A Conjecture on the Stability of the Periodic Solutions of Ricker's Equation With Periodic Parameters

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Abstract

In this work we propose a conjecture about the stability of the periodic solutions of the Ricker equation with periodic parameters, which goes beyond the existing theory, and for the special case of period-two parameters we analytically show the conjecture is true. For this case we show that the stability region in parameter space obtained from the conjecture is larger than a previously proposed stability region. The period-three case is investigated numerically and similar extensions are realized. This suggests that the current theory cited in this paper, while giving sufficient conditions for stability is far from optimal.

Keywords: Periodic difference equation, global stability, Ricker equation *AMS 2000 Subject Classification:* 39A11, 92D25

1 Introduction

Recent advances in genetics have allowed scientists to genetically modify mosquitoes in the laboratory to hinder or block parasite transmission, thus making the mosquitoes refractory. This opens the possibility to release genetically modified mosquitoes into the wild with the objective to reduce the spread of mosquito borne diseases like Malaria. The progress in this area is fairly recent [1] and only few mathematical models for the population dynamics of wild and genetically modified mosquitoes are available in the literature. Jia Li [2] proposed a discrete-time mathematical model for populations consisting of wild and genetically

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altered mosquitoes, and in [3] presented an extension of his model where the zygocity of the mosquitos is considered. See Ackleh et al [4] for a three-stage discrete-time population model with continuous versus seasonal reproduction, and Jang [5] for a discrete-time model consisting of two interacting populations. Ratio dynamics was used in [7, 8] to decouple the mosquitoes population dynamics equations introduced in [2] into two Ricker equations. In [6], the idea of dynamic reduction was introduced, and a model where the seasonal variability was taken into account by allowing the birth and survival functions to have periodic parameters was presented. Through the use of dynamic reduction the equations governing the population of the wild and genetically modified mosquitoes become a set of decoupled Ricker equations with periodic parameters. So the study of the stability of the periodic solutions of the mosquitoes model in [6] reduces to study the stability of periodic solutions of the Ricker equation with periodic parameters. This highlights the importance of understanding the stability of the periodic solutions of Ricker's equation with periodic parameters and the motivation for the work here presented.

Here we propose a conjecture about the stability of the periodic solutions of the Ricker equation with periodic parameters, and for the special case of period-two parameters we analytically show the conjecture is valid. Furthermore, for this case we show that the stability region in parameter space obtained from the conjecture is larger than the one obtained by applying the results of Z. Zhou and X. Zou [9]. Results from numerical solutions suggest that for some regions in parameter space, it may be possible to attain stability of the periodic solutions for parameter values beyond the ones constrained by the conjecture.

The conjecture on the stability of periodic solutions of Ricker's equation with periodic parameters is given in Section 2, there it is also shown that the conjecture is valid for periodic parameters with period two. In Section 3 the conjecture is explored numerically for parameters with periods 2, 3 and 4. Conclusions are given in Section 4.

2 The Conjecture

In this section we present a conjecture on the stability of the periodic orbit for the Ricker equation with periodic parameters. The stability of such equations has been studied by Zhou and Zou [9], Kon [10], and Sacker [**?**].

Consider the Ricker equation given by

$$z_{n+1} = z_n e^{p(n)-z_n}, \tag{2.1}$$

where the parameter $p(n)$ is periodic with period k. In the trivial case of $k = 1$ we have that $p(n) = p$ for all n, and (2.1) has the unique positive fixed point $\hat{z} = p$, that is globally asymptotically stable with respect to initial conditions $z_0 \in \mathbb{R}_0^+ \doteq (0, \infty)$ if $0 < p < 2$. The upper bound is significant since as p increases across 2, $\hat{z}$ loses its stability and bifurcates into a stable period-two solution. In the general case, with $p(n)$ k-periodic, it is shown in [13] that for $0 < p(n) < 2$, there is a periodic solution that is globally asymptotically stable with respect to $\mathbb{R}_0^+$. We propose the following conjecture.

Conjecture *Suppose $p(n) > 0$ and $p(n)$ is periodic with period k. Then the Ricker equation (2.1) has a globally asymptotically stable periodic orbit if $\frac{p(0)+\ldots+p(k-1)}{k} < 2$, and $0 < p(j) < 2 + \epsilon_k$, where ϵ_k is a positive number depending on k, and $0 \le j \le k-1$.*

We first consider the special case of $k = 2$. Suppose $p(n)$ is period two, that is, $p(n+2) = p(n)$ for all $n \geq 0$. Let $p_0 = p(0)$ and $p_1 = p(1)$ with both p_0 and p_1 being positive, then to study the 2-periodic solutions of (2.1) we can consider

$$z_{n+1} = g(z_n), \quad \text{where} \quad g(z_n) = (g_1 \circ g_0)(z_n), \tag{2.2}$$

with $g_i(z) = ze^{p_i - z}$, and $i = 0, 1$. The fixed points of (2.2) are the period-2 solutions of (2.1).

A direct computation gives

$$\begin{aligned} g(z) &= ze^{p_0+p_1-z-ze^{p_0-z}}, \quad \text{and} \\ g'(z) &= (1-z)(1-ze^{p_0-z})e^{p_0+p_1-z-ze^{p_0-z}}. \end{aligned} \tag{2.3}$$

In order to show local exponential asymptotic stability of a fixed point z^* we must show $|f'(z^*)| < 1$.

From (2.3) we have that $z = 0$ is a fixed point and for positive p_0 and p_1, $g'(0) = e^{p_0+p_1} > 1$. Thus the zero fixed point is unstable. If we consider only positive fixed points of (2.2), using (2.3) we can get the following two equivalent conditions

$$\begin{aligned} e^{p_0+p_1-z-ze^{p_0-z}} &= 1, \quad \text{and} \\ p_0 + p_1 - z - ze^{p_0-z} &= 0. \end{aligned} \tag{2.4}$$

Using (2.4) in (2.3) we obtain the following expression,

$$g'(z) = (1-z)(1+z-(p_0+p_1)). \tag{2.5}$$

Figure 1 shows a contour plot of the derivative given by (2.5) as a function of z and the sum $p_0 + p_1$. The thick solid lines mark the line where $|g'(z)| = 1$. Note that on this plot the parameters p_0 and p_1 are not restricted by the conjecture and the values of z shown may not be fixed points of (2.2). The plot shows the derivative of the composite map (2.2) at values of z that are *candidates* for fixed points of (2.2). The graph shows that there is a region where the magnitude of the derivative of the composite map is less than unity.

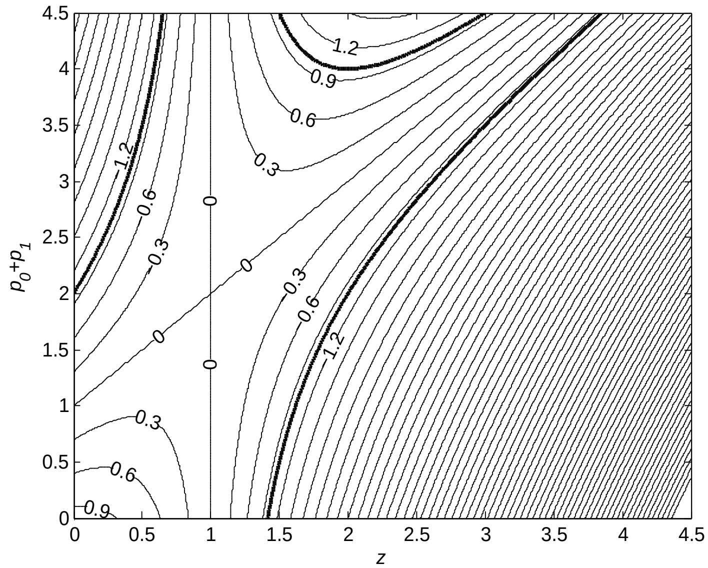

Figure 1: Level curves of the derivative of the composite map at candidates for a fixed point z

The right-hand side of (2.5) is a quadratic expression in z. If one bounds p_0 and p_1 by $0 < p_0+p_1 < 4$, and additionally constrains p_0 and p_1 with $0 < p_0 < 2+\epsilon_2$ and $0 < p_1 < 2+\epsilon_2$, then a direct computation shows that for $\epsilon_2 \approx 0.2845$ we have that $|g'(z)| < 1$, for all z that are fixed points of g in (2.3). Furthermore, $\epsilon_2 \approx 0.2845$ is approximately the largest value of ϵ_2 for which the conjecture holds with period two parameters.

Figure 2 shows a contour plot of level p_0 lines at the fixed points of (2.2) as a function of the sum $p_0 + p_1$. Here, as in Figure 1, the thick solid lines mark the boundary of stability (i.e., the line where $|g'(x)| = 1$). In contrast with Figure 1, here only values at actual fixed points are represented. For a known sum of the parameters $(p_0 + p_1)$ and a given p_0, Figure 2 can be used to determine the location of the fixed-point. With the location of the fixed point, Figure 1 can be used to determine the stability of that fixed point.

The following Lemma found in [11] and [12] will be used to show that the conjecture holds for the $k = 2$ case.

Lemma *Let z^* be a fixed point of a continuous map on the closed interval $I = [a, b]$. Then z^* is globally asymptotically stable relative to (a, b) if and only if $f^2(z) > z$ for $z < z^*$ and $f^2(z) < z$ for $z > z^*$ for all z in $(a, b)\backslash\{z^*\}$, and a, b are not periodic points.*

Recall that $g(z)$ in (2.3) is such that $g'(0) = e^{p_0+p_1} > 1$, and due to the exponential, $g(z)$ will tend to zero as z increases. By continuity, g will have at least one positive fixed point for all

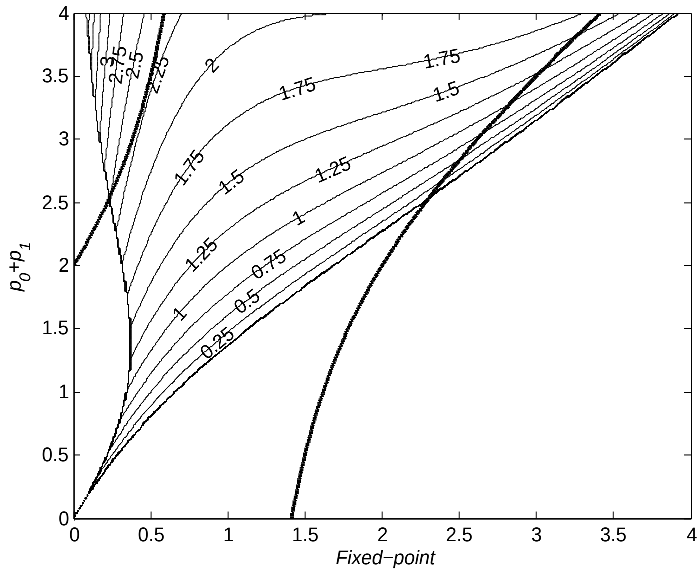

Figure 2: Level curves of p_0

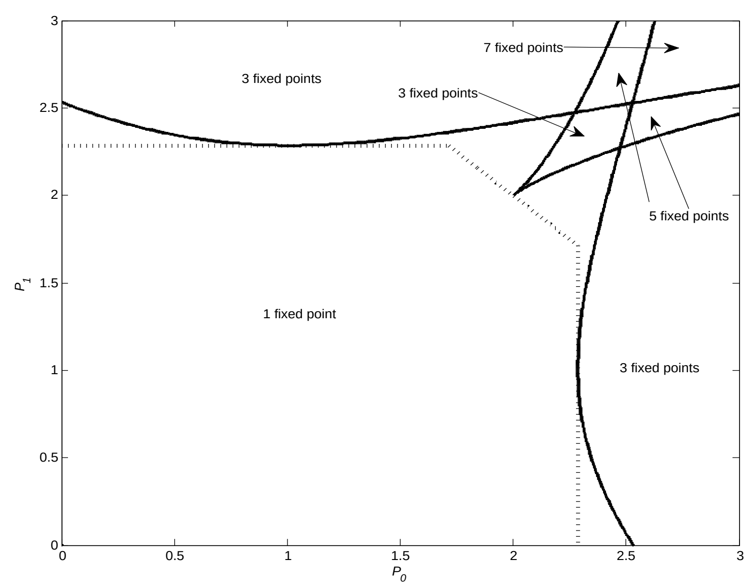

Figure 3: Number of positive fixed points of $g^2(z)$ as a function of p_0 and p_0

nonzero p_0 and p_1. The number of positive fixed points of the iterated map $g^2(z) = g(g(z))$ are shown on Figure 3 as a function of p_0 and p_0. The values shown on the figure are the ones that were found computationally. The solid lines represent the boundaries between the regions with different number of fixed points. The region enclosed by the dashed lines represents the region in parameter space where the assumptions of the conjecture hold.

It is clear that the region in which only one fixed point $g^2(z)$ occurs, that fixed point is the unique positive fixed point of $g(z)$. Note that $d(g^2(0))/dz = e^{2(p_0+p_1)} > 1$ so for values of z close to 0, we have that $g^2(z) > z$. In fact, for the region where only one fixed point occurs, by continuity, the inequality holds for $z < z^*$, where z^* is the positive fixed point of g and $z > 0$. Due to the exponential decay term of $g^2(z)$, for large values of z, we have that $g^2(z) < z$. Again, in particular, for the region where only one fixed point occurs, by continuity, the inequality holds for $z > z^*$. In [13] it was shown that there exists and interval $I = [a, b]$, with $a > 0$, which is invariant under the application of g and into which all points of $\mathbb{R}_0^+$ are mapped in a finite number of applications of g. The endpoints of the invariant interval are $b = \exp(\max(p_i) - 1)$ and $a = \min\{\min(p_i), b\exp(\min(p_i) - b)\}$. Hence by the Lemma 2, the region on Figure 3 where $g^2(z)$ has only one fixed point, corresponds to the region in parameter space (p_0, p_1) where the unique positive fixed points of g are globally asymptotically stable. This shows that the conjecture actually holds for $k = 2$. Figure 3 actually shows that the region in parameter space where (2.1) has a globally asymptotically stable orbit is larger than the one given in the conjecture.

In [9], Z. Zhou and X. Zou provide a condition (Theorem 3.2) on the parameter values for a Ricker type difference equation that guarantees the stability of periodic orbits. In the period-2 case, the condition can be used to establish a stability region in the p_0, p_1 parameter space. On Figure 4 the region labeled A is the stability region obtained by using the result from Zhou and Zou. Our approach yields a significantly larger stability region $A \bigcup B$ shown on Figure 4.

In general, for the periodic-k solutions of (2.1) we can consider

$$\begin{aligned} z_{n+1} &= g(z_n), \text{and} \\ g(z_n) &= (g_{k-1} \circ g_{k-2} \circ ... \circ g_1 \circ g_0)(z_n), \end{aligned} \tag{2.6}$$

with $g_i(z) = ze^{p_i - z}$, and $i = 0, 1, ..., k-1$. The fixed points of (2.6) give the period-k solutions of (2.1). In the next section we explore the conjecture numerically for period 2, 3 and 4 solutions.

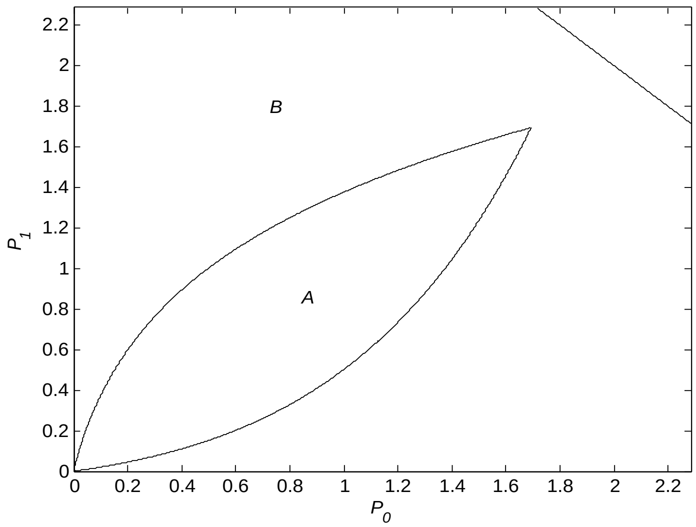

Figure 4: Stability regions of periodic orbits in the p_0, p_1 parameter space

3 Results from numerical simulations

Numerical simulations were conducted in order to estimate the values of ϵ_k for $k = 2, 3, 4$. For each k a large set of randomly selected parameters $p(0), p(1), ..., p(k-1)$ satisfying the conditions of the conjecture was selected. At every combination of the parameters the fixed point of (2.1) was computed together with the derivative at the fixed point. The largest possible values of ϵ_k for which all parameter combinations yielded a stable fixed point were selected. The numerical simulations gave as estimates $\epsilon_2 \approx 0.285$, $\epsilon_3 \approx 0.383$ and $\epsilon_4 \approx 0.15$

The $k = 2$ case. A large set of random values of the parameters p_0 and p_1 were used to numerically estimate the largest value of ϵ_2 that satisfies the conditions of the conjecture. The numerical simulations suggested a value of $\epsilon_2 \approx 0.285$. Using this value of ϵ_2 as guide, ϵ_2 was set to equal 0.28, and a large set of such parameters p_0 and p_1, satisfying the conjecture, were randomly chosen. The fixed point and the derivative of the composite map at the fixed point were computed for each case of the randomly selected parameters p_0 and p_1.

Figure 5 shows a plot of the fixed point in terms of the sum of the parameters $p_0 + p_1$ for the above-mentioned set of randomly chosen parameters. The solid lines correspond to the points where the magnitude of the derivative of the composite map at the fixed point is unity. The general shape of the region on Figure 5 is very similar to the one on Figure 2. The main difference between the plots is that on Figure 2 values of fixed points corresponding to all possible parameter combinations are shown, including values with $\epsilon_2 > 0.285$, while

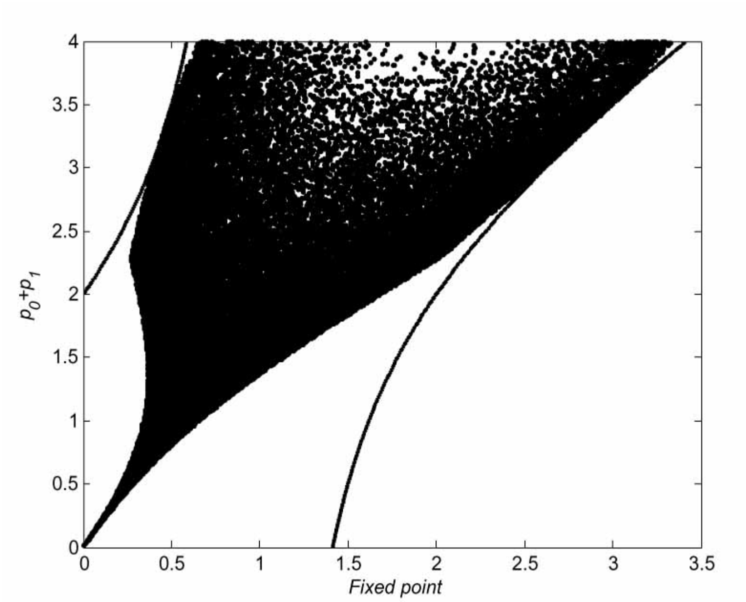

Figure 5: Sum of the parameter values $(p_0 + p_1)$ in terms of the fixed point

on Figure 5 only parameter combinations that satisfy the conjecture are shown.

Figure 6 shows a plot of the derivative at the fixed point in terms of the sum of the parameters $p_0 + p_1$ for the above-mentioned set of randomly chosen parameters. The figure shows that for the chosen set of parameters that satisfy the conjecture, the magnitude of the derivative of the composite map evaluated at the fixed point of the composite map (2.2) is always less than unity, numerically confirming the assertion given in the conjecture.

The $k = 3$ case.

Figure 7 shows a plot of of the sum $p_0 + p_1 + p_2$ in terms of the derivative of the composite map evaluated at the fixed point of (2.6) for a set of randomly chosen parameters, and corresponds to Figure 6 in the $k = 3$ case. For the simulations a large set of parameter values that satisfy the conditions of the conjecture with $\epsilon_3 = 0.383$ were chosen.

The magnitude of the derivative at the fixed point approaches one when the sum $p_0+p_1+p_2$ is close to 4 and 5. On the other hand, if the sum of the parameters is fixed at a large value of say, $p_0 + p_1 + p_2 = 5.75$, one can observe that the derivative of the composite map evaluated at the fixed point is relatively smaller than one. This suggests that one may be able to choose one of the $p_i = 2 + \epsilon > 2 + \epsilon_3$ and still have a stable periodic solution. This is actually the case. Figure 8 shows a plot of the derivative at the fixed point of the composite map (2.6) as a function of p_1, for various values of ϵ. Here p_0 was taken to be $p_0 = 2+\epsilon$, and $p_0+p_1+p_2 = 5.75$. As previously observed, the plot suggests that it is possible

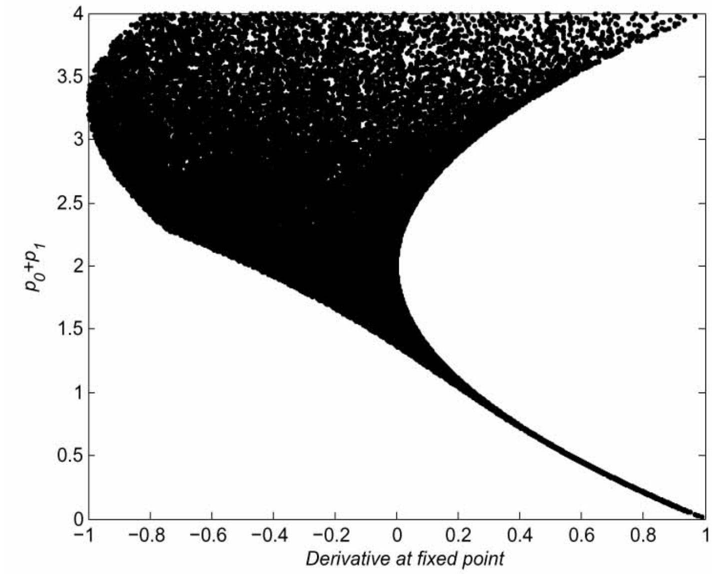

Figure 6: Sum of the parameter values $(p_0 + p_1)$ in terms of the derivative at the fixed point

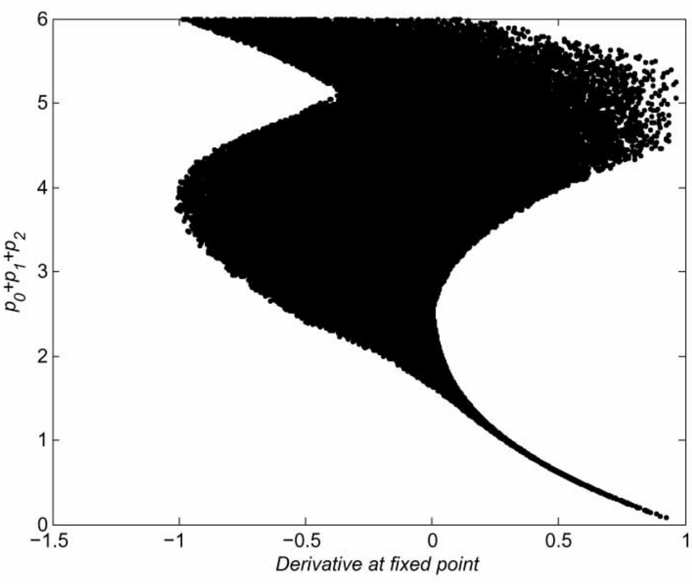

Figure 7: Sum of the parameter values $(p_0 + p_1 + p_2)$ in terms of the derivative at the fixed point

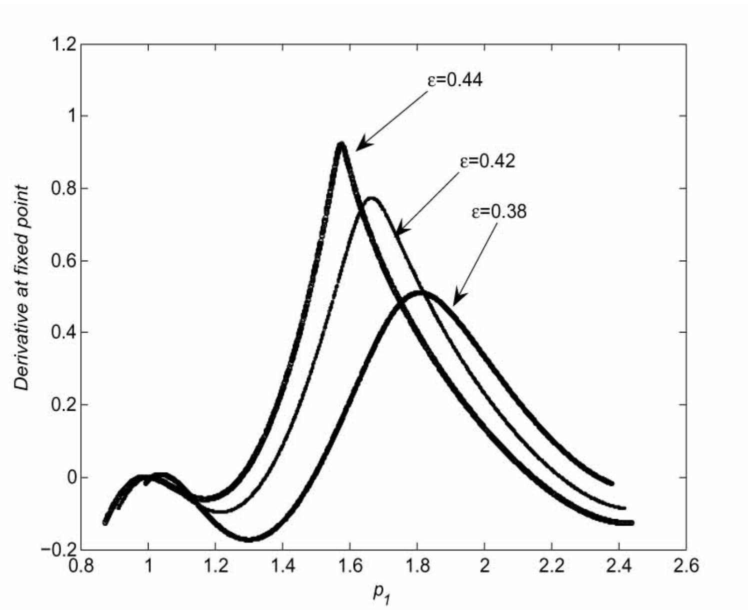

Figure 8: Derivative at the fixed point as a function of p_1 for various values of ϵ

to obtain stable solutions at some combinations of parameters where the conditions of the conjecture are not satisfied. In this example p_0 was allowed to exceed the values allowed by the conjecture. The example suggests that by appropriately *increasing* a parameter it may be possible to change from an unstable solution to a stable solution. This behavior is also suggested in the $k = 2$ case. Figures 2 and 6 show that for values of $p_0 + p_1$ slightly less than 4 it is possible to chose one of the parameters with a value exceeding $2 + \epsilon_2$ and still obtain a stable solution.

4 Conclusions

A conjecture about the stability of periodic solutions of the Ricker equation with periodic parameters is given. For the special case of period-two parameters the conjecture is analytically shown to be valid. The region in parameter space (for period-two parameters) given by the conjecture where the periodic solutions of the equation are asymptotically stable is compared with the region obtained by a result from Z. Zhou and X. Zou [9]. The stability region obtained by using the conjecture encompasses the one obtained in [9], and it actually is significantly larger. It is also larger than that obtained in [13]. In addition, in the period-two case we find numerically the maximal parameter region in which the equation has a globally asymptotically stable periodic solution.

Results from numerical explorations for parameters with period-three suggest that in parameter space, it may be possible to attain stability for parameter values in regions larger than the ones conjectured. The simulations also suggest that by appropriately increasing a parameter it may be possible to move from a region of unstable solutions to one with stable solutions.

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