.535 P_n \mathrm{e}^{-0.000783 P_n}.$$

It is easy to show the Ricker map in the form

$$x_{n+1} = x_n \mathrm{e}^{p - x_n}, \tag{1}$$

exhibits global asymptotic stability of the fixed point p provided $0 < p < 2$ and this can be extended to $p = 2$ using the Schwartzian derivative Elaydi [4, p. 52]. When p in (1) is periodic with period k then there is a globally asymptotically stable k-periodic solution provided $0 < p_n < 2$, Sacker [11]. Much attention has been given to 1D maps, Cull [2], Devaney [3], Elaydi [5], Liz [8] and Sharkovsky et al. [14] while 2D maps have understandably been given much less attention, Smith [16], Sacker [12].

Coupled Ricker equations occur in studying the behaviour of populations of genetically altered mosquitoes versus the wild types Li [7], Sacker and Von Bremen [13] where the Ricker functions are multiplied by a Ricatti functions to form a hybrid model. For further results and references to coupled Ricker maps see the book of Mira et al. [10]. See also Bartha, et. al. [17] where global asymptotic stability is proved for a narrow class of 2D maps obtained from a Ricker map with one unit time delay.

In this short paper we revisit the *basic question*, 'Under what conditions does local asymptotic stability of a fixed point of two coupled Ricker maps imply global attraction of the fixed point'. In a groundbreaking paper by Balreira et al. [1] a first attempt at solving this problem was carried out and it was that paper that inspired this work.

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This basic question could arise for two population species each governed by a Ricker model but exhibiting interspecies competition. Since the Columbia River salmon and the steelhead trout share the same environment it is not inconceivable that the coupled Ricker equations may one day be worthy of attention by marine biologists.

2. Summary of prior work and improvements

The 2D Ricker Map consists of two 1D Ricker Maps coupled together. That is, we will consider the map $T : [0, \infty) \times [0, \infty) \longrightarrow [0, \infty) \times [0, \infty)$ given by $T(x, y) = (f(x, y), g(x, y))$, where

$$f(x, y) = x\mathrm{e}^{p-x-ay}, g(x, y) = y\mathrm{e}^{q-y-bx}, \tag{2}$$

with four parameters a, b, p, q, and throughout we will refer to this map as the 'Ricker Map' without explicitly mentioning the dimension. In this paper, we focus on the case where the couplings are small, namely $a, b \in (0,\ 1)$. We will also assume that the carrying capacities satisfy $p, q \in (1,\ 2)$ so that the map behaves the same as the 1D Ricker Map when uncoupled. This map has three fixed points on the coordinate axes at (0, 0), $(p, 0)$, and $(0, q)$. It is easy to see that under the above constraints that the origin is always repelling and the other two fixed points are saddle points.

In addition, one has a coexistence fixed point when the two isoclines

$$L_p = \{(x, y) \,|\, p - x - ay = 0\},$$

and

$$L_q = \{(x, y) \,|\, q - y - bx = 0\},$$

intersect in the first quadrant. Intersections elsewhere are biologically meaningless and are not considered. A calculation shows that the intersection is

$$(x^*, y^*) = \left(\frac{p - aq}{1 - ab}, \frac{q - bp}{1 - ab}\right),$$

so a necessary and sufficient condition for the existence of a fourth fixed point is $p > aq$ and $q > bp$ (Note that $1 - ab$ is always positive). A necessary and sufficient condition that (x^*, y^*) is (locally) asymptotically stable is given by [4],

$$\left|\operatorname{tr} DT(x^*, y^*)\right| < \det DT(x^*, y^*) + 1 < 2.$$

In [1] they study images of a critical curve to show attraction to the coexistence fixed point. To understand the idea, it is helpful to first consider the 1D Ricker Map $T : [0, \infty) : \longrightarrow [0, \infty)$ given by $T_r(x) = x\mathrm{e}^{r-x}$ when $1 < r < 2$. This map has two fixed points, $x = 0$ and $x = r$, the former being unstable and the latter being stable. The map also has a single critical point at $x = 1$.

What follows will be an informal description of the dynamics. It is easy to see that for $x < 1$, T always maps points forward, and eventually there must be an integer m such that $T^m(x) > 1$. For points larger than 1, something quite different happens. Points now oscillate around the fixed point, with their images alternating between being below r and then above r, eventually converging to the fixed point. That is, we have global attraction to the fixed point; points smaller than 1 march forward, while points larger than 1 'dance' around the fixed point, getting closer on each iterate (see Figure 1).

The main idea of the prior work [1] is to generalize the argument given above for the 1D Ricker Map to two dimensions. Instead of a critical point, there is now a critical curve C that comprises the lower branch of a hyperbola; this hyperbola is where the Jacobian of the map has determinant zero. They then consider the sequence of curves given by C, $T(C)$, $T^2(C)$, $T^3(C)$, and show that, under certain topological and algebraic conditions, these curves 'dance' around an invariant curve which they argue is comprised of the two unstable manifolds of the saddle points on the x and y axis (see Figure 2).

We will now switch to a more rigorous framework. To state their Theorem precisely, we need to compare the relative position of the curves C, $T(C)$, $T^2(C)$, Informally, the notation $C < T(C)$ means that the curve C lies to the lower left of $T(C)$ (see Figure 2); we can make this precise by noting this implies C is inside the domain bounded by $T(C)$ and the coordinate axes.

Definition 2.1. *By a* Jordan curve joining the axes *we mean a simple Jordan curve* $\gamma : [0,\ 1] \to \mathbb{R}^2$ *with* $\gamma(0) = (0, y)$, $\gamma(1) = (x, 0)$, *and* $\gamma(t) \in (0, \infty) \times (0, \infty)$ *for* $0 < t < 1$. *Define the segments* $\gamma_x = \{[0,\ x] \times \{0\}\}$ *and* $\gamma_y = \{\{0\} \times [0, y]\}$ *and let S denote the interior of the set whose boundary is:*

$$\partial S = \{\gamma \cup \gamma_x \cup \gamma_y\},$$

where γ is a Jordan curve joining the axes. We say a set $A \leq \gamma$ if $A \subset \bar{S}$ and $A < \gamma$ if $A \subset S$. We also say $A \geq \gamma$ if $A \subset S^C$ and $A > \gamma$ if $A \subset \bar{S}^C$ where complements are relative to $[0, \infty) \times [0, \infty)$.

Thus given two Jordan curves joining the axes $\gamma_1 < \gamma_2$, the set $\mathcal{U}$,

$$\gamma_1 \leq \mathcal{U} \leq \gamma_2,$$

is well defined to be the closure of the region between the two bounding curves.

The critical curve C is defined by

$$C = \{(x, y) \in [0,\ 1] \times [0,\ 1] \,|\, x + y - 1 - (1 - ab)xy = 0\}.$$

We now state their Theorem, using the notation of this paper.

Theorem 2.2. [1]

Let T be the Ricker competition model with $p, q \in (1,\ 2)$ and $ab < 1$. Suppose that the coexistence fixed point (x^*, y^*) is localy asymptotically stable. Assume the following conditions:

1. *The critical curve C satisfies $C < L_p$ and $C < L_q$.*
2. *For all $m \neq n$, $T^n(C) \cap T^m(C) = \emptyset$.*

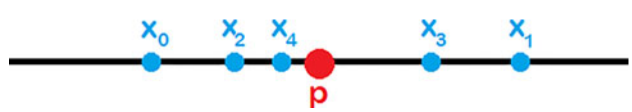

Figure 1. The dynamics of the 1D Ricker Map for $x_0 > 1$ and $1 < p < 2$. Points oscillate around the globally attracting fixed point, getting closer on each iteration.

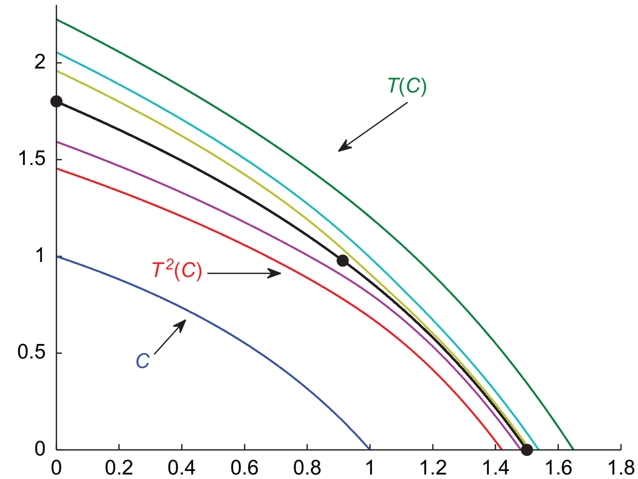

Figure 2. Example plot of the curves $T(C) \cap C^+ = \emptyset$ for $m = 0, 1, \ldots, 4$. The bottom curve is C, the top is $T(C)$, and they continue to oscillate around the invariant curve in a nested fashion, just as in the 1D case. The invariant curve is shown as the union of the unstable manifolds of the boundary fixed points that terminates on the interior fixed point.

Then (x^*, y^*) is globally asymptotically stable with respect to the interior of the first quadrant.

Note: The assumption $ab < 1$ is actually stated earlier in their paper.

Here condition (1) ensures that points inside the region bounded by C eventually map outside this region; they proved this condition is equivalent to

$$p \in \left(\frac{a+1-2a\sqrt{b}}{1-ab}, \frac{a+1+2a\sqrt{b}}{1-ab}\right) \quad \text{and} \quad q \in \left(\frac{b+1-2b\sqrt{a}}{1-ab}, \frac{b+1+2b\sqrt{a}}{1-ab}\right).$$

Condition (2), combined with the fact that the dynamics on the axes are known, ensures that the curves $T^m(C)$ are nested as in Figure 2. In fact, there is a third condition implicitly used in their proof, namely that the $T^m(C)$ are simple Jordan arcs.

Note that, with this Theorem, given a fixed a, b, p, q, one cannot determine if it applies, for one has to check that for any $m \neq n$ one has that $T^n(C) \cap T^m(C) = \emptyset$, which requires checking an infinite number of intersections. Our aim is to replace this statement with a finite number of conditions in the cases $0 < a, b < 1$. As a first step we will need the following lemma whose proof is in Section 5,

Lemma 2.3. *The curves C and $T(C)$ are simple Jordan arcs that can be represented as functions that are monotonically decreasing in x and concave.*

The next key element will depend on the outer branch of the singular hyperbola that is shown in Figures 7 and 8. We denote this outer branch by C^+ where

$$C^+ = \left\{ (x, y) \in \left(\frac{1}{1-ab}, \infty\right) \times \left(\frac{1}{1-ab}, \infty\right) \middle| x + y - 1 - (1-ab)x = 0 \right\}.$$

We will prove the following Theorem

Theorem 2.4. *Let* (1) $C < T^2(C) < T(C)$ *and* (2) $T(C) \cap C^+ = \emptyset$. *Then* $T^n(C) \cap T^m(C) = \emptyset$ *for all positive integers* $m \neq n$ *and each* $T^m(C)$ *is a simple Jordan arc.*

Sufficient conditions guaranteeing condition (2) are given in Theorem 5.3. Before giving the proof of the Theorem, we will show some examples to illustrate the inherent difficulties in analysing this map.

3. Examples

In this section, we show some numerically computed plots of the first few images of the critical curve C. As a prelimary example, we show why one might not expect to always have the nice alternating behaviour that is present in the 1D Ricker Map.

Consider the parameters $a = 0.3$, $b = 0.7$, $p = 1.6$, and $q = 1.9$. A plot of the first several images of C are shown in Figure 3. We observe that here, as before, the curves do oscillate around the fixed point, getting closer on each iteration, and do not intersect. However, only C and $T(C)$ are monotone and concave. The other curves bend, and it is not inconceivable that they could, for other parameters, intersect other curves or even themselves. In fact, we will give examples to show both are possible and that the dynamics of this map are not to be taken lightly.

One way to have the critical curves intersect each other is to take small carrying capacities p, q. Even for modest values of the coupling parameters a, b, one observes intersections when p, q are small. For instance, consider $p = 1.1$, $q = 1.05$, $a = 0.2$, $b = 0.9$ (see Figure 4). Note that $C \cap T(C) \neq \emptyset$ and that it appears future iterates will also always intersect the critical curve C.

This picture can be exaggerated by pulling the couplings further apart. Doing so widens the range of 'bad' p, q. See Figure 5 where $p = 1.25$, $q = 1.35$, $a = 0.0001$, $b = 0.99$. Indeed, even outside C, higher iterates intersect each other in this example.

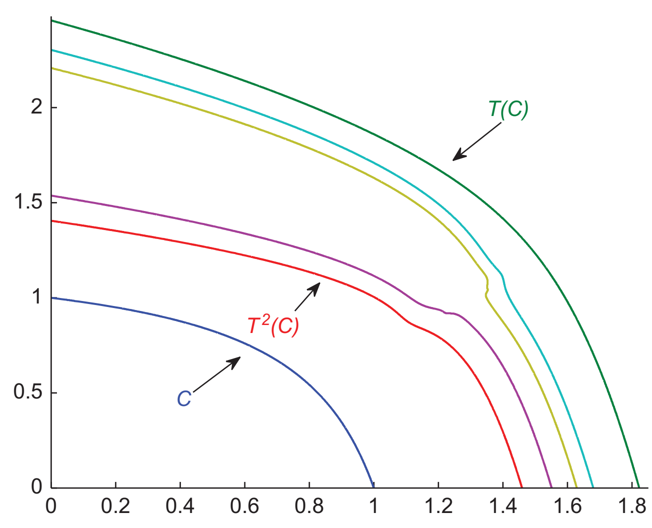

Figure 3. Example plot of the curves $T^m(C)$ for $m = 0, 1, \ldots, 5$ for the parameters $a = 0.3$, $b = 0.7$, $p = 1.6$, $q = 1.9$. The curves oscillate around the fixed point and do not intersect in this example. However, $T^m(C)$ for $m \geq 2$ are not monotone or concave.

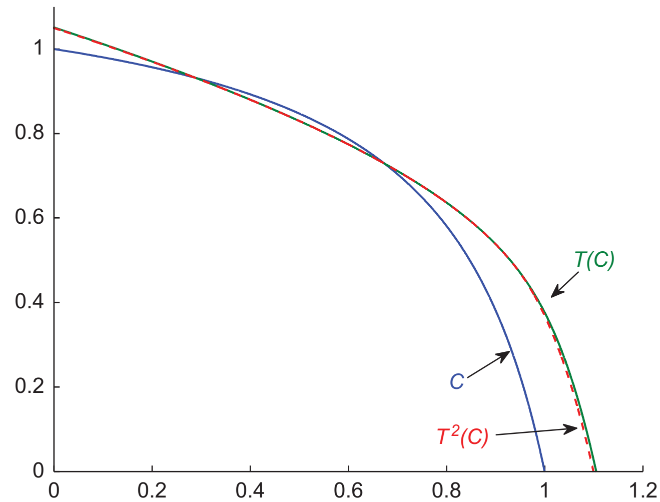

Figure 4. Example plot of the curves $T^m(C)$ for $m = 0, 1, 2$ for the parameters $a = 0.2$, $b = 0.9$, $p = 1.1$, $q = 1.05$. The curves $T(C)$ and $T^2(C)$ both intersect C. $T^2(C)$ is shown dashed.

It is also not hard to find examples where self intersections occur. Consider, for instance, $p = 1.6$, $q = 1.9$, $a = 0.2$, $b = 0.4$. A plot is shown in Figure 6. Here the second iterate $T^2(C)$ has a 'loop' where it self intersects. In addition, it also intersects $T^4(C)$. However, one does have that the first few iterates are separated (that is, $C < T^2(C) < T(C)$). This shows that one is unlikely to rule out non intersections with any kind of induction argument on the first few C iterates alone.

This example can also be exaggerated by reducing the coupling. A plot of $p = 1.6$, $q = 1.9$, $a = 0.1$, $b = 0.3$ is shown in Figure 7. Here the second iterate $T^2(C)$ has multiple self-intersections. The critical feature omitted in Figure 6 that explains why $T(C)$ can

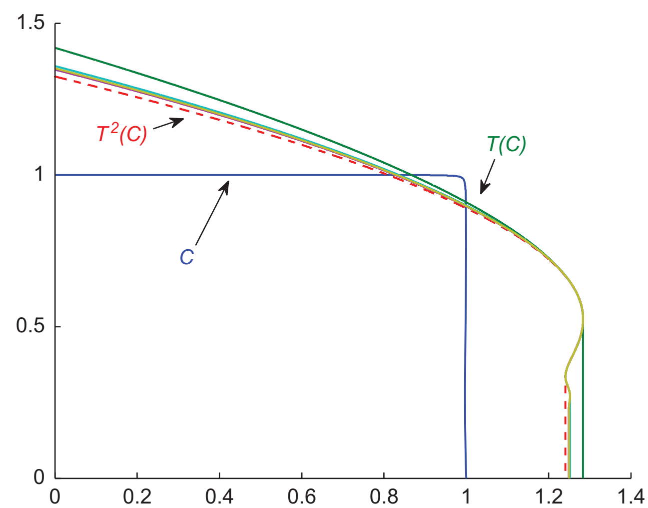

Figure 5. Example plot of the curves $T^m(C) \cap T^n(C) = \emptyset$ for $m = 0, 1, \ldots, 5$ for the parameters $a = 0.0001$, $b = 0.99$, $p = 1.25$, $q = 1.35$. The curves C and also intersect in the region to the right of C. $T^2(C)$ is shown dashed.

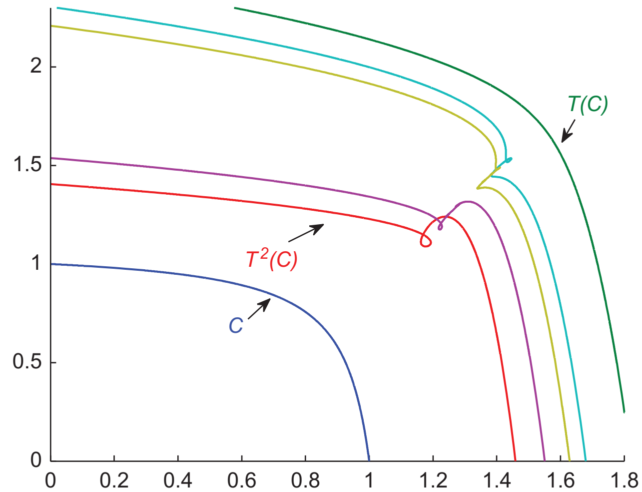

Figure 6. Example plot of the curves $T^m(C)$ for $m = 0, 1, \ldots, 4$ for the parameters $a = 0.2$, $b = 0.4$, $p = 1.6$, and $q = 1.9$. The curve $T^2(C)$ intersects itself and $T^4(C)$. In addition, higher iterates also have self intersections.

behave nicely but $T^2(C)$ does not, is the fact that the outer branch of the hyperbola C^+ shown in Figure 7 intersects $T(C)$. This causes the 'wrapping' of $T^2(C)$ that forces it to intersect itself due to loss of injectivity, Lemma 4.3.

Now the expression for C^+ only depends on a, b. Furthermore, for fixed a, b, larger p, q moves $T(C)$ further up and to the right. This shows that for fixed a, b, one should expect an upper bound on p, q for which the behaviour of the curves is well behaved like

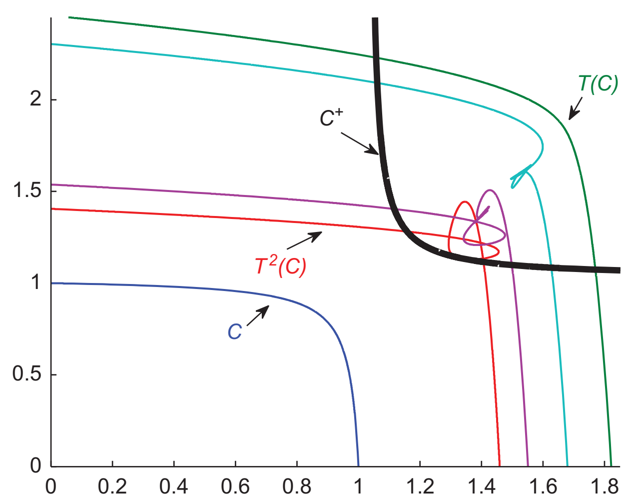

Figure 7. Example plot of the curves $T^m(C)$ for $m = 0, 1, \ldots, 4$ for the parameters $a = 0.1$, $b = 0.3$, $p = 1.6$, and $q = 1.9$. The curve $T^2(C)$ intersects itself and $T^4(C)$. In addition, higher iterates also have self intersections. The thick curve joining the upper to the right boundaries of the figure is the outer branch of the singular curve C^+ whose intersection with $T(C)$ is responsible for the self intersections as Lemma 4.3 shows.

the 1D map. In addition, the first two examples with the small p, q show that one should also expect a lower bound for fixed a, b.

In the next section, we give a proof of the Theorem that shows one only needs to check four curves $(C, T(C), T^2(C), C^+)$ to determine the non-intersection of the remaining curves. Then we will give some sufficient conditions on the parameters for which some of these are satisfied. Not surprisingly, these conditions will involve upper and lower bounds on p and q in terms of a and b.

4. Proof of theorem

The main ingredient in passing from an infinite number of conditions to a finite number is injectivity. This will come from the Lemma 4.3.

Definition 4.1. *A mapping between topological space $f : X \to Y$ is called* proper *if the inverse image of each compact subset of Y is compact in X.*

Theorem 4.2. ([6*, p.* 240]) *Let X be pathwise connected and Y be simply connected Hausdorff spaces. A local homeomorphism $f : X \to Y$ is a global homeomorphism of X onto Y if and only if f is proper.*

Lemma 4.3. *Let $C < T^2(C) < T(C)$ and define the set $\mathcal{U}$ (see Definition* 2.1)

$$C \leq \mathcal{U} \leq T(C),$$

and let $\mathcal{V} = T(\mathcal{U})$. If $T(C) \cap C^+ = \emptyset$ then $T : \mathcal{U} \to \mathcal{V}$ is a homeomorphism.

Proof of Lemma 4.3. Since $(x_1, y_1) \in T^{m-1}(C)$ we have that T is nonsingular on the interior $\mathring{\mathcal{U}}$ of $\mathcal{U}$ and therefore locally injective on $\mathring{\mathcal{U}}$. Since the zero-eigenspaces along C are transverse to C the local injectivity extends to include C, i.e. each point of C has a small compact neighbourhood relative to $\mathcal{U}$ on which the mapping is injective and hence a homeomorphism. Thus T is a local homeomorphism on $\mathcal{U}$. To see that T is proper note that for any compact $K \subset \mathcal{V}$, one has $T^{-1}(K)$ is closed. Since $K \cap (0,0) = \emptyset$, $T^{-1}(K)$ is bounded. Finally, $\mathcal{U}$ is clearly connected and $\mathcal{V}$ is clearly simply connected. □

Thus the location of the curve C^+ plays a critical role in the dynamics. The main idea is as follows: With a homeomorphism, all the images $T^n(C)$ are necessarily simple Jordan arcs, as a self intersection violates injectivity. Furthermore, the first few curves being separated is enough for an induction argument when combined with the homeomorphism.

Proof of Theorem 2.4. Since $C < T^2(C) < T(C)$, we have that for $n \geq 1$ that $T^n(C) \in \mathcal{V}$ by Lemma 4.3. We now proceed inductively. Assume that $T^m(C) \cap T^n(C) = \emptyset$ for all $M \geq m > n \geq 1$. This holds for $M = 2$ by assumption. If $T^{M+1}(C) \cap T^n(C) \neq \emptyset$ for some $1 \leq n \leq M$, then there exists $(x_0, y_0) \in T^M(C)$ and $(x_1, y_1) \in T^{n-1}(C)$ such that $T(x_0, y_0) = T(x_1, y_1)$. By the inductive hypothesis, $(x_0, y_0) \neq (x_1, y_1)$, so by Lemma 4.3 we have that $T^{M+1}(C) \cap T^n(C) = \emptyset$.

Likewise, if any $T^m(C)$ is not simple, then there exists a parametrization $\gamma(t)$ with $\gamma(t_0) = \gamma(t_1)$ for $t_0 < t_1$. But then there exists two distinct points $(x_0, y_0) \in T^{m-1}(C)$ and

$(x_1, y_1) \in T^{m-1}(C)$ such that $T(x_0, y_0) = \gamma(t_0) = \gamma(t_1) = T(x_1, y_1)$, which violates injectivity. $\square$

The statement of Theorem 2.4 has topological conditions on the curves C, $T(C)$, $T^2(C)$, and C^+. A natural question to ask is for which parameters a, b, p, q these conditions are satisfied. This will be explored in the next section.

5. Parameter values

Theorem 2.4 essentially has four topological conditions. Namely, $C < T(C)$, $C < T^2(C)$, $T^2(C) < T(C)$, and $T(C) \cap C^+ = \emptyset$. These in turn imply that all future iterates are nonintersecting and simple Jordan arcs. Ideally, one would have parameter values for which these four conditions are satisfied.

In [1] it was shown that an algebraic condition that ensures that C satisfies $C < L_p$ and $C < L_q$ is

$$p \in \left(\frac{a + 1 - 2a\sqrt{b}}{1 - ab}, \frac{a + 1 + 2a\sqrt{b}}{1 - ab} \right) \quad \text{and} \quad q \in \left(\frac{b + 1 - 2b\sqrt{a}}{1 - ab}, \frac{b + 1 + 2b\sqrt{a}}{1 - ab} \right).$$

For these same parameters, we claim it follows that $C < T(C)$ and $T^2(C) < T(C)$. The first of these claims will follow from the fact that C is monotone and concave as well as the fact that, under these conditions, points on C are mapped to the upper right. The second claim will follow from the fact that $T(C)$ is an upper bound for the map T [1].

We also claim that under the algebraic conditions $p < a + 1$, $q < b + 1$, it follows that $T(C) \cap C^+ = \emptyset$. This will make use of the concavity of $T(C)$ stated in Lemma 2.3 which we now prove. Note that these conditions are only sufficient.

Proof of Lemma 2.3. We can express C by the curve $y = \phi(x)$ where

$$\phi(x) = \frac{1 - x}{1 - (1 - ab)x} \text{ for } 0 \leq x \leq 1.$$

We note that $\phi(x) \in [0, 1]$ and that

$$\phi'(x) = \frac{-ab}{(1 - (1 - ab)x)^2} < 0.$$

Now we parametrize C by $(t, \phi(t))$ with $0 \leq t \leq 1$.

Then $T(C)$ is parametrized by $(x(t), y(t))$ with $x(t) = t\mathrm{e}^{p-t-a\phi(t)}$ and $y(t) = \phi(t)\mathrm{e}^{q-bt-\phi(t)}$. Then

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \phi'(t)\mathrm{e}^{q-bt-\phi(t)} - (b + \phi'(t))\phi(t)\mathrm{e}^{q-bt-\phi(t)} = \mathrm{e}^{q-bt-\phi(t)}(\phi'(t)(1 - \phi(t)) - b\phi(t)) < 0$$

and

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \mathrm{e}^{p-t-a\phi(t)} + t(-1 - a\phi'(t))\mathrm{e}^{p-t-a\phi(t)} = \mathrm{e}^{p-t-a\phi(t)}((1 - t) - at\phi'(t)) > 0$$

so that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x} < 0$.

For concavity, we aim to show that $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} < 0$. A computation shows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{be^{E(t)}p(t)}{(1 - (1 - ab)t)^3},$$

where $p(t) = At^2 + Bt + C$ with $A = -1 + b - a^2b^2 + 2ab - 2ab^2 + a^2b^3$, $B = 3 - 2b - 4ab + 2ab^2 + a^2b^2$, $C = -2 + b + a^2b$ and $E(t) = q - p + t(1 - b) + \phi(t)(a - 1)$. Since

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\frac{dy}{dx} \Big/ \frac{dx}{dt}$$

it suffices to show that $p(t) < 0$ on [0, 1].

Since $A = (b-1)(1-ab)^2$ is always negative, $p(t)$ has an absolute maximum at $t^* = -\frac{B}{2A}$. Noting that $B = (3 - ab - 2b)(1 - ab)$, we have that this value of t^* is given by

$$t^* = \frac{3 - ab - 2b}{2(1-b)(1-ab)} .$$

However, $t^* \leq 1$ would imply that $3 - ab - 2b \leq 2 - 2b - 2ab + 2ab^2$ or equivalently $1 \leq ab(2b - 1)$, which is impossible for $a, b \in (0, 1)$. Hence $t^* > 1$ so the absolute maximum on the interval [0, 1] occurs at $t = 1$. A computation shows that

$$p(1) = ab(ab^2 + a - 2)$$

which is clearly negative for $a, b \in (0, 1)$.

Hence $\frac{d^2y}{dx^2} < 0$ as claimed. □

Lemma 5.1. [1] *Let* $(x, y) \in [0, \infty) \times [0, \infty)$. *Then* $T(x, y) \leq T(C)$.

Proof of Lemma 5.1. A topological proof is given in [1]. We give a different algebraic proof here.

Let $(x, y) \in [0, \infty) \times [0, \infty)$. We first claim that there exists $(x_0, y_0) \in C$ and $t, s \in \mathbb{R}$ such that

$$x = x_0 + x_0at = x_0 + (1 - y_0)s, \tag{3}$$

and

$$y = y_0 + (1 - x_0)t = y_0 + y_0bs . \tag{4}$$

We address the trivial cases first. If $x = 0$ then one takes $x_0 = 0$, $y_0 = 1$, $t = y - 1$, $s = \frac{1-y}{b}$. If $x \neq 0$ and $y = 0$, then one takes $x_0 = 1$, $y_0 = 0$, $t = \frac{1-x}{a}$, $s = x - 1$.

Now assume $0 < x < \infty$ and $0 < y < \infty$. We have $y_0 = \frac{1-x_0}{1-(1-ab)x_0}$ (since $(x_0, y_0) \in C$). The first equalities in (3) and (4) will follow if by eliminating t, the resulting equations have a solution $(x_0, y_0) \in C$. Thus, putting $t = \frac{x-x_0}{x_0a}$, (3) into (4), we obtain:

$$y = \frac{1 - x_0}{1 - (1-ab)x_0} + (1 - x_0)\frac{x - x_0}{x_0a} .$$

This is equivalent to the cubic

$$h(x_0) = y(1 - (1-ab)x_0)(x_0a) - (1 - x_0)(x_0a) - (1 - x_0)(x - x_0)(1 - (1-ab)x_0)$$

having a root in the interval (0, 1). This follows immediately since

$$h(0) = -x < 0$$

and

$$h(1) = ay(1 - (1 - ab)) = a^2\, by > 0\,.$$

Taking this toot x_0, as well as taking $y_0 = \frac{1-x_0}{1-(1-ab)x_0}$ and $t = \frac{x-x_0}{x_0 a}$ gives the first of the equalities in (3) and (4).

To obtain the second equalities in (3) and (4), we note that for $(x_0, y_0) \in C$, one has that

$$(1 - x_0)(1 - y_0) = abx_0y_0\,.$$

Thus taking $s = \frac{x_0 at}{1-y_0}$ we have that $x_0at = (1 - y_0)s$ and $(1 - x_0)t = \frac{abx_0y_0}{1-y_0}t = y_0bs$ as claimed.

Now

$$T(x, y) = \left((x_0 + x_0at)\mathrm{e}^{p-x_0-x_0at-ay_0-at+ax_0t}, (y_0 + y_0bs)\mathrm{e}^{q-bx_0-bs+by_0s-y_0-y_0bs}\right)$$

which simplifies to

$$T(x, y) = \left(x_0(1 + at)\mathrm{e}^{p-x_0-ay_0}\mathrm{e}^{-at}, y_0(1 + bs)\mathrm{e}^{q-bx_0-y_0}\mathrm{e}^{-bs}\right).$$

Then referring to (2),

$$f(x, y) = x_0(1 + at)\mathrm{e}^{p-x_0-ay_0}\mathrm{e}^{-at} = f(x_0, y_0)(1 + at)\mathrm{e}^{-at} \le f(x_0, y_0),$$

and

$$g(x, y) = y_0(1 + bs)\mathrm{e}^{q-bx_0-y_0}\mathrm{e}^{-bs} = g(x_0, y_0))(1 + bs)\mathrm{e}^{-bs} \le g(x_0, y_0)),$$

where we have made use of the inequality $1 + x \le \mathrm{e}^x$.

Thus $T(x, y)$ is in the negative cone of the point $T(x_0, y_0)$ on $T(C)$. This implies $T(x, y) \le T(C)$ since $T(C)$ is a monotonically decreasing concave Jordan arc from Lemma 2.3. □

Theorem 5.2. Let

$$p \in \left(\frac{a + 1 - 2a\sqrt{b}}{1 - ab}, \frac{a + 1 + 2a\sqrt{b}}{1 - ab}\right) \quad \text{and} \quad q \in \left(\frac{b + 1 - 2b\sqrt{a}}{1 - ab}, \frac{b + 1 + 2b\sqrt{a}}{1 - ab}\right).$$

Then $C < T(C)$ and $T^2(C) < T(C)$

Proof of Theorem 5.2. We first consider C and $T(C)$. As the endpoints of $T(C)$ lie outside the region bounded by C, it suffices to show that $C \cap T(C) = \emptyset$. For the parameters given, we note that for any $(x, y) \in C$, one has that $f(x, y) > x$ and $g(x, y) > y$. Thus if at some point C intersected $T(C)$, there would be a point on C that maps to another point on C. But then there exists two points $(x_0, y_0), (x_1, y_1) \in C$ with $x_0 < x_1$ and $y_0 < y_1$, contradicting the fact that C is monotone decreasing (Lemma 2.3).

To see that $T^2(C) < T(C)$, note that by lemma 5.1 if they intersected, there would be an intersection point of C and $T(C)$, which was shown to not occur. □

We now give sufficient conditions for disjointness of $T(C)$ and C^+

THEOREM 5.3. *If $a+1 > p$ and $b+1 > q$ then $T(C) \cap C^+ = \phi$.*

Proof of Theorem 5.3. We will first give a bounding curve C^* for $T(C)$ such that $T(C) < C^*$ and then will argue that $C^* \cap C^+ = \varnothing$ (see Figure 8).

First, we parametrize C by $(t, \phi(t))$, $0 \le t \le 1$ where $\phi(t) = \frac{1-t}{1-(1-ab)t}$. Note that then $T(C)$ is parametrized by $\left(t\mathrm{e}^{p-t-a\phi(t)}, \phi(t)\mathrm{e}^{q-bt-\phi(t)}\right)$.

Define C^* by the parametrization

$$(x(t), y(t)) = \left(t\mathrm{e}^{m(1-\phi(t))+1-t}, \phi(t)\mathrm{e}^{m(1-t)+1-\phi(t)}\right)$$

where $m = \max\{a, b\}$ We observe that

$$t\mathrm{e}^{p-t-a\phi(t)} < t\mathrm{e}^{a+1-t-a\phi(t)} \le t\mathrm{e}^{m(1-\phi(t))+1-t}$$

and

$$\phi(t)\mathrm{e}^{q-bt-\phi(t)} < \phi(t)\mathrm{e}^{b+1-bt-\phi(t)} \le \phi(t)\mathrm{e}^{m(1-t)+1-\phi(t)}$$

so that $T(C) < C^*$.

The advantage of C^* is that it is symmetric with respect to t and $\phi(t)$; that is we have that $x(t) = y(\phi(t))$ and $x(\phi(t)) = y(t)$. Note that $C^* = T(C)$ for a special set of parameters (which do not apply to this lemma, namely when $p = q = m+1$ and $a = b = m$) so that C^* is monotone and concave by Lemma 2.2.

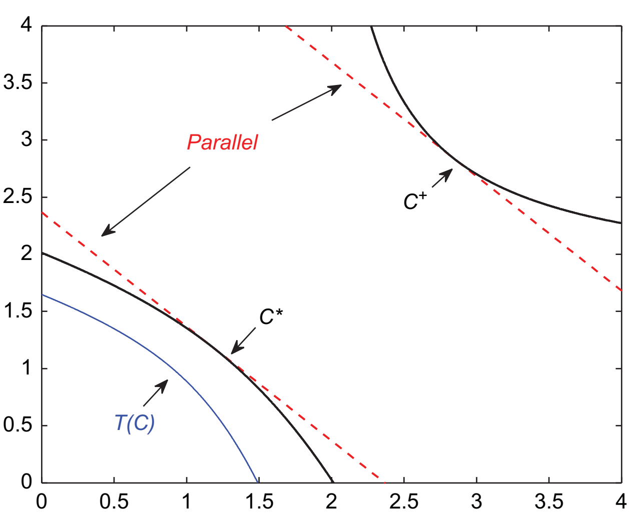

Figure 8. The bounding curve C^* (thick concave curve) of $T(C)$ (thin curve) is shown, along with the outer branch of the hyperbola C^+ (thick convex curve). Both C^* and C^+ have a tangent line with slope -1 (shown as dashed lines), so in the proof of the Theorem it suffices to show that the point of tangency on C^* is to the lower left of the tangent point on C^+ whenever $a+1 > p$ and $b+1 > q$.

A computation or geometric reasoning shows that $dy/dx = -1$ at the point where C^* crosses the line $y = x$, which occurs when $t = \phi(t)$. Solving $t = \phi(t)$ gives $t = 1/(1 \pm \sqrt{ab})$ and it is $t^* = 1/(1 + \sqrt{ab})$ that lies in the interval [0, 1]. Computing $x(t), y(t)$ at this point gives the coordinate as $(t^* e^{(m+1)(1-t^*)}, t^* e^{(m+1)(1-t^*)})$. Since C^* is concave, the tangent line at this point lies above C^* (and thus the tangent line lies above $T(C)$ as in Figure 8). Now on C^+, the curve also has slope -1 at the point

$$\left(\frac{1}{1-\sqrt{ab}}, \frac{1}{1-\sqrt{ab}}\right)$$

and since C^+ is convex, this tangent line lies below C^+. As these two tangents are parallel, the proof will be complete if

$$t^* e^{(m+1)(1-t^*)} < \frac{1}{1-\sqrt{ab}}$$

This is equivalent to showing $\frac{1}{t^*} e^{(t^*-1)(m+1)} - 1 + \sqrt{ab} > 0$. We compute directly:

$$\begin{aligned}
\frac{1}{t^*} e^{(t^*-1)(m+1)} - 1 + \sqrt{ab} &\geq \frac{1}{t^*}(1 + (t^* - 1)(m+1)) - 1 + \sqrt{ab} \\
&= m - \frac{m}{t^*} + \sqrt{ab} \\
&= m - m(1 + \sqrt{ab}) + \sqrt{ab} \\
&= (1-m)\sqrt{ab}
\end{aligned}$$

This is positive since $0 < m < 1$ by definition. □

Remark Trivially one has

$$a + 1 < \frac{a + 1 + 2a\sqrt{b}}{1 - ab} \quad \text{and} \quad b + 1 < \frac{b + 1 + 2b\sqrt{a}}{1 - ab}$$

so the parameter conditions given in Theorem 5.3 imply the upper bound part of the conditions in Theorem 5.2.

In summary, we have sufficient (but not necessary) parameter values for three of the four topological conditions. We comment that we lack parameter values for the final topological condition $C < T^2(C)$. This is likely to be difficult because of the nature of the curve $T^2(C)$. We can find conditions for the curves C, $T(C)$, and C^+ because the curves have simple first and second derivatives; they are always monotone decreasing and either always concave as with C and $T(C)$ or always convex as with C^+. The same is not true of $T^2(C)$. In general it is both not monotone and the second derivative changes sign.

6. Conclusion

Our main points and results are as follows. We reduced an infinite number of topological conditions for global attraction to the fixed point to a finite number. We also give upper and lower bounds on p and q in terms of a and b which guarantee most of these topological conditions. We are able to do this because the curves C, $T(C)$, and $C^+$are well behaved.

Finally, we believe the Ricker Map is worth revisiting to tackle some of the more pathological cases with intersections.

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Disclosure statement

No potential conflict of interest was reported by the authors.

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