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RESEARCH ARTICLE

A Note: An Invariance Theorem for Mappings

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If a continuous mapping f carries the boundary of a set D into the closure of D it may not be true that f maps D into its closure, even if f is injective on D. Examples are discussed and conditions are given under which it is true. An simplified application is given to a Biological migration-selection model.

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1. Introduction

The study of long time behavior of solutions of difference equations is the study of iterations of mappings $f : X \to X$. In some cases, the first step in showing the existence of a *globally* attracting fixed point is to show that for each $x \in X$ there is a positive integer k such that $f^k(x) \doteq f \circ f \circ \cdots \circ f(x)$ (k-times) lies in some compact set K, the next step being to show that K is *invariant* under the action of f, $f(K) \subset K$, i.e. the mapping is *dissipative*. Finally, using the compactness and other properties, show the existence of a fixed point that attracts points of K. Only the invariance issue will be discussed here.

For an autonomous ordinary differential equation in $\mathbb{R}^n$ the invariance problem is solved by showing that the vector field on the boundary of K is never pointing into the complement, $K^C = \mathbb{R}^n \backslash K$. For mappings the problem is a bit more tricky since points of K can be mapped outside K even in the case f is injective (one-to-one), see examples in Section 3.

Many interesting difference equations give rise to mappings that are not injective when considered on their total domain of definition. Nevertheless, under certain reasonable conditions sub-domains can be found where the mapping is injective and indeed the Theorem given below applies, viz. the example in $\mathbb{R}^2$ given in Section 4 of a simplified migration-selection model.

2. The Invariance Theorem

By *invariance* here we always mean forward invariance, i.e. D is invariant under the action of f if $f(D) \subset D$. For a set $D \subset \mathbb{R}^n$, let $\overline{D}$ denote the closure of D, $\mathring{D}$ the interior of D, $\partial D = \overline{D} \backslash \mathring{D}$ the boundary of D and $D^C = \mathbb{R}^n \backslash D$, the complement of D. By a *path* we mean a homeomorphic image of $[0, 1]$.

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Theorem 2.1 :

Let $D \subset \mathbb{R}^n$ be a bounded subset and $f : \overline{D} \to \mathbb{R}^n$ continuous. Suppose $f : \mathring{D} \to \mathbb{R}^n$ is injective (one-to-one) and $f(\partial D) \subset \overline{D}$.

If $\overline{D}^C = \mathbb{R}^n \backslash \overline{D}$ has no bounded components, then $f(\overline{D}) \subset \overline{D}$.

Remarks:

(1) The function f, while continuous on $\overline{D}$ is assumed to be injective only on the interior of D.

(2) For $n \geq 2$ the "no bounded components" assumption on $\overline{D}^C$ together with D bounded imply $\overline{D}^C$ is connected and in fact path connected.

Proof: For $n = 1$ the conditions imply D is an interval I and the theorem follows easily from the Intermediate Value Theorem. Under the mapping the left end point moves to the right and the right endpoint to the left. Injectivity implies f is strictly monotonic and therefore its range lies in $[f(b), f(a)]$ or $[f(a), f(b)]$, where $a \leq b$ are the endpoints of the interval I.

For $n \geq 2$ assume the theorem is not true. Then $G \doteq f(\mathring{D}) \cap \overline{D}^C \neq \varnothing$. By Brouwer's Invariance of Domain Theorem ([1], [2]) $f|\mathring{D}$ is an open map (and hence a homeomorphism) onto $f(\mathring{D})$. Thus G is open. Let $\mathcal{B}$ be an open ball about the origin so large that $\overline{D} \subset \mathcal{B}$.

Let $y \in G$ and from path connectedness of $\overline{D}^C$, let $\gamma : [0, 1] \to \overline{D}^C$ be a path with $\gamma(0) = y$ and $\gamma(1) \in \partial\mathcal{B}$. Next define $t_0 = \min\{t \in [0, 1] \mid \gamma(t) \in \partial G\}$ and set $\eta = \gamma(t_0) \in \gamma \cap \partial G$.

Now choose a sequence $\eta_j \in G \cap \gamma$ such that $\eta_j \to \eta$. Then $x_j = f^{-1}(\eta_j) \in \mathring{D}$. By compactness of $\overline{D}$ there is a subsequence, that we again call x_j that converges, $x_j \to x$. If $x \in \partial\mathring{D}$ then $\eta \in f(\partial\mathring{D}) \subset f(\partial D) \subset \overline{D}$. But $\eta \in \gamma$ and thus $\eta \in \overline{D} \cap \gamma = \varnothing$, a contradiction.

Thus, $x \in \mathring{D}$ and $\eta = f(x) \in f(\mathring{D}) \cap \gamma$. Now $\gamma \subset \overline{D}^C$ so that $\eta \in f(\mathring{D}) \cap \gamma \cap \overline{D}^C = \gamma \cap G$, and G is an open subset of $\mathbb{R}^n$. But $\eta \in \gamma \cap \partial G$, another contradiction and thus the proof is complete. ■

3. A Discussion of the Hypotheses

We next construct counterexamples after dropping each hypothesis one at a time, while keeping the other hypotheses intact.

(A). If the set D is unbounded, the conclusion of the Theorem is false without further assumptions. To see this let $D = (0, \infty)$ and define $f : \overline{D} \to \mathbb{R}$ by $f(x) = -x$.

(B). If $\overline{D}^C$ is allowed to have a bounded component then let $D = (0, 1) \cup (2, 3)$ and

$$f(x) = \begin{cases} 0.5x & x \in [0, 1] \\ 2x - 3 & x \in [2, 3] \end{cases}$$

and again the conclusion of the Theorem is false since $f(\overline{D}) = [0, 0.5] \cup [1, 3]$.

(C). If we replace $\mathbb{R}^n$ by a path connected metric space X then for $X =$ the unit circle in $\mathbb{R}^2$, let

$$D = \{(x, y) \in \mathbb{R}^2 \mid y = \sqrt{1 - x^2},\ |x| \le 1\}, \qquad \text{the upper half-circle,}$$

and define $f : \overline{D} \to X$ by $f(x, y) = (x, -y)$ and again the conclusion of the Theorem is false.

(D). The injective assumption cannot be dropped due to the quadratic map f on $D \doteq [0, 1]$, where

$$f(x) = ax(1 - x), \qquad a > 4,$$

since $f(\frac{1}{2}) > 1$.

(E). The following rather bizarre example (a sliding window) shows that requiring only $\overline{\mathring{D}^C}$ to be connected is not sufficient (see Remark 2). Let $D = D_1 \cup L_1 \cup L_2 \cup L_3 \subset \mathbb{R}^2$ and $f : \overline{D} \to \mathbb{R}^2$ be defined as follows,

$$\begin{array}{lll} D_1: & 0 \le x \le 1, \quad 0 \le y \le 1, & (x, y) \mapsto (2x, y) \\ L_1: & 1 \le x \le 2, y = 1, & (x, 1) \mapsto (2, 1) \\ L_2: & x = 2, 0 \le y \le 1, & (2, y) \mapsto (2, y) \\ L_3: & 1 \le x \le 2, y = 0, & (x, 0) \mapsto (2, 0). \end{array}$$

It is easily checked that f satisfies all the remaining conditions, but $f(\overline{D}) \not\subseteq \overline{D}$. Note that f is not injective on the boundary, but even requiring that does not help as the next example shows.

(F). In $\mathbb{R}$ define $D = [2, 3] \cup \{\frac{1}{k}, k = 1, 2, \ldots\} \cup \{0\}$, and $f(0) = 0, f(\frac{1}{k}) = \frac{1}{k+1}$ and on $[2, 3]$ let f be linear with $f(3) = 3, f(2) = 1$.

4. An Application

The following is a simplified version of a migration-selection model to be considered further in a forthcoming study. Consider the following mapping $f : \mathbb{R}^2_+ \to \mathbb{R}^2_+$,

$$f_i(x_1, x_2) = \frac{(1 + a_i)x_i}{1 + b_i(x_1 + x_2)}\phi(x_1 + x_2), \qquad a_i > 0,\ b_i > 0, \quad i = 1, 2, \tag{1}$$

where $\frac{a_1}{b_1} > \frac{a_2}{b_2}$ and for *fixed* $\tau > \frac{a_1}{b_1}$,

$$\phi(s) = \begin{cases} 1, & 0 \le s \le \tau \\ e^{-(s-\tau)}, & s > \tau. \end{cases}$$

It is easily verified that $P_1 = (\frac{a_1}{b_1}, 0)$ and $P_2 = (0, \frac{a_2}{b_2})$ are fixed points, the first being asymptotically stable, the second unstable for the difference equation

$$x(t + 1) = f(x(t)). \tag{2}$$

We wish to show that P_1 *globally* attracts all orbits of (2) having initial conditions in the *open* first quadrant. Define $S(t) = x_1(t) + x_2(t)$.

The following are easily established:

(a) $S(t) > \dfrac{a_1}{b_1} \implies S(t+1) < S(t)$,

(b) $S(t) = \dfrac{a_1}{b_1} \implies S(t+1) \leq S(t)$, with " = " $\iff x_2(t) = 0$,

(c) $S(t) < \dfrac{a_2}{b_2} \implies S(t+1) > S(t)$,

(d) $S(t) = \dfrac{a_2}{b_2} \implies S(t+1) \geq S(t)$, with " = " $\iff x_1(t) = 0$.

Next consider the open trapezoidal region D in the first quadrant bounded by the two parallel lines $L_1 : S = \dfrac{a_1}{b_1}$, $L_2 : S = \dfrac{a_2}{b_2}$ and the axes. By (a) and (c), each point in the open first quadrant is mapped into $\overline{D}$ by sufficiently many iterations of f. By (b) and (d), L_1 and L_2 map into $\overline{D}$, while the bounding segments in the axes are themselves invariant. Thus $f(\partial D) \subset \overline{D}$.

It only remains to show that f is injective on D. Assume $f(x_1, x_2) = f(\xi_1, \xi_2) = (u_1, u_2)$, and define $\Sigma = \xi_1 + \xi_2$ and $\mathcal{U} = \dfrac{u_1}{(1+a_1)} + \dfrac{u_2}{(1+a_2)}$. From

$$\frac{(1+a_i)x_i}{1+b_i S} = \frac{(1+a_i)\xi_i}{1+b_i \Sigma} = u_i, \quad i = 1, 2, \tag{3}$$

it follows that

$$\begin{aligned} (1+a_1)x_1 &= u_1 + u_1 b_1 S, & (1+a_2)x_2 &= u_2 + u_2 b_2 S \\ (1+a_1)\xi_1 &= u_1 + u_1 b_1 \Sigma & (1+a_2)\xi_2 &= u_2 + u_2 b_2 \Sigma\ . \end{aligned}$$

Dividing by the $1 + a_i$, adding horizontally and setting $p = \dfrac{u_1 b_1}{1+a_1} + \dfrac{u_2 b_2}{1+a_2}$,

$$S = \mathcal{U} + pS \quad \text{and} \quad \Sigma = \mathcal{U} + p\Sigma.$$

Solving for $\mathcal{U}$, $\mathcal{U} = (1-p)S = (1-p)\Sigma$.

Since $f : \mathbb{R}^2_+ \to \mathbb{R}^2_+$, $\mathcal{U} = 0$ is impossible and therefore $p \neq 1$, from which it follows that $S = \Sigma$. Finally from (3), $x_1 = \xi_1$ and $x_2 = \xi_2$ and thus f is injective and Theorem 2.1 tells us $f(\overline{D}) \subset \overline{D}$.

It remains to show points in $\overline{D}$ are attracted to $(a_1/b_1, 0)$. This issue and others for a more general form of (1) will be addressed in a forthcoming manuscript.

Remarks:

(3) It was pointed out to the author by Christian Pötzsche that the Invariance of Domain Theorem and hence Theorem 2.1 can be extended to Banach space if D is assumed relatively compact and $f \mid \mathring{D}$ is continuous, locally a compact perturbation of the identity and injective, [2, p. 705].

References

[1] James Dugundji. *Topology.* Allyn and Bacon, Boston, 1966.

[2] Eberhard Zeidler. *Nonlinear Functional Analysis and its Applications*, volume 1. Springer Verlag, 2nd corrected printing edition, 1986.